

Numerical Stochastic Perturbation Theory (NSPT)

Francesco Di Renzo



UNIVERSITÀ DI PARMA



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A (by now) quite long story
started with Pino and Enrico as supervisors

In early 90's Parma was provided an APE100 prototype (tubo). Only a few Gflops computing power, but something very intriguing at that time (well... by now a piece of cake...)

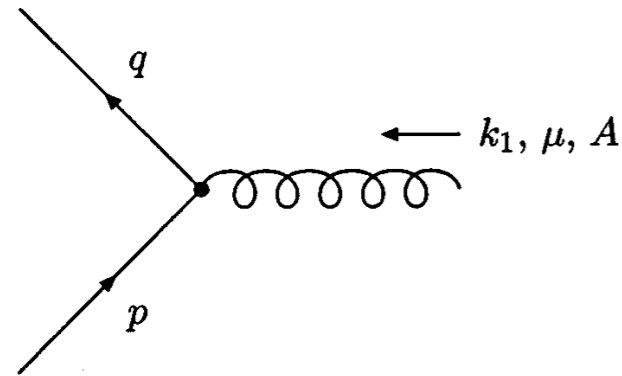
QUESTION: what to do? From my point of view: I was starting my PhD

Pino and Enrico were very keen on numerically implementing Stochastic Perturbation Theory.

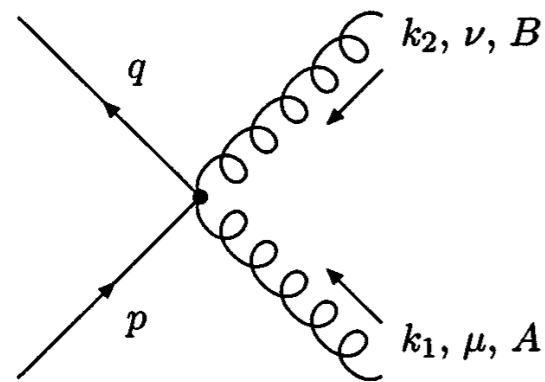
From my point of view: the start of a career in research, under the supervision of people who have always been firm reference points.

An invitation

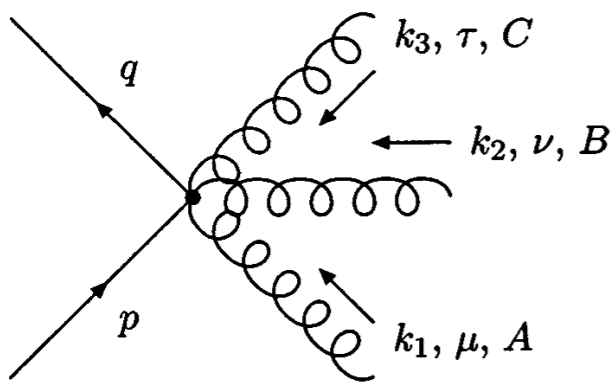
Perturbation Theory (PT) is nothing less than ubiquitous in Field Theory. In principle the lattice is a regulator among the others ... in practice it is a dreadful one so that when it comes to compute something in **Lattice Perturbation Theory** (LPT) you will probably start to get nervous ...



(a)



(b)



(c)

In particular for **LGT**:

lot of **vertices** (not given once and for all)

Sums and/or integrals ... a lot of **trigonometrics** ...

A variety of **actions** (both for glue and for quarks)

and as an extra bonus ... often **bad convergence** properties

$$V_{1\mu}^A(p, q) = -gT^A \left[i\gamma_\mu \cos\left(\frac{p_\mu + q_\mu}{2}\right) + r \sin\left(\frac{p_\mu + q_\mu}{2}\right) \right]$$

$$V_{c1\mu}^A(p, q) = -gT^A c_{sw} \frac{r}{2} \sum_\nu \sigma_{\mu\nu} \times \cos\left(\frac{p_\mu - q_\mu}{2}\right) \sin(p_\nu - q_\nu)$$

$$V_{2\mu\nu}^{AB}(p, q) = \frac{a}{2} g^2 \frac{1}{2} \{T^A, T^B\} \delta_{\mu\nu} \times \left[i\gamma_\mu \sin\left(\frac{p_\mu + q_\mu}{2}\right) - r \cos\left(\frac{p_\mu + q_\mu}{2}\right) \right]$$

$$V_{c2\mu\nu}^{AB}(p, q, k_1, k_2) = -\frac{a}{2} g^2 i f_{ABC} T^C c_{sw} \frac{r}{4} \left\{ \sigma_{\mu\nu} \left[4 \cos\left(\frac{k_{1\nu}}{2}\right) \cos\left(\frac{k_{2\mu}}{2}\right) \cos\left(\frac{q_\mu - p_\mu}{2}\right) \times \cos\left(\frac{q_\nu - p_\nu}{2}\right) - 2 \cos\left(\frac{k_{1\mu}}{2}\right) \cos\left(\frac{k_{2\nu}}{2}\right) \right] + \delta_{\mu\nu} \sum_\rho \sigma_{\mu\rho} \sin\left(\frac{q_\mu - p_\mu}{2}\right) [\sin(k_{2\rho}) - \sin(k_{1\rho})] \right\}$$

$$V_{3\mu\nu\tau}^{ABC}(p, q) = \frac{a^2}{6} g^3 \frac{1}{6} [T^A \{T^B, T^C\} + T^B \{T^C, T^A\} + T^C \{T^A, T^B\}] \delta_{\mu\nu} \delta_{\mu\tau} \left[i\gamma_\mu \cos\left(\frac{p_\mu + q_\mu}{2}\right) + r \sin\left(\frac{p_\mu + q_\mu}{2}\right) \right],$$

$$V_{c3\mu\nu\tau}^{ABC}(p, q, k_1, k_2, k_3) = -3ig^3 \frac{a^2}{6} c_{sw} r \left\{ T^A T^B T^C \delta_{\mu\nu} \delta_{\mu\tau} \sum_\rho i\sigma_{\mu\rho} \left[-\frac{1}{6} \cos\left(\frac{q_\mu - p_\mu}{2}\right) \sin(q_\rho - p_\rho) + \cos\left(\frac{q_\mu - p_\mu}{2}\right) \cos\left(\frac{q_\rho - p_\rho}{2}\right) \cos\left(\frac{k_{3\rho} - k_{1\rho}}{2}\right) \sin\left(\frac{k_{2\rho}}{2}\right) \right] - \frac{1}{2} [T^A T^B T^C + T^C T^B T^A] i\sigma_{\mu\nu} \left[\delta_{\nu\tau} 2 \cos\left(\frac{q_\mu - p_\mu}{2}\right) \cos\left(\frac{q_\nu - p_\nu}{2}\right) \cos\left(\frac{k_{3\mu} + k_{2\mu}}{2}\right) \sin\left(\frac{k_{1\nu}}{2}\right) + \delta_{\nu\tau} \sin\left(\frac{k_{3\nu} + k_{2\nu}}{2}\right) \cos\left(\frac{k_{1\mu} + k_{2\mu}}{2}\right) + \delta_{\mu\tau} \sin\left(\frac{k_{1\mu} + 2k_{2\mu} + k_{3\mu}}{2}\right) \cos\left(\frac{q_\nu - p_\nu}{2}\right) \cos\left(\frac{k_{3\nu} - k_{1\nu}}{2}\right) \right] \right\}$$

FIG. 1. Momentum assignments for the quark-antiquark-gluon vertices.

Despite this ...



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NUCLEAR
PHYSICS B

Renormalons from eight-loop expansion of the gluon condensate in lattice gauge theory[★]

F. Di Renzo^a, E. Onofri^a, G. Marchesini^b

^a *Dipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, Italy*

^b *Dipartimento di Fisica, Università di Milano and INFN, Sezione di Milano, Italy*

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Compelling Evidence of Renormalons in QCD from High Order Perturbative Expansions

Clemens Bauer,¹ Gunnar S. Bali,¹ and Antonio Pineda²

¹*Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*

²*Grup de Física Teòrica, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain*

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We compute the static self-energy of SU(3) gauge theory in four spacetime dimensions to order α^{20} in the strong coupling constant α . We employ lattice regularization to enable a numerical simulation within the framework of stochastic perturbation theory. We find perfect agreement with the factorial growth of high order coefficients predicted by the conjectured renormalon picture based on the operator product expansion.

