Moduli Anomalies and Local Terms in the Operator Product Expansion

Stefan Theisen

Albert-Einstein-Institut Golm

GGI, May 7, 2018

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with Adam Schwimmer (to be published)

This talk is about certain aspects of superconformal field theories with moduli.

The work I will report here is an extension of earlier work done in collaboration with Gomis, Hsin, Komargodski, Seiberg and Schwimmer [1509.08511].

There we exploited extended supersymmetry to compute certain local terms in the generating functional which allowed us to determine the sphere partition function as a function of the data (Kählerpotential) of the conformal manifold.

We extended this to semi-local terms, in the way which will be described in some detail.

As in the earlier paper (to be partially reviewed shortly) this will be based on a study of the (Super-Weyl) anomaly polynomial of a generic SCFT with extended SUSY in even dimensions ($\mathcal{N} = (2,2)$ in $d = 2$ and $\mathcal{N} = 2$ in $d = 4$).

The results obtained are very general and I think they fit well with the general theme of the workshop

Supersymmetric Quantum Field Theories in the Non-perturbative Regime

Outline:

- CFTs with moduli
- Their Weyl anomalies
- SCFTs and their Super-Weyl anomalies
- Lessons from the anomaly polynomial
- Illustrative example: $\mathcal{N} = 2$ SUSY Maxwell theory
- Further comments, summary, conclusions

CFTs with Moduli or Exactly Marginal Deformations

Given a fiducial CFT $\mathcal{S}^*$, we can perturb it by operators $\mathcal{O}_i \subset$ CFT

$$\mathcal{S} = \mathcal{S}^* + \sum_i \lambda^i \int \mathcal{O}_i(x)\, d^d x$$

the deformed CFT is generally not a CFT ...

- this is obvious for relevant, i.e. dim $\mathcal{O}_i < d$, and irrelevant, i.e. dim $\mathcal{O}_i > d$ operators:

  in these cases dim $\lambda^i > 0$ and $\lambda^i < 0$ and we have an explicit mass scale which breaks scale invariance classically

- for marginal perturbations with dim $\mathcal{O}_i = d \Rightarrow \dim \lambda^i = 0$, the situation is more subtle:
  - for $\mathcal{O}_i$ marginal but not exactly marginal, $\beta_i \neq 0$ and scale invariance is broken quantum mechanically
  - the perturbed theory stays conformal, i.e. $\beta_i = 0$, if the $\mathcal{O}_i$ are exactly marginal operators, called moduli and denoted in the following M_i.

    This implies additional conditions (besides dim $M_i = d$)

    One necessary condition is vanishing 3-point functions at separate points $x \neq y \neq z \neq x$
    $$\langle M_i(x)\, M_j(y)\, M_k(z)\rangle = 0$$
    this guarantees $\beta_i = 0$ at lowest non-trivial order in λ_i

    i.e. the operator product coefficients c_{ijk} which involve three moduli
    $$M_i(x)\, M_j(y) = \frac{c_{ijk}}{|x-y|^d} M_k(y) + \ldots$$
    vanish.

From now on: we consider only exactly marginal perturbations, i.e. we deal with CFTs with free parameters λ^i.

They parametrize families of CFTs and are local coordinates – in the neighbourhood of the reference CFT $\mathcal{S}^*$ – on the conformal manifold or moduli space $\mathcal{M}_{con}$.

Even though $\beta = 0$, scale and therefore conformal invariance is broken in a subtle way by the conformal or Weyl anomaly (cf. below).

In unitary theories this is unavoidable and, in fact, offers a tool for further analysis of unitary CFTs.

Examples of CFTs with marginal deformations:

- $d = 2$: the world-sheet theories of compactified string theory
  - String on a torus T^n: (2d sigma-model)

    $S = \int \partial_\alpha X^i \partial^\alpha X^i + g_{ij} \partial_\alpha X^i \partial^\alpha X^j + b_{ij}\epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j$

    n^2 marginal perturbations: the (constant) components of g_{ij} and b_{ij}

    $$\mathcal{M}_{con} = \Gamma \backslash O(n,n)/O(n) \times O(n)$$

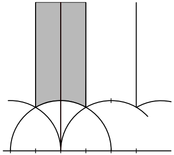
  - Type II string on CY: $\mathcal{N} = (2,2)$ SCFTs on world-sheet

    moduli are in 1-1 correspondence with Ricci flat deformations of the CY metric and the B-field: complexified Kähler and complex structures deformations

    $\mathcal{N}$ =(2,2) SCFTs, dim$(\mathcal{M}_{con}) = h^{1,1}_{\rm CY} + h^{2,1}_{\rm CY}$,

    $$\mathcal{M}_{con} = \mathcal{M}_{\rm Kahler} \times \mathcal{M}_{\rm c.s.}$$

- $d = 4$ superconformal field theories



  - $\mathcal{N} = 4$ SYM: $\lambda \equiv \tau = \theta + \frac{i}{g^2_{\rm YM}}$, $M = \mathcal{L}_{\rm YM}$
  - $\mathcal{N} = 2$ superconformal Seiberg-Witten theories: SYM with $N_f = 2\, N_c$
  - $\mathcal{N} = 2$ Maxwell . . . this will play a role later to check our claims
  - $\mathcal{N} = 1$ superconformal theories:

    all chiral operators $\mathcal{O}$ with dim$(\mathcal{O}) = 3$ are marginal operators

    superpotential deformations $W = \sum \lambda^i\, \mathcal{O}_i$

    - if there is no global symmetry other than $U(1)_R$:
      they are all exactly marginal
    - if there is additional global symmetry G : $\mathcal{M}_{con} = \{\lambda^i\}/G^{\mathbb{C}}$

      the remaining couplings are marginally irrelevant

Leigh-Strassler; Kol; Green-Komargodski-Seiberg-Tachikawa-Wecht

The conformal manifold $\mathcal{M}_{con}$ can be endowed with a natural Riemannian structure:

A metric $G_{ij}(\lambda)$ on $\mathcal{M}_{con}$ was proposed by Zamolodchikov

$$\langle M_i(x)\, M_j(y)\rangle_\lambda = \frac{G_{ij}(\lambda)}{|x-y|^{2d}}$$

The Zamolodchikov metric G_{ij} is positive definite for unitary theories.

It is of great interest, one reason being that in string compactifications the Zamolodchikov metric of the world-sheet CFT determines to a large extend the low energy effective action Dixon-Kaplunovsky-Louis,. . .

The geometric structure on the conformal manifold in terms of higher point correlation functions of moduli was analysed a long time ago by Kutasov. We will return to it later.

But let us first consider the above two-point function somewhat closer.

For $x \neq y$ the space-time dependence of

$$\langle M_i(x)\, M_j(y) \rangle_\lambda = \frac{G_{ij}(\lambda)}{|x-y|^{2d}}$$

is completely fixed by conformal symmetry ...

... but for $x = y$ it is not defined, even in a distributional sense, as it has no Fourier transform.

To define it requires regularization, leading to (for even d)

$$\langle M_i(p)\, M_j(-p) \rangle \propto G_{ij}\, (p^2)^{d/2} \log \Lambda^2/p^2$$

This has an explicit scale Λ and therefore violates scale invariance:

under rescaling of momenta $p \to e^{-\lambda} p \mathrel{\widehat{=}}$ dilations $x \to e^{\lambda} x$ in position space:

$$\delta_\lambda^{(\text{anom})} \left((p^2)^{d/2} \, \log \Lambda^2/p^2 \right) = 2\,\lambda\, (p^2)^{d/2} = 2\,\lambda\, \text{F.T.} \left(\Box^{d/2} \delta(x) \right)$$

This reflects an anomaly in the conservation Ward identity of the dilatation current

$$\partial_\mu \langle j_D^\mu(x)\, M_i(y)\, M_j(z) \rangle = \langle T^\mu_\mu(x)\, M_i(y)\, M_j(z) \rangle \neq 0$$

Weyl or Trace Anomalies in CFTs ... and some consequences

In even dimensions the two Ward identities following from

conservation and tracelessness of $T_{\mu\nu}$

cannot be maintained simultaneously.

Counterterms needed to regularize the theory necessarily break one of the symmetries. Usually one chooses to give up $T^{\mu}_{\mu} = 0$. Either way it leads to

anomalous Ward identities in correlators involving the em-tensor

The above was just one example involving the correlator

$$\langle T_{\mu\nu}(x)\, M_i(y)\, M_j(z)\rangle$$

Following the classification of Deser and Schwimmer, this is a type B anomaly, which is characterized by the appearance of an explicit scale Λ in a counter term.