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Agenda

- Basics of Stochastic Quantization and Stochastic Perturbation Theory
- From Stochastic Perturbation Theory to NSPT (moving straight to LGT)
- Stochastic Gauge Fixing
- Fermionic loops in NSPT
- A few different frameworks for NSPT (i.e. a few handles to possibly improve it)
- A canonical application: renormalization constants
- Something maybe more field-theoretic (numerics stumbles on fundamental QFT...)
 - a. Renormalons
 - b. Resurgence?
- (*) Of course I had to make a selection, with Pino on my mind
- Conclusions

Basics of Stochastic Quantization and Stochastic Perturbation Theory

Basics of Stochastic Quantization and Stochastic Perturbation Theory

You start with a field theory you want to solve

$$\langle O[\phi] \rangle = \frac{\int D\phi O[\phi] e^{-S[\phi]}}{\int D\phi e^{-S[\phi]}}$$

Parisi-Wu, *Sci. Sinica* 24 (1981) 35, Damgaard-Huffel, *Phys Rept* 152 (1987) 227

You now want an extra degree of freedom which you will think of as a **stochastic time** in which an evolution takes place according to the **Langevin equation**

$$\phi(x) \mapsto \phi_\eta(x; t)$$

$$\frac{d\phi_\eta(x; t)}{dt} = -\frac{\partial S[\phi]}{\partial \phi_\eta(x; t)} + \eta(x; t)$$

The **drift term** is given by the **equations of motion**...

... but beware! This is a stochastic differential equation due to the presence of the **gaussian noise**

$$\eta(x; t) : \quad \langle \eta(x, t) \eta(x', t') \rangle_\eta = 2 \delta(x - x') \delta(t - t')$$

Noise expectation values are now naturally defined

$$\langle \dots \rangle_\eta = \frac{\int D\eta(z, \tau) \dots e^{-\frac{1}{4} \int dz d\tau \eta^2(z, \tau)}}{\int D\eta(z, \tau) e^{-\frac{1}{4} \int dz d\tau \eta^2(z, \tau)}}$$

The key assertion of Stochastic Quantization can be now simply stated

$$\langle O[\phi_\eta(x_1; t) \dots \phi_\eta(x_n; t)] \rangle_\eta \xrightarrow{t \rightarrow \infty} \langle O[\phi(x_1) \dots \phi(x_n)] \rangle$$

A conceptually simple proof comes from the **Fokker Planck equation** formalism

$$\langle O[\phi_\eta(t)] \rangle_\eta = \frac{\int D\eta O[\phi_\eta(t)] e^{-\frac{1}{4} \int dz d\tau \eta^2(z, \tau)}}{\int D\eta e^{-\frac{1}{4} \int dz d\tau \eta^2(z, \tau)}} = \int D\phi O[\phi] P[\phi, t]$$

$$\dot{P}[\phi, t] = \int dx \frac{\delta}{\delta\phi(x)} \left(\frac{\delta S[\phi]}{\delta\phi(x)} + \frac{\delta}{\delta\phi(x)} \right) P[\phi, t]$$

Floratos-Iliopoulos, Nucl.Phys. B 214 (1983) 392

for the solution of which we can introduce a **perturbative expansion** which generates a **hierarchy of equations**

$$P[\phi, t] = \sum_{k=0} g^k P_k[\phi, t]$$

Leading order is easy to solve and admits an infinite time (equilibrium) limit such that

$$P_0[\phi, t] \xrightarrow{t \rightarrow \infty} P_0^{eq}[\phi] = \frac{e^{-S_0[\phi]}}{Z_0}$$

In a convenient weak sense at every order one gets equilibrium $P_k[\phi, t] \xrightarrow{t \rightarrow \infty} P_k^{eq}[\phi]$

in terms of quantities which are interrelated by a set of relations in which one recognizes the **Schwinger-Dyson** equations ... i.e. we are done!

We want to go via another expansion, i.e. the **expansion of the solution of Langevin equation** in power of the **coupling constant**

$$\phi_\eta(x; t) = \phi_\eta^{(0)}(x; t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x; t)$$

Parisi-Wu, Damgaard-Huffel

Langevin equation for the free scalar field (momentum space) $\frac{\partial}{\partial t} \phi_{\eta}^{(0)}(k, t) = -(k^2 + m^2) \phi_{\eta}^{(0)}(k, t) + \eta(k, t)$

Look for **(propagator)** $\phi(k, t) = \int_0^t d\tau G(k, t - \tau) \eta(k, \tau)$ $\frac{\partial}{\partial t} G^{(0)}(k, t) = -(k^2 + m^2) G^{(0)}(k, t) + \delta(t)$

i.e. $G^{(0)}(k, t) = \theta(t) \exp(-(k^2 + m^2)t)$

$$\phi^{(0)}(k, t) = \phi^{(0)}(k, 0) \exp(-(k^2 + m^2)t) + \int_0^t d\tau \exp(-(k^2 + m^2)(t - \tau)) \eta(k, \tau)$$

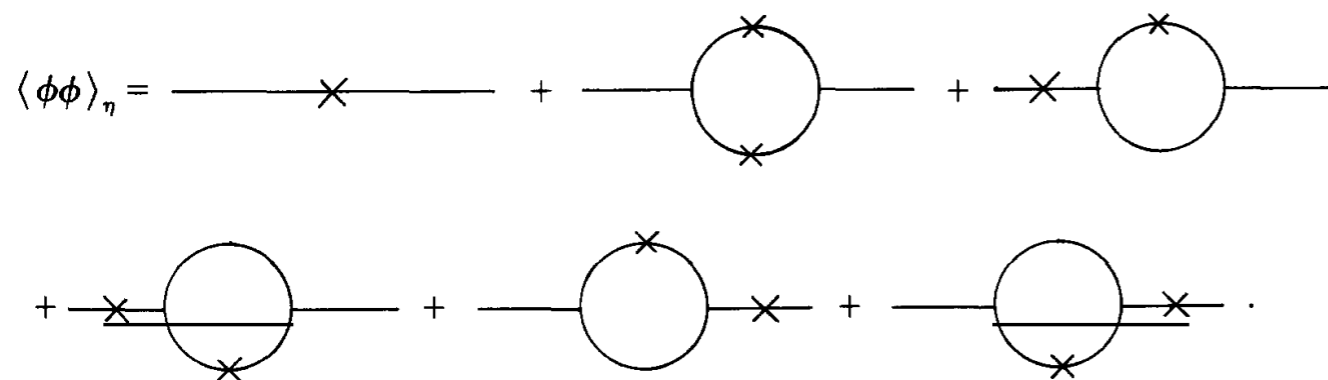
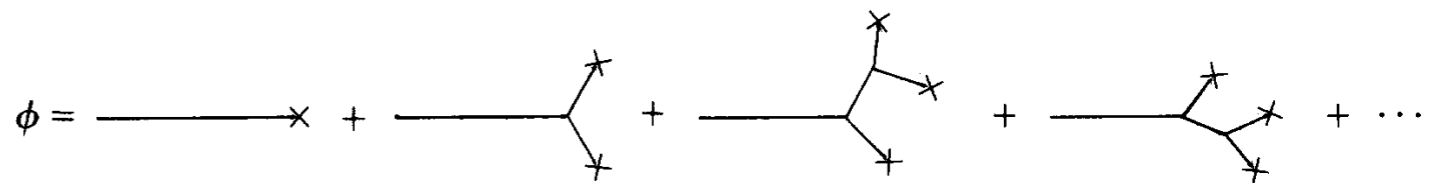
Interacting case (cubic interaction in the following) is solved by **superposition** ...

$$\phi(k, t) = \int_0^t d\tau \exp(-(k^2 + m^2)(t - \tau)) \left[\eta(k, \tau) - \frac{\lambda}{2!} \int \frac{dpdq}{(2\pi)^{2n}} \phi(p, \tau) \phi(q, \tau) \delta(k - p - q) \right]$$

... which leaves the solution in a form which is ready for **iteration**. It is actually also ready for a graphical interpretation and for the formulation of a

diagrammatic Stochastic Perturbation Theory

$$\phi = \int G\eta - \frac{\lambda}{3!} \int \int \int \int G(G\eta)(G\eta) + \dots$$



The **stochastic diagrams** one obtains when averaging over the noise (contractions!) reconstruct, in a convenient **infinite time** limit, the contributions of the (topologically) correspondent **Feynman diagrams** ...

but we do not want to go this way ...