To put (anomalous) Ward identities into evidence, introduce space-time dependent sources for the composite operators:

$$\lambda^i \to J^i(x)\,, \qquad \eta_{\mu\nu} \to g_{\mu\nu}(x)$$

$$\uparrow \qquad\qquad \uparrow$$

source for $\qquad M_i \qquad\qquad T^{\mu\nu}$

- Poincaré invariance $\quad (\partial^\mu T_{\mu\nu} = 0) \quad \Leftrightarrow \quad$ diffeo invariance

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \qquad\qquad \delta_\xi J^i = \xi^\mu\, \partial_\mu J^i$$

- conformal invariance $\quad (T^\mu_\mu = 0) \quad \Leftrightarrow \quad$ Weyl invariance

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma(x)\, g_{\mu\nu} \qquad\qquad \delta_\sigma J^i = 0$$

of the generating functional $W[g,J]$

$$Z[g,J] = e^{-W[g,J]} = \int \mathcal{D}[CFT]\, e^{-(\mathcal{S}^*[g] + \int \sqrt{g}\, J^i(x)\, M_i(x) + \dots\,)}$$

...... up to anomalies

…the non-invariance of the generating functional under Weyl transformations

$$\delta_\sigma W[g,J] = \mathcal{A}[g,J] = \int \sqrt{g}\,\sigma\, a(g,J)$$

where

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma g_{\mu\nu} \qquad\qquad \delta_\sigma J^i = 0$$

A priori conditions on the anomaly $\mathcal{A}$:

- solves the Wess-Zumino consistency condition

$$\delta_{\sigma_2}\mathcal{A}_1 = \delta_{\sigma_1}\mathcal{A}_2$$

- $\mathcal{A}[g,J]$ is a local functional
- diffeo invariant (in space-time and in $\mathcal{M}_{con}$)
- non-trivial: i.e. $\mathcal{A} \neq \delta_\sigma \int$ local $\Rightarrow$ cannot be removed by adding a local counterterm

This is a cohomology problem which can be solved in any dimension (non-trivial solutions only exist for even d).

If the metric is the only source, the general solution is known up to $d = 8$; e.g.

- $d = 2$

$$\mathcal{A} = c \int \sqrt{g}\,\sigma\,R$$

- $d = 4$

$$\mathcal{A} = a \int \sqrt{g}\,\sigma\,E_4 + c \int \sqrt{g}\,\sigma\,C^2$$

In $d = 4$ there is also the trivial solution $\int \sqrt{g}\,\sigma\,\Box R \propto \delta_\sigma \int \sqrt{g}\,R^2$

In the presence of moduli the cohomology problem was studied by Osborn. Additional Weyl anomalies i.e. non-trivial solution of WZ consistency, are e.g.

- $d = 2$

$$\mathcal{A} = \int \sqrt{g}\,\sigma\, G_{ij}(\lambda)\,\partial^\mu J^i\,\partial_\mu J^j \qquad G_{ij} \quad \text{the Zamolodchikov metric}$$

- $d = 4$

$$\mathcal{A} = \int \sqrt{g}\,\sigma\left(G_{ij}(\lambda)\,\hat{\Box} J^i\,\hat{\Box} J^j - 2G_{ij}(\lambda)\partial_\mu J^i\left(R^{\mu\nu} - \tfrac{1}{3}g^{\mu\nu}R\right)\partial_\nu J^j\right)$$

$$\mathcal{A} = \int \sqrt{g}\,\sigma\, c_{ijkl}(J)\,\partial^\mu J^i \partial_\mu J^j\,\partial^\nu J^k \partial_\nu J^l \qquad \text{‘Osborn Anomaly’}$$

where c_{ijkl} is a tensor on $\mathcal{M}_{conv.}$.

There are also trivial solutions, e.g. in $d = 2$

$$\mathcal{A} = \int \sqrt{g}\,\Box\sigma K(J) \propto \delta_\sigma \int \sqrt{g}\,K(J)\,R$$

where K is an arbitrary function on $\mathcal{M}_{conv.}$.

These are type B, i.e.