From Stochastic Perturbation Theory to NSPT (directly for LGT...)

Stochastic Quantization for LGT Batrouni et al (Cornell group) PRD 32 (1985)

We now start with the Wilson action $S_G = -\frac{\beta}{2N_c} \sum_P \text{Tr} (U_P + U_P^\dagger)$

We now deal with a theory formulated in terms of group variables and Langevin equation reads

$$U_{\mu x} = e^{A_\mu(x)}$$

$$\frac{\partial}{\partial t} U_{x\mu}(t; \eta) = (-i\nabla_{x\mu} S_G[U] - i\eta_{x\mu}(t)) U_{x\mu}(t; \eta)$$

where the Lie derivative is in place

$$\nabla_{x\mu} = T^a \nabla_{x\mu}^a = T^a \nabla_{U_{x\mu}}^a \quad \nabla_V^a f(V) = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} (f(e^{i\alpha T^a} V) - f(V))$$

This is again a stochastic differential equation with (gaussian) noise averages satisfying

$$\lim_{t \rightarrow \infty} \langle O[U(t; \eta)] \rangle_\eta = \frac{1}{Z} \int DU e^{-S_G[U]} O[U]$$

In order to proceed we now need a (numerical) integration scheme to simulate, e.g. Euler

$$U_{x\mu}(n+1; \eta) = e^{-F_{x\mu}[U, \eta]} U_{x\mu}(n; \eta)$$

$$F_{x\mu}[U, \eta] = \epsilon \nabla_{x\mu} S_G[U] + \sqrt{\epsilon} \eta_{x\mu}$$

$$F_{x\mu}[U, \eta] = \frac{\epsilon\beta}{4N_c} \sum_{U_P \supset U_{x\mu}} \left[(U_P - U_P^\dagger) - \frac{1}{N_c} \text{Tr} (U_P - U_P^\dagger) \right] + \sqrt{\epsilon} \eta_{x\mu}$$

$$\langle \eta_{i,k}(z) \eta_{l,m}(w) \rangle_\eta = \left[\delta_{il} \delta_{km} - \frac{1}{N_c} \delta_{ik} \delta_{lm} \right] \delta_{zw}$$

Now we look for a **solution** in the form of a **perturbative expansion**

$$U_{x\mu}(t; \eta) \rightarrow 1 + \sum_{k=1} \beta^{-k/2} U_{x\mu}^{(k)}(t; \eta)$$

then we **plug it** into the (numerical scheme!) **Langevin equation** and get a **hierarchy of equations!**

$$U^{(1)'} = U^{(1)} - F^{(1)}$$

$$U^{(2)'} = U^{(2)} - F^{(2)} + \frac{1}{2} F^{(1)2} - F^{(1)} U^{(1)}$$

$$U^{(3)'} = U^{(3)} - F^{(3)} + \frac{1}{2} (F^{(2)} F^{(1)} + F^{(1)} F^{(2)}) - \frac{1}{3!} F^{(1)3} - (F^{(2)} - \frac{1}{2} F^{(1)2}) U^{(1)} - F^{(1)} U^{(2)}$$

...

In practice: we do not look closely at the (underlying) Stochastic Perturbation Theory because the computer is going to (numerically) take care of it and all that you are interested in are the **observables**, for which

$$\langle O[\sum_k g^k \phi_\eta^{(k)}(t)] \rangle_\eta = \sum_k g^k \langle O_k(t) \rangle_\eta \quad \lim_{t \rightarrow \infty} \langle O_k(t) \rangle_\eta = \lim_{T \rightarrow \infty} 1/T \sum_{j=1}^T O_k(jn)$$

Beware! Lattice PT is (always!) a **decompactification** of lattice formulation, so that ultimately one should be able to make contact with the **continuum Langevin equation**, i.e.

$$\frac{\partial}{\partial t} A_\mu^a(\eta, x; t) = D_\nu^{ab} F_{\nu\mu}^b(\eta, x; t) + \eta_\mu^a(x; t)$$

Where has this gone?

We did not lose anything, since we can always **think** of all this **in the algebra**

$$A_{x\mu}(t; \eta) \rightarrow \sum_{k=1} \beta^{-k/2} A_{x\mu}^{(k)}(t; \eta)$$

$$A = \log(U) = \log \left(1 + \sum_{k>0} \beta^{-\frac{k}{2}} U^{(k)} \right)$$

$$= \frac{1}{\sqrt{\beta}} U^{(1)} + \frac{1}{\beta} \left(U^{(2)} - \frac{1}{2} U^{(1)2} \right) + \left(\frac{1}{\beta} \right)^{\frac{3}{2}} \left(U^{(3)} - \frac{1}{2} \left(U^{(1)} U^{(2)} + U^{(2)} U^{(1)} \right) + \frac{1}{3} U^{(1)3} \right) + \dots$$

$$= \frac{1}{\sqrt{\beta}} A^{(1)} + \frac{1}{\beta} A^{(2)} + \left(\frac{1}{\beta} \right)^{\frac{3}{2}} A^{(3)} + \dots$$

$$A^{(k)\dagger} = -A^{(k)} \quad \text{Tr} A^{(k)} = 0 \quad \forall k$$

and the (expanded) Langevin equation now reads

$$A^{(1)'} = A^{(1)} - F^{(1)}$$

$$A^{(2)'} = A^{(2)} - F^{(2)} - \frac{1}{2} [F^{(1)}, A^{(1)}]$$

$$A^{(3)'} = A^{(3)} - F^{(3)} - \frac{1}{2} [F^{(1)}, A^{(2)}] - \frac{1}{2} [F^{(2)}, A^{(1)}] + \frac{1}{12} [F^{(1)}, [F^{(1)}, A^{(1)}]] + \frac{1}{12} [A^{(1)}, [F^{(1)}, A^{(1)}]]$$

... which I wanted to specify because it is an effective way of preparing for the fact that **this is not the end of the story!** Problems are going to pop out which we have to take care of ...

Stochastic Gauge Fixing

Stochastic Gauge Fixing D. Zwanziger, Nucl.Phys. B 192 (1981) 259

Let's go back to the **continuum**

$$\frac{\partial}{\partial t} A_\mu^a(\eta, x; t) = D_\nu^{ab} F_{\nu\mu}^b(\eta, x; t) + \eta_\mu^a(x; t)$$

whose expanded version has a (momentum space) solution

$$A_\mu^{(n)a}(k; t) = T_{\mu\nu}^{ab} \int_0^t ds e^{-k^2(t-s)} f_\nu^{(n)b}(k, s) + L_{\mu\nu}^{ab} \int_0^t ds f_\nu^{(n)b}(k, s)$$

in which **vertices** pop in (as they should ...)

$$f_\nu^{(0)a}(k; t) = \eta_\nu(k; t)^a \quad f_\nu^{(n)a}(k; t) = g I_\mu^{(3)(n-1)a}(k; t) + g^2 I_\mu^{(4)(n-2)a}(k; t)$$

$$g I_\mu^{(3)a}(k; t) = \frac{igf^{abc}}{2(2\pi)^n} \int dpdq \delta(k+p+q) A_\nu^b(-p; t) A_\sigma^c(-q; t) v_{\mu\nu\sigma}^{(3)}(k, p, q)$$

$$v_{\mu\nu\sigma}^{(3)}(k, p, q) = \delta_{\mu\nu}(k-p)_\sigma + \text{cyclic permutations}$$

Remember the scalar case ...
$$\phi(k, t) = \int_0^t d\tau \exp -(k^2 + m^2)(t - \tau) \left[\eta(k, \tau) - \frac{\lambda}{3!} \int \frac{dpdqds}{(2\pi)^{2n}} \phi(p, \tau) \phi(q, \tau) \phi(s, \tau) \delta(k - p - q - s) \right]$$

BUT ALL THIS IS GOING TO BE ONLY FORMAL ... WE WILL NOT OBTAIN LONG TIME CONVERGENCE BECAUSE OF THE **LOSS OF DAMPING IN THE LONGITUDINAL** (NON-gauge-invariant) SECTOR

SOLUTION: add an **extra piece**

$$\dot{A}_\mu^a(x; t) = -\frac{\delta S[A]}{\delta A_\mu^a(x; t)} - D_\mu^{ab} V^b[A, t] + \eta_\mu^a(x; t)$$

Any **functional** evolves like

$$\frac{\partial F[A]}{\partial t} = \int dx \frac{\delta F[A]}{\delta A_\mu^a(x; t)} \frac{\partial A_\mu^a(x; t)}{\partial t}$$

but **GAUGE INVARIANT** ones are such that

$$D_\mu^{ab} \frac{\delta F[A]}{\delta A_\mu^b(x)} = 0$$

and thus **physics is unaffected!** (integration by parts ...) ... while if we make a convenient choice for the extra term we have **new damping factors** in place!

$$-D_\mu^{ab} V^b = \frac{1}{\alpha} D_\mu^{ab} \partial_\nu A_\nu^b \quad A_\mu^{a(n)}(k; t) = T_{\mu\nu} \int_0^t ds e^{-k^2(t-s)} f_\nu^{a(n)}(k, s) + L_{\mu\nu} \int_0^t ds e^{-\frac{k^2}{\alpha}(t-s)} f_\nu^{a(n)}(k, s)$$

On the lattice we **interleave a gauge fixing step** to the Langevin evolution

$$U'_{x\mu} = e^{-F_{x\mu}[U, \eta]} U_{x\mu}(n)$$

$$U_{x\mu}(n+1) = e^{w_x[U']} U'_{x\mu} e^{-w_{x+\hat{\mu}}[U']}$$

which has by the way an obvious interpretation

$$U_{x\mu}(n+1) = e^{-F_{x\mu}[U^G, G\eta G^\dagger]} U_{x\mu}^G(n)$$

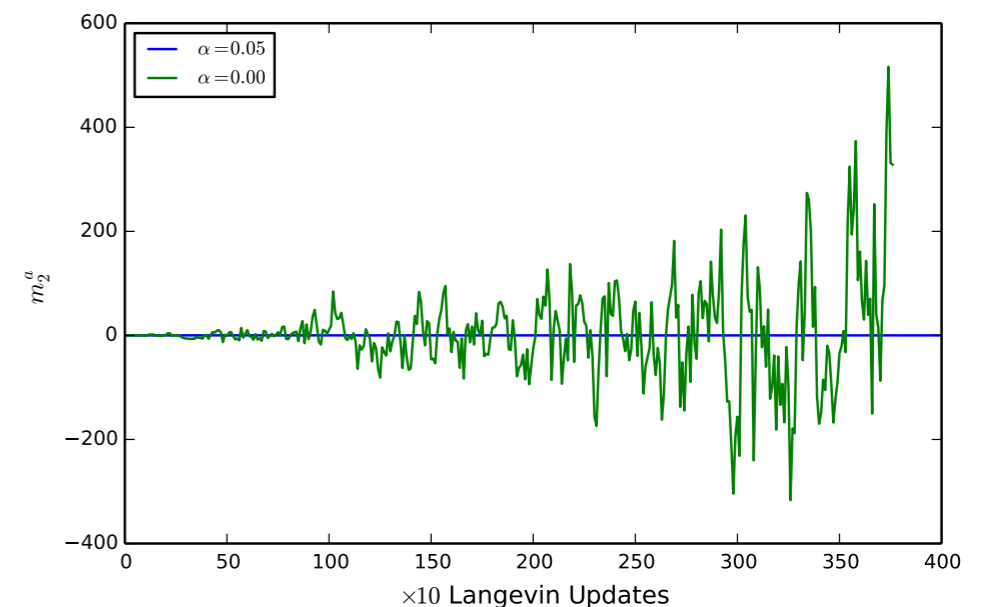


Figure 1. The effect of stochastic gauge fixing.

Fermionic loops in NSPT

FERMIONIC LOOPS in NSPT Di Renzo, Scorzato 2001

Let's add **fermions** (Wilson fermions, in this case) in the Langevin equation

$$\begin{aligned}
 S_F^{(W)} &= \sum_{xy} \bar{\psi}_x M_{xy}[U] \psi_y \\
 &= \sum_x (m + 4) \bar{\psi}_x \psi_x - \frac{1}{2} \sum_{x\mu} \left(\bar{\psi}_{x+\hat{\mu}} (1 + \gamma_\mu) U_{x\mu}^\dagger \psi_x + \bar{\psi}_x (1 - \gamma_\mu) U_{x\mu} \psi_{x+\hat{\mu}} \right)
 \end{aligned}$$

From the point of view of the functional integral measure $e^{-S_G} \det M = e^{-S_{eff}} = e^{-(S_G - \text{Tr} \ln M)}$

and in turns $\nabla_{x\mu}^a S_G \mapsto \nabla_{x\mu}^a S_{eff} = \nabla_{x\mu}^a S_G - \nabla_{x\mu}^a \text{Tr} \ln M \equiv \nabla_{x\mu}^a S_G - \text{Tr} \left((\nabla_{x\mu}^a M) M^{-1} \right)$

Batrouni et al (Cornell group) PRD 32 (1985)

In $U_{x\mu}(n+1; \eta) = e^{-F_{x\mu}[U, \eta]} U_{x\mu}(n; \eta)$ we now write

$$F = T^a (\epsilon \Phi^a + \sqrt{\epsilon} \eta^a) \quad \Phi^a = \left[\nabla_{x\mu}^a S_G - \text{Re} \left(\xi_k^\dagger (\nabla_{x\mu}^a M)_{kl} (M^{-1})_{ln} \xi_n \right) \right]$$

where $\langle \xi_i \xi_j \rangle_\xi = \delta_{ij}$ or (this is what we always do)

$$\Phi^a = \left[\nabla_{x\mu}^a S_G - \text{Re} \left(\xi_l^\dagger (\nabla_{x\mu}^a M)_{ln} \psi_n \right) \right] \quad M_{kl} \psi_l = \xi_k$$

From a numerical point of view this boils down to the (technically challenging) problem of inverting the Dirac operator efficiently. This is a heavy task, making unquenched simulations much more demanding in terms of computer time.

But we have not **put our expansion in the coupling in place!** Once we do it, we find **much less problems** than expected from the non-perturbative simulations point of view!

$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)} \quad M^{-1} = M^{(0)^{-1}} + \sum_{k>0} \beta^{-k/2} M^{-1(k)}$$

In NSPT we have to deal with only one inverse (known once and for all: the Feynman free propagator) plus a tower of **recursive relations**

$$M^{-1(1)} = -M^{(0)^{-1}} M^{(1)} M^{(0)^{-1}}$$

$$M^{-1(2)} = -M^{(0)^{-1}} M^{(2)} M^{(0)^{-1}} - M^{(0)^{-1}} M^{(1)} M^{-1(1)}$$

$$M^{-1(3)} = -M^{(0)^{-1}} M^{(3)} M^{(0)^{-1}} - M^{(0)^{-1}} M^{(2)} M^{-1(1)} - M^{(0)^{-1}} M^{(1)} M^{-1(2)}$$

i.e.

$$M^{-1(n)} = -M^{(0)^{-1}} \sum_{j=0}^{n-1} M^{(n-j)} M^{(j)^{-1}}$$

This has a direct counterpart in the solution of the linear system we have to face, which is also translated into a perturbative version (beware! the noise source is 0-th order)

$$\psi^{(j)} \equiv M^{-1(j)} \xi$$

$$\psi^{(0)} = M^{(0)^{-1}} \xi$$

$$\psi^{(1)} = -M^{(0)^{-1}} M^{(1)} \psi^{(0)}$$

$$\psi^{(2)} = -M^{(0)^{-1}} \left[M^{(2)} \psi^{(0)} + M^{(1)} \psi^{(1)} \right]$$

$$\psi^{(3)} = -M^{(0)^{-1}} \left[M^{(3)} \psi^{(0)} + M^{(2)} \psi^{(1)} + M^{(1)} \psi^{(2)} \right]$$

i.e.

$$\psi^{(n)} = -M^{(0)^{-1}} \sum_{j=0}^{n-1} M^{(n-j)} \psi^{(j)}$$

with $M^{(0)^{-1}}$ the (tree-level, field independent)
Feynman propagator

which is particularly nice, since it can be solved by going **back and forth from momentum to coordinate representation!**

A few different frameworks for NSPT (i.e. a few handles to possibly improve it)

(there are projects going on this!)