- they do not vanish for constant σ

or, equivalently,

- they arise from a log-divergent counterterm Deser-Duff-Isham

  e.g. for the $2d$ example

$$\log\Lambda^2 \int G_{ij}(J)\,\partial^\mu J^i\,\partial_\mu J^j \qquad (*)$$

and encode the two-point function

$$\langle M(p)_i\, M(-p)_j\rangle_\lambda = G_{ij}\, p^2 \log(\Lambda^2/p^2)$$

Taking three functional derivatives of $(*)$ gives

$$\langle M_i(p_1)M_j(p_2)M_k(p_3)\rangle_\lambda = \log\Lambda^2\ \left(p_3^2\,\Gamma_{ij,k} + \text{cyclic permutations}\right)$$

with $\Gamma_{ij,k}$ the Christoffel connection for G_{ij}

In position space this yields the semi-local expression

$$\langle M_i(x)\, M_j(y)\, M_k(z)\rangle_\lambda = \Gamma_{ij,k}\, \delta^{(d)}(x-y)\ \left(|y-z|^{-2d}\right)_{\text{reg.}} + \text{cyclic permutations}$$

which is now valid in any even d.

It implies a local term in the OPE of two moduli

$$M_i(x)\, M_j(y) \sim \delta^{(d)}(x-y)\, \Gamma^k_{ij}\, M_k(y)$$

Note that while the OPE coefficient c_{ijk} vanishes by the property of M_i being moduli, local terms are allowed. However ...

...this local term in the OPE is not universal. It can be removed by a coordinate change on $\mathcal{M}_{conf.}$, i.e. by redefining the sources:

$$J^i \to J^i + \Gamma^i_{jk} J^j J^k + \dots \qquad \text{Riemann normal coordinates}$$

But the four point function contains universal data of $\mathcal{M}_{conf.}$, the Riemann tensor

Kutasov; Friedan-Konechny,...

In the remaining time we will apply the same logic to identify new local contributions to the OPE of moduli and currents which cannot be removed by source redefinitions and are therefore universal:

They are normalized by the Zamolodchikov metric, which is a tensor on $\mathcal{M}_{conf}$ and can hence cannot be transformed away.

This will be a consequence of SUSY and applies to $\mathcal{N} = 2$ theories in $d = 4$ and to $\mathcal{N} = (2, 2)$ theories in $d = 2$. I will mainly discuss the former.

I will start with a discussion of their Super-Weyl Anomalies and then exploit them along the above lines.

Super-Weyl Anomalies

here for $\mathcal{N} = 2$ in $d = 4$.

As often with SUSY theories, it is convenient to start in superspace where SUSY is manifest. But many details are hidden in the compact notation and they become manifest only in the component field expansion.

The purely gravitational Weyl anomalies have been known for some time

Kuzenko; de Wit et al,...

$$\mathcal{A}_g = \int d^4x\, d^4\theta\, \mathcal{E}\, \Sigma\Big(a\,\Xi + (c-a)W^{\alpha\beta}W_{\alpha\beta}\Big) + \text{c.c.}$$

Here

- the integral is over one chiral half of superspace and $\mathcal{E}$ is the chiral density
- Ξ and $W^{\alpha\beta}$ are gravitational (chiral) superfields
- Σ is a chiral superfield with $\Sigma| = \sigma + i\alpha$ where α is the gauge parameter of the anomalous $U(1)_\mathcal{R} \subset SU(2)_\mathcal{R} \times U(1)_\mathcal{R}$ part of the $\mathcal{R}$-symmetry
- a and c are, as before, parameters which are characteristic of a given SCFT

The moduli source dependent part is   Gomis et al.

$$\mathcal{A}_J = \int d^4x\, d^4\theta\, d^4\bar{\theta}\, E\, (\Sigma + \bar{\Sigma}) K(J, \bar{J})$$

- an integral over full $\mathcal{N} = 2$ superspace, where E is the density
- J^i and $\bar{J}^i$ are neutral chiral superfields with $J^i| = J^i$ and Weyl weight zero
- $K(J, \bar{J})$ is the Kähler potential for the Zamolodchikov metric

It is normalized to the $\langle M_i\, \bar{M}_j \rangle \sim G_{i\bar{j}}$ two-point function.

These are the three irreducible (in the sense of $\mathcal{N} = 2$ SUSY) non-trivial solutions to WZ consistency.