There are various formulations of NSPT one can think of ...

(1) Is Langevin the only stochastic equation one can play with in NSPT?

NO! e.g. **Stochastic Molecular Dynamics** (SMD Horowitz 1985 ...) (not to mention **ISPT Luescher 2014**)

I have been talking to Pino on this several times...

$$\frac{d\phi(x; t)}{dt} = \pi(x; t)$$

$$\frac{d\pi(x; t)}{dt} = -\frac{\partial S[\phi]}{\partial \phi(x; t)} - 2\mu_0\pi(x; t) + \eta(x; t)$$

$$\eta(x; t) : \quad \langle \eta(x, t) \eta(x', t') \rangle_\eta = 4\mu_0 \delta(x - x') \delta(t - t')$$

which is Langevin for $\mu_0 \rightarrow \infty$

Notice that one can tune the lattice parameter $\gamma = 2\mu_0 a$ to minimize errors!

(which depend on both **autocorrelation** times and **standard deviations (*)**!)

(*) **subtle** issues in the **continuum limit**! Beware! **Variances are not intrinsic properties of QFT...**

Beware! Quite often gains in autocorrelation times come with losses on the variance side...

Dalla Brida Kennedy Garofalo 2015

Dalla Brida Luescher 2016 (Gradient Flow!)

(2) **Numerical integrators** (numerical integration schemes) DO MATTER!

... and of course various combinations are possible ... e.g.

(2a) **Langevin with 2nd order integrator**

(2b) **Stochastic Molecular Dynamics with 4th order OMF integrator**

Bali Bauer Torrero 2008

Dalla Brida Kennedy Garofalo 2015

Dalla Brida Luescher 2016

- A canonical application: renormalization constants

Renormalization constants used to be the realm of LPT ...

... but these days this is NOT the case. A **non-perturbative** determination (where possible) is now the preferred choice (**RI-MOM Rome group**, **SF ALPHA 90s**). Still,

Renormalization is strictly speaking proved in PT
 There are different systematics involved in PT and non-PT
 ... and at some point PT is supposed to converge (this is a UV problem ...)

The **RI-MOM** schemes (**Rome group 1994**) are a good framework (in the **massless limit**). Being the scheme **Regulator Independent**, the coefficients of the **logs are known!** ... and the finite parts are the easy part in NSPT ...

Let's see how it works for quark bilinear (**currents**)

$$G_{\Gamma}(p) = \int dx \langle p | \bar{\psi}(x) \Gamma \psi(x) | p \rangle \quad \Gamma_{\Gamma}(p) = S^{-1}(p) G_{\Gamma}(p) S^{-1}(p) \quad O_{\Gamma}(p) = \text{Tr} \left(\hat{P}_{O_{\Gamma}} \Gamma_{\Gamma}(p) \right)$$

$$Z_{O_{\Gamma}}(\mu, \alpha) Z_q^{-1}(\mu, \alpha) O_{\Gamma}(p) |_{p^2=\mu^2} = 1 \quad Z_q(\mu, \alpha) = -i \frac{1}{12} \frac{\text{Tr}(\not{p} S^{-1}(p))}{p^2} |_{p^2=\mu^2}$$

We know what to expect

$$Z(\mu, \alpha_0) = 1 + \sum_{n>0} d_n(l) \alpha_0^n \quad d_n(l) = \sum_{i=0}^n d_n^{(i)} l^i \quad l \equiv \log(\mu a)^2$$

A key ingredient is the quark 2-points function (beware! we will work with **Wilson fermions**...)

$$a\Gamma_2(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = i\hat{p} + \hat{m}_W(\hat{p}) - \hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1})$$

$$\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_c(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_{\gamma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_{\text{other}}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) \quad \left(\beta^{-1} \equiv \frac{2\pi\alpha_0}{3} \right)$$

$$\frac{1}{4} \sum_{\mu} \gamma_{\mu} \text{Tr}_{\text{spin}}(\gamma_{\mu} \hat{\Sigma}) = \hat{\Sigma}_{\gamma}$$

What one really computes is

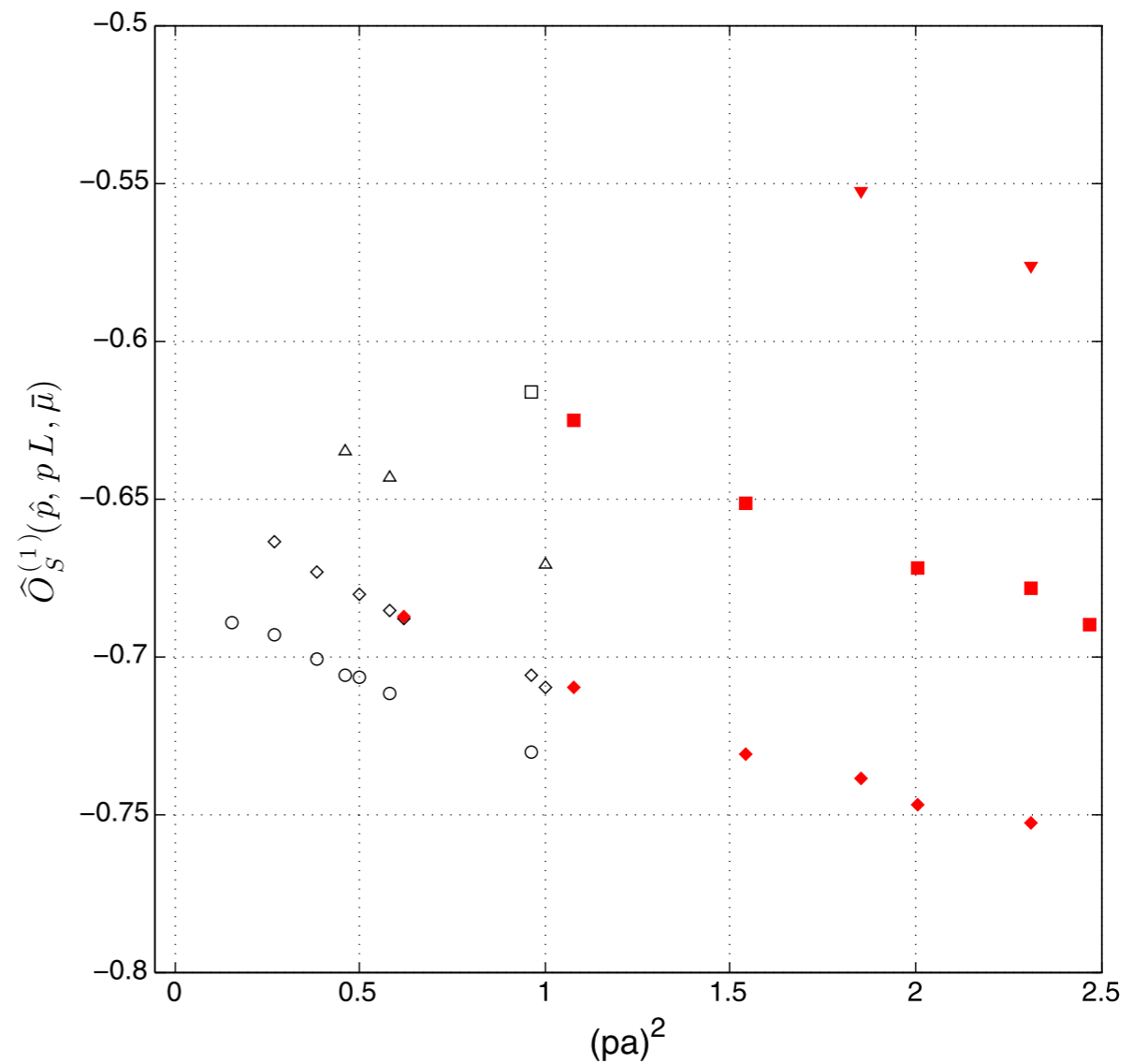
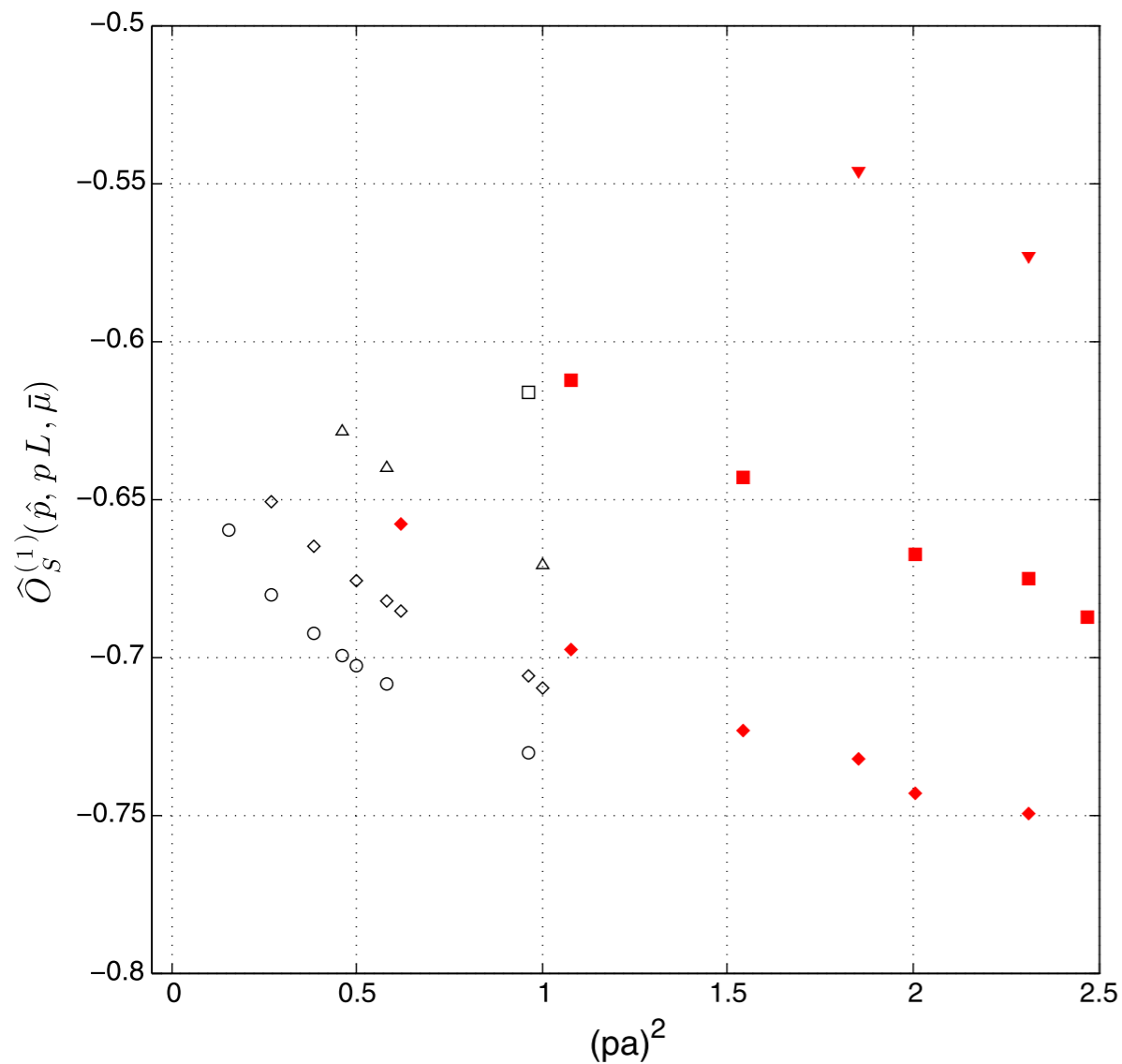
$$Z_{O_\Gamma}(\mu = p, \beta^{-1})|_{\text{finite part}} = \lim_{\substack{a \rightarrow 0 \\ L \rightarrow \infty}} \frac{\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu})}{\widehat{O}_\Gamma(\hat{p}, pL)}|_{\text{log subtr}}$$