When expanded in components, they contain many terms

- some of them true Weyl anomalies, parametrized by σ
- some of them true $U(1)_{\mathcal{R}}$ chiral anomalies, parametrized by α
- some of them trivial if it were not for SUSY, which demands them, i.e. there is no local superspace counterterm to remove them

Explicit calculation yields, relying on Butter-de Wit-Kuzenko-Lodato

$$
\begin{aligned}
\mathcal{A} = \int \sqrt{g}\Big\{ & - a\,\sigma\,\big(E_4 - \tfrac{2}{3}\,\Box R\big) + c\,\sigma\, C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - 2\,c\,\sigma\, F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} c\,\sigma\,\mathrm{tr}\big(F^{\mu\nu} F_{\mu\nu}\big) \\
& + (a-c)\,\alpha\, R^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma} + 2(c-a)\,\alpha\, F_{\mu\nu}\tilde{F}^{\mu\nu} + \tfrac{1}{2}(2\,a-c)\,\alpha\,\mathrm{tr}\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big) \\
& + 4\,a\,\nabla^\mu A_\mu\,\Box\alpha - 8\,\alpha\, A^\mu\Big(R_{\mu\nu} - \tfrac{1}{3} R\, g_{\mu\nu}\Big)\nabla^\nu\alpha - 8\,a\,F_{\mu\nu}\,A^\mu\,\nabla^\nu\sigma\Big\}
\end{aligned}
$$

$$
\begin{aligned}
& +\tfrac{1}{6}\int\sqrt{g}\,\Big\{\sigma\mathcal{R}_{i\bar{k}j\bar{l}}\nabla^\mu J^i \nabla_\mu J^j\,\nabla^\nu \bar{J}^k \nabla_\nu \bar{J}^l + \sigma\, G_{i\bar{\jmath}}\Big(\hat{\Box} J^i\,\hat{\Box}\bar{J}^j - 2\big(R^{\mu\nu} - \tfrac{1}{3} R\, g^{\mu\nu}\big)\partial_\mu J^i\,\partial_\nu \bar{J}^j\Big) \\
& + \tfrac{1}{2} K\,\Box^2\sigma + \tfrac{1}{6} K\,\partial^\mu R\,\partial_\mu\sigma + K\,\big(R^{\mu\nu} - \tfrac{1}{3} R\, g^{\mu\nu}\big)\,\nabla_\mu\nabla_\nu\sigma - 2\,G_{i\bar{\jmath}}\,\nabla^\mu J^i\,\nabla^\nu \bar{J}^j\,\nabla_\mu\nabla_\nu\sigma \\
& + i\,G_{i\bar{\jmath}}\,\Big(\hat{\nabla}^\mu\hat{\nabla}^\nu J^i\,\nabla_\nu \bar{J}^j - \hat{\nabla}^\mu\hat{\nabla}^\nu \bar{J}^j\,\nabla_\nu J^i\Big)\partial_\mu\alpha - \nabla^\mu\mathcal{A}_\mu\,\Box\alpha + 2\,\mathcal{A}^\mu\,\big(R_{\mu\nu} - \tfrac{1}{3} R\, g_{\mu\nu}\big)\nabla^\nu\alpha \\
& - \sigma\, F_{\mu\nu}\,\mathcal{F}^{\mu\nu} + 2\,F_{\mu\nu}\,\mathcal{A}^\mu\,\nabla^\nu\sigma + F_{\mu\nu}\,\nabla^\mu K\,\nabla^\nu\alpha\Big\}
\end{aligned}
$$

Here A_μ is the gauge field contained in the SUGRA multiplet which couples to the $U(1)_{\mathcal{R}}$ current j^μ and $\mathcal{A}_\mu$ is the Kähler connection

$$
\mathcal{A}_\mu = \frac{i}{2}\Big(\partial_i K\,\partial_\mu J^i - \partial_{\bar{\jmath}} K\,\partial_\mu \bar{J^{\imath}}\Big)
$$

and $\mathcal{F}_{\mu\nu}$ its field strength which depends on K through $G_{i\bar{\jmath}}$.

Lessons from the anomaly polynomial

In Gomis et al we used the cohomologically trivial term

$$\mathcal{A} = \int \sqrt{g} \Big\{ - a\,\sigma \left(E_4 - \tfrac{2}{3}\,\Box R\right) + c\,\sigma\, C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - 2\,c\,\sigma\, F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} c\,\sigma\, \mathrm{tr}\big(F^{\mu\nu} F_{\mu\nu}\big)$$
$$+ (a-c)\,\alpha\, R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} + 2(c-a)\,\alpha\, F_{\mu\nu} \tilde{F}^{\mu\nu} + \tfrac{1}{2}(2\,a - c)\,\alpha\, \mathrm{tr}\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big)$$
$$+ 4\,a\,\nabla^\mu A_\mu\, \Box\alpha - 8\,\alpha\, A^\mu \Big(R_{\mu\nu} - \tfrac{1}{3} R\, g_{\mu\nu}\Big) \nabla^\nu \alpha - 8\,a\, F_{\mu\nu}\, A^\mu\, \nabla^\nu \sigma \Big\}$$

$$+ \tfrac{1}{6} \int \sqrt{g}\, \Big\{ \sigma \mathcal{R}_{i\bar{k}j\bar{l}} \nabla^\mu J^i \nabla_\mu J^j\, \nabla^\nu \bar{J}^k \nabla_\nu \bar{J}^l + \sigma\, G_{i\bar{j}} \Big(\hat{\Box} J^i\, \hat{\Box} \bar{J}^j - 2\big(R^{\mu\nu} - \tfrac{1}{3} R\, g^{\mu\nu}\big) \partial_\mu J^i\, \partial_\nu \bar{J}^j \Big)$$
$$+ \tfrac{1}{2} K\, \Box^2 \sigma + \tfrac{1}{6} K\, \partial^\mu R\, \partial_\mu \sigma + K\, \left(R^{\mu\nu} - \tfrac{1}{3} R\, g^{\mu\nu}\right) \nabla_\mu \nabla_\nu \sigma - 2\, G_{i\bar{j}}\, \nabla^\mu J^i\, \nabla^\nu \bar{J}^j\, \nabla_\mu \nabla_\nu \sigma$$
$$+ i\, G_{i\bar{j}} \left(\hat{\nabla}^\mu \hat{\nabla}^\nu J^i\, \nabla_\nu \bar{J}^j - \hat{\nabla}^\mu \hat{\nabla}^\nu \bar{J}^j\, \nabla_\nu J^i \right) \partial_\mu \alpha - \nabla^\mu \mathcal{A}_\mu\, \Box\alpha + 2\, \mathcal{A}^\mu \left(R_{\mu\nu} - \tfrac{1}{3} R\, g_{\mu\nu}\right) \nabla^\nu \alpha$$
$$- \sigma\, F_{\mu\nu}\, \mathcal{F}^{\mu\nu} + 2\, F_{\mu\nu}\, \mathcal{A}^\mu\, \nabla^\nu \sigma + F_{\mu\nu}\, \nabla^\mu K\, \nabla^\nu \alpha \Big\}$$

to establish the relation between the S^4 partition function and the Kähler potential Z.

For later reference we also point out the Osborn anomaly. SUSY requires that the a priori arbitrary tensor is the Riemann tensor on the conformal manifold; i.e. it is no longer an independent anomaly

Here we will use

$$\mathcal{A} = \int\sqrt{g}\Big\{ - a\,\sigma\left(E_4 - \tfrac{2}{3}\Box R\right) + c\,\sigma\, C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - 2\,c\,\sigma\, F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}c\,\sigma\,\mathrm{tr}\big(F^{\mu\nu}F_{\mu\nu}\big)$$

$$+ (a-c)\,\alpha\, R^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma} + 2(c-a)\,\alpha\, F_{\mu\nu}\tilde{F}^{\mu\nu} + \tfrac{1}{2}(2\,a - c)\,\alpha\,\mathrm{tr}\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big)$$

$$+ 4\,a\,\nabla^\mu A_\mu\,\Box\alpha - 8\,\alpha\, A^\mu\Big(R_{\mu\nu} - \tfrac{1}{3}R\, g_{\mu\nu}\Big)\nabla^\nu\alpha - 8\,a\, F_{\mu\nu}\,A^\mu\,\nabla^\nu\sigma\Big\}$$