where the **limits** are encoded in **expansions**, e.g.

$$\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu})|_{\text{log subtr}} = c_1^{(0)} + c_2^{(0)} \sum_\nu \hat{p}_\nu^2 + c_3^{(0)} \frac{\sum_\nu \hat{p}_\nu^4}{\sum_\nu \hat{p}_\nu^2} + c_1^{(1)} p_{\bar{\mu}}^2 + \Delta \widehat{\Sigma}_\gamma(pL) + \mathcal{O}(a^4)$$

and **finite size** effects come from

$$\widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \equiv \widehat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) + \Delta \widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \quad \Delta \widehat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \sim \Delta \widehat{\Sigma}_\gamma(pL)$$



Three-loop computations of RI-MOM renormalization constants (*)

Parma group 2007, 2013, 2014

(*) for different glue action

Something maybe more field-theoretic (numerics stumbles on fundamental QFT...)

a. Renormalons

An old goal: a lattice determination of the gluon condensate ...

... where an **OPE** is in place ...

$$W = \langle \alpha_s F^2 \rangle / Q^4 = W_0 + (\Lambda^4 / Q^4) W_4 + \dots$$

... now the **plaquette** is our observable

$$W(N) = 1 - \frac{1}{3} \langle \text{Tr} U_p \rangle$$

... unavoidably computed on a lattice of **finite extent** Na

Perturbative (**PT**) contribution (associated to the identity) should be **subtracted from** Non-Perturbative (**NPT**) Monte Carlo (**MC**) data measured at various values of the lattice coupling β , looking for the signature dictated by **asymptotic scaling**, i.e. $\Lambda a \sim e^{-\beta/12b_0}$

$$W_{\text{MC}} - W_{\text{pert}} = (\Lambda^4 / Q^4) W_4 + \dots$$

PROBLEM: expect **RENORMALONS!**

From dimensional and RG arguments

$$W^{\text{ren}} = C \int_{r\Lambda^2}^{Q^2} \frac{k^2 dk^2}{Q^4} \alpha_s(k^2)$$

by changing variable $z \equiv z_0 (1 - \alpha_s(Q^2) / \alpha_s(k^2))$ $z_0 \equiv \frac{1}{3b_0}$

$$W^{\text{ren}} = \mathcal{N} \int_0^{z_{0-}} dz e^{-\beta z} (z_0 - z)^{-1-\gamma}$$

The experts will recognize a **Borel** integral ...

$$4\pi\alpha_s(Q^2) \equiv 6/\beta \quad \gamma \equiv 2 \frac{b_1}{b_0^2} \quad 0 < z < z_{0-} \equiv z_0(1 - \alpha_s(Q^2) / \alpha_s(r\Lambda^2))$$

$$W^{\text{ren}} = \sum_{\ell=1} \beta^{-\ell} \{c_\ell^{\text{ren}} + \mathcal{O}(e^{-z_0\beta})\} \quad c_\ell^{\text{ren}} = \mathcal{N}' \Gamma(\ell + \gamma) z_0^{-\ell}$$