$$+\tfrac{1}{6}\int\sqrt{g}\,\Big\{\sigma\mathcal{R}_{i\bar{k}j\bar{l}}\nabla^\mu J^i\nabla_\mu J^j\,\nabla^\nu\bar{J}^k\nabla_\nu\bar{J}^l + \sigma\, G_{i\bar{\jmath}}\Big(\hat{\Box}J^i\,\hat{\Box}\bar{J}^j - 2\big(R^{\mu\nu} - \tfrac{1}{3}R\, g^{\mu\nu}\big)\partial_\mu J^i\,\partial_\nu\bar{J}^j\Big)$$

$$+ \tfrac{1}{2}K\,\Box^2\sigma + \tfrac{1}{6}K\,\partial^\mu R\,\partial_\mu\sigma + K\left(R^{\mu\nu} - \tfrac{1}{3}R\, g^{\mu\nu}\right)\nabla_\mu\nabla_\nu\sigma - 2\,G_{i\bar{\jmath}}\,\nabla^\mu J^i\,\nabla^\nu\bar{J}^j\,\nabla_\mu\nabla_\nu\sigma$$

$$+ i\,G_{i\bar{\jmath}}\Big(\hat{\nabla}^\mu\hat{\nabla}^\nu J^i\,\nabla_\nu\bar{J}^j - \hat{\nabla}^\mu\hat{\nabla}^\nu\bar{J}^j\,\nabla_\nu J^i\Big)\partial_\mu\alpha - \nabla^\mu\mathcal{A}_\mu\,\Box\alpha + 2\,\mathcal{A}^\mu\left(R_{\mu\nu} - \tfrac{1}{3}R\, g_{\mu\nu}\right)\nabla^\nu\alpha$$

$$- \sigma\, F_{\mu\nu}\,\mathcal{F}^{\mu\nu} + 2\, F_{\mu\nu}\,\mathcal{A}^\mu\,\nabla^\nu\sigma + F_{\mu\nu}\,\nabla^\mu K\,\nabla^\nu\alpha\Big\}$$

which is a type B Weyl anomaly and therefore tells us that the generating functional contains the counterterm

$$\log\Lambda^2\int F_{\mu\nu}\,\mathcal{F}^{\mu\nu}$$

which encodes non-local information in certain correlation functions.

Taking functional derivatives with respect to J^i, $\bar{J}^{\bar{\jmath}}$ and A_μ gives

$$\langle M_i(k_1)\,\overline{M}_{\bar{\jmath}}(k_2)\,j_\mu(-k_1-k_2)\rangle = G_{i\bar{\jmath}}(q^2\,r_\mu - q\cdot r\,q_\mu)\log\Lambda^2$$

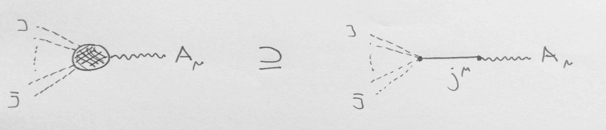
$$q = k_1 + k_2\,, \qquad\qquad r = k_1 - k_2$$

- This cannot originate from an ordinary three point function: the moduli are neutral under $U(1)_R$ and therefore the structure constant $c_{M\bar{M}j}$ vanishes
- This indicates that the $U(1)$ $\mathcal{R}$-current j_μ appears in a contact term in the $M\,\overline{M}$ operator product

$$M_i(x)\,\overline{M}_j(y) \sim G_{i\bar{\jmath}}\Big(\partial_\mu^{(x)}\delta^4(x-y)\,j^\mu(y) - \partial_\mu^{(y)}\delta(x-y)\,j^\mu(y)\Big) + \ldots$$

- It is proportional to the Zamolodchikov metric $\Rightarrow$ cannot be removed by a reparametrization of the sources
- It is a consequence of SUSY
- There could be other local terms in the OPE, but they do not couple to j_μ

- The same counterterm generates correlation functions of an arbitrary number of moduli and one current via $\langle j_\mu \, j_\nu \rangle$



- The local term in the OPE will give a contribution to any correlator involving moduli by coupling the moduli to the $U(1)_{\mathcal{R}}$ current j_μ.

The correlators of $\mathcal{R}$-currents are represented by terms in the effective action containing its source A_μ

$\Rightarrow$ the contribution of the local term in the OPE to correlators with moduli is obtained by replacing A_μ in any term in the generating