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PROBLEMS

1. Computing power ...

2. The **IR** renormalon deserves its name and relevant momenta go like $k^* \sim s^{-1} e^{-(\ell-1)/2}$

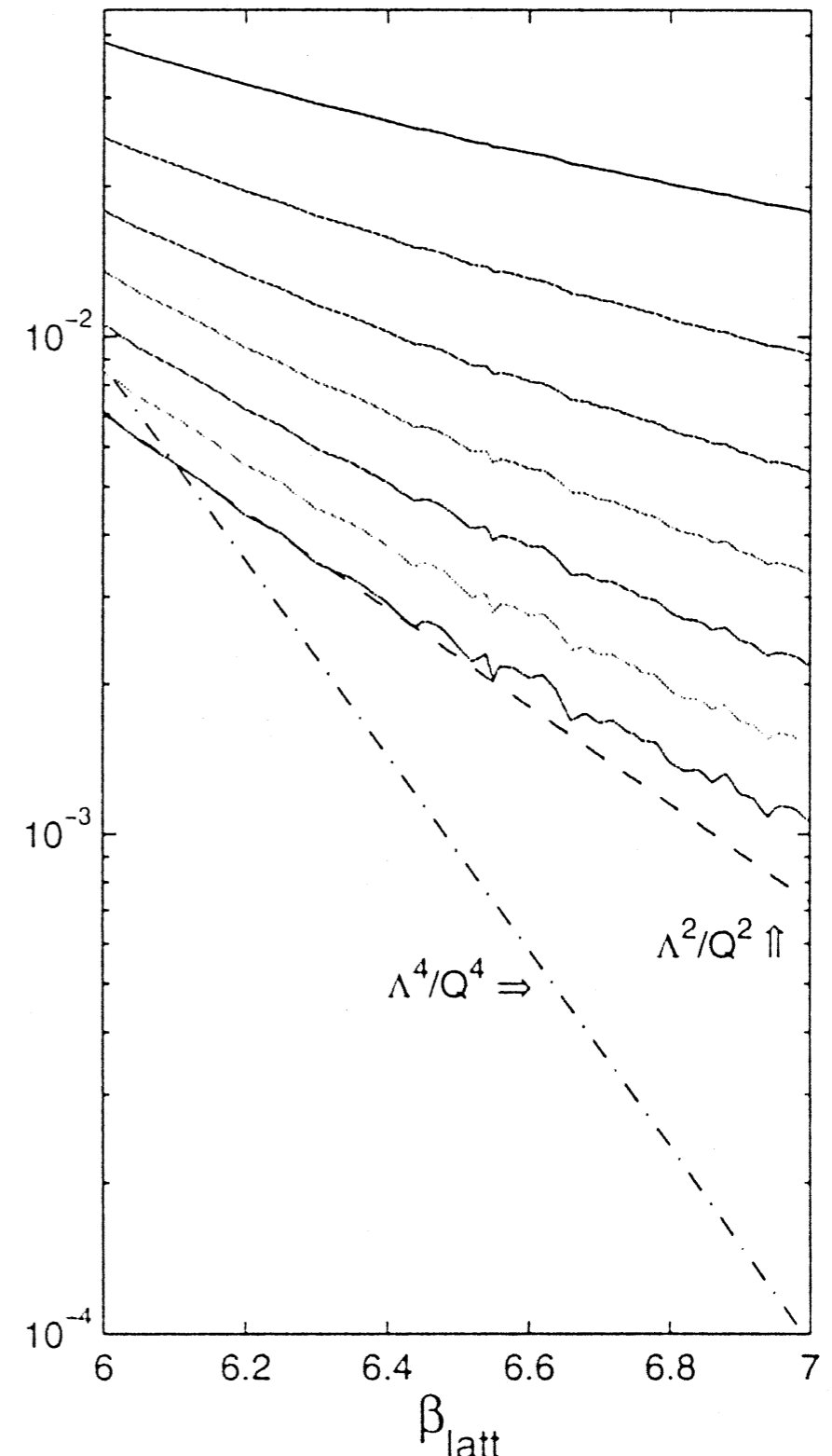
... rather study
$$W^{\text{ren}}(N) = C \int_{Q_0^2(N)}^{Q^2} \frac{k^2 dk^2}{Q^4} \alpha_s(sk^2) \rightarrow \sum_{\ell=1} \beta^{-\ell} c_\ell^{\text{ren}}(N; s, C)$$

... where the finite lattice has been explicitly taken into account, while the change of scale can be reabsorbed in a change of scheme (i.e., look for a scheme in which renormalon is better described...)

... but all in all the final result for the subtraction was signaling something odd going on ... **WRONG SCALING!**

Burgio Di Renzo Marchesini Onofri 1998

We now know that NSPT CAN ACTUALLY DIRECTLY INSPECT RENORMALONS, but one has to go to HIGHER ORDERS ... (at the time the first 8 orders had been computed)



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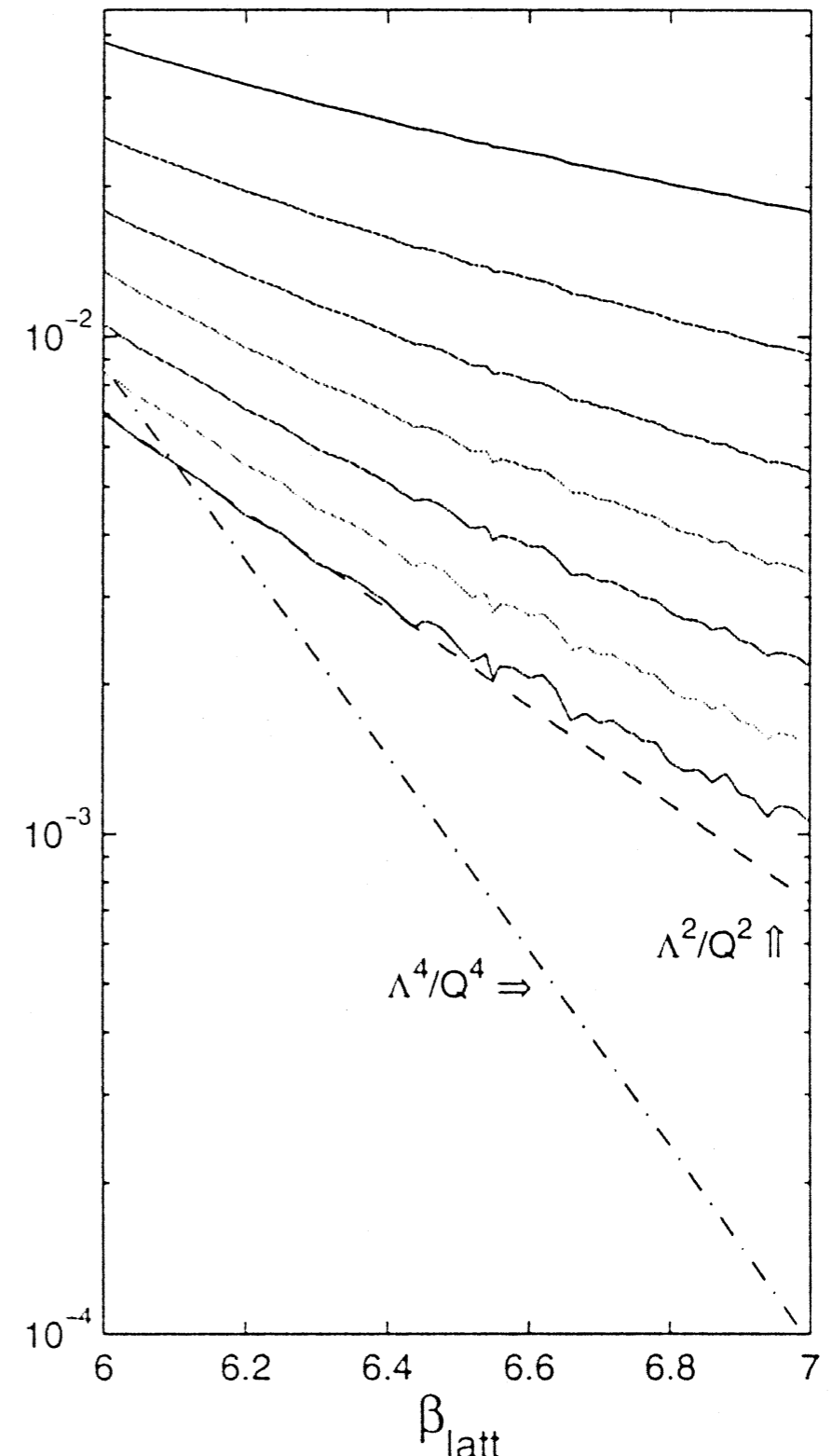
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(Pino so excited of this...)

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Solution of the puzzle and direct inspection of renormalons Bali Bauer Pineda 2014

In 2012, **Horsley et al** computed the first 20 orders.

In 2013 **Bali** and **Pineda** detected the renormalon in the **HQET/pole mass** framework: dimensions do matter! **The order at which renormalons show up increases with the dimension of the operator!**

Improvements (**Bali Pineda**) for the plaquette case (**2014**):

1. **Twisted BCs** (which **kill zero modes**; I have cheated a little bit about those till now...)
2. **2nd order integrator** for Langevin equation(s)
3. computer power (well ... it was 20 years later ...)
4. careful treatment of **finite size** effects by **perturbative OPE** (separation of scales!)

$$\langle P \rangle_{\text{pert}}(N) = P_{\text{pert}}(\alpha) \langle \mathbb{1} \rangle + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle O_G \rangle_{\text{soft}} + \mathcal{O}\left(\frac{1}{N^6}\right) \quad \frac{1}{a} \gg \frac{1}{Na}$$

$$P_{\text{pert}}(\alpha) = \sum_{n \geq 0} p_n \alpha^{n+1} \quad \frac{\pi^2}{36} a^4 \langle O_G \rangle_{\text{soft}} = -\frac{1}{N^4} \sum_{n \geq 0} f_n \alpha^{n+1} ((Na)^{-1})$$

with both p_n and f_n asymptotically dominated by the IR renormalon!

Normalization for Wilson action is fixed by $C_G(\alpha) = 1 + \sum_{k \geq 0} c_k \alpha^{k+1} = -\frac{\beta_0 \alpha^2}{2\pi\beta(\alpha)}$

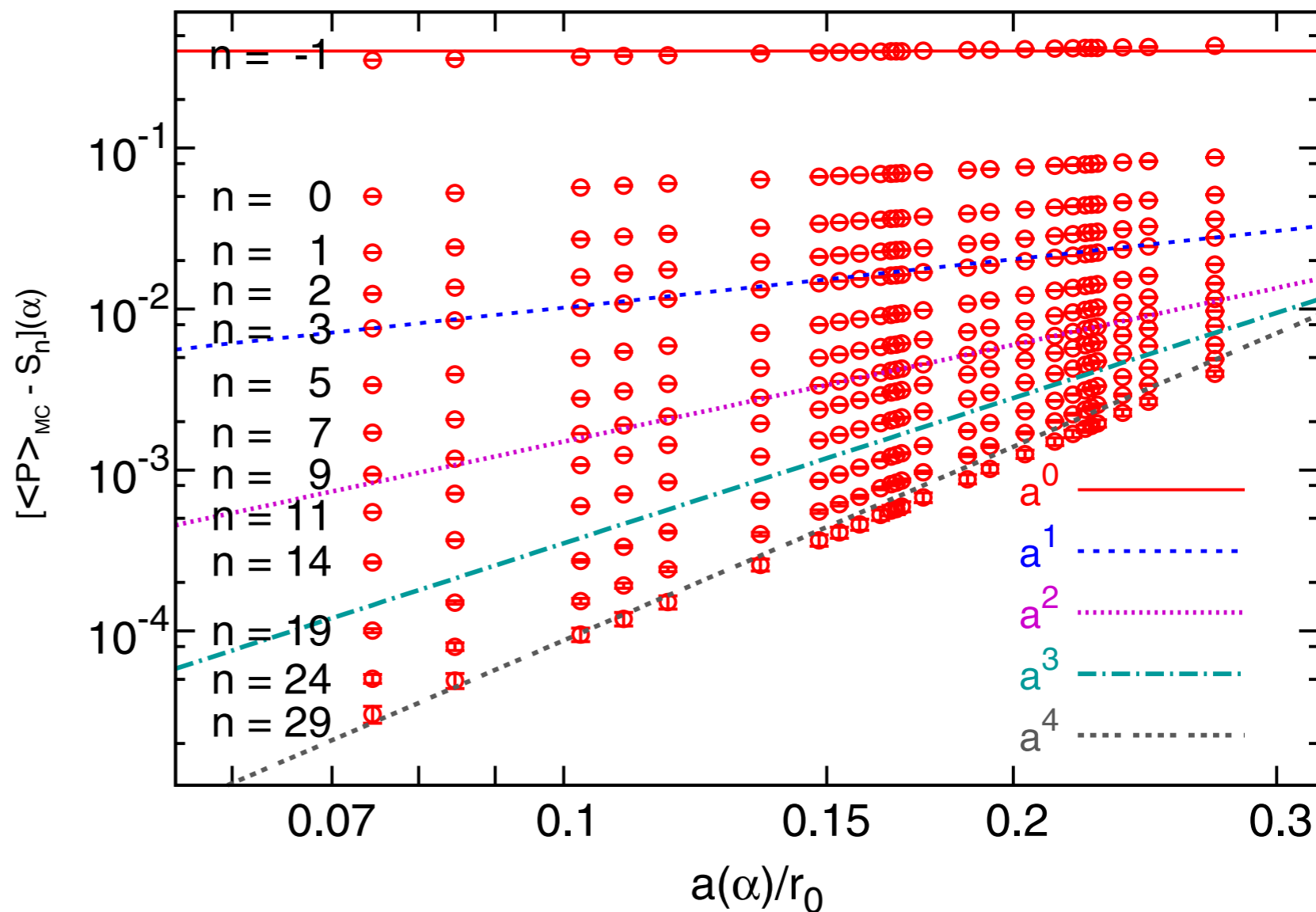
... and one can finally fit the computed $\langle P \rangle_{\text{pert}}(N) = \sum_{n \geq 0} \left[p_n - \frac{f_n(N)}{N^4} \right] \alpha^{n+1}$

IT WORKS!

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

... and they could finally determine the gluon condensate



Model Independent Determination of the Gluon Condensate in Four Dimensional SU(3) Gauge Theory

Gunnar S. Bali,^{1,2} Clemens Bauer,¹ and Antonio Pineda³

¹*Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*

²*Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India*

³*Grup de Física Teòrica, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain*

(Received 25 March 2014; revised manuscript received 7 May 2014; published 25 August 2014)

We determine the nonperturbative gluon condensate of four-dimensional SU(3) gauge theory in a model-independent way. This is achieved by carefully subtracting high-order perturbation theory results from nonperturbative lattice QCD determinations of the average plaquette. No indications of dimension-two condensates are found. The value of the gluon condensate turns out to be of a similar size as the intrinsic ambiguity inherent to its definition. We also determine the binding energy of a B meson in the heavy quark mass limit.

Something maybe more field-theoretic (numerics stumbles on fundamental QFT...)

b. Resurgence?

Resurgence, trans-series and all that

From **Gerald Dunne's** lectures at the Parma School 2016 (Decoding the path integral: resurgence, Lefschetz thimbles, non-perturbative physics)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever ... That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.



N. Abel, 1802 – 1829

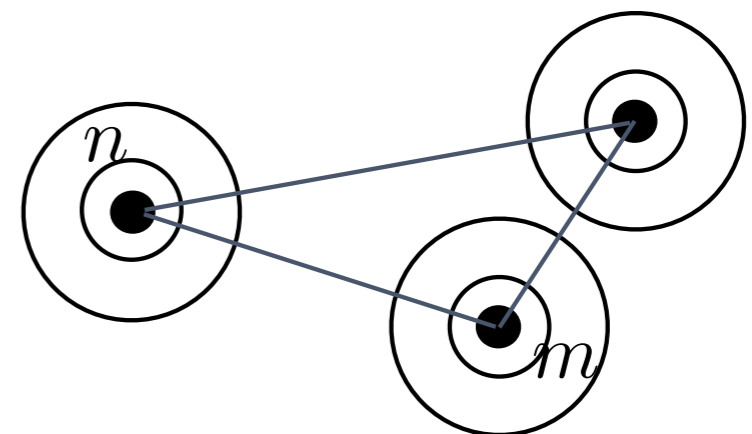
The series is divergent; therefore we may be able to do something with it

O. Heaviside, 1850 – 1925



*resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or **surge up** - in a slightly different guise, as it were - at their singularities*

J. Écalle, 1980



From [Mitat Unsal](#)'s presentation at LATTICE2015

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ,
Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)} (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n} e^{-n A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)} (\lambda\hbar)^k$$

pert. th.

n-instanton factor

pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called **trans-series following resurgence** terminology.

Actually the “Resurgence people” have quite a number of predictions for perturbative expansions and they went many steps further than the typical claim for QM cases (double-well potential ...)

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle.

There quite a lot of work we can do! ... The main point is that **NSPT can compute expansions in the background of (in principle) any classical solution!** In particular, for field theories the resurgence scenario needs to be tested ...

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(I am pretty sure Pino would be curious on this...)

Conclusions

- NSPT has been around for roughly 20 years. Maybe it has never been extremely popular, but it is fair to say that it has never stopped attracting attentions ... even more in recent times.
- I think there are many applications one can think of, but what I am really interested in is how (brute...) numerics can lead you to tackle fundamental issues in field theories (it has been the case for renormalons; now ... tackle Resurgence!)
- I would say this is one (of the) thing(s) I owe Pino a great debt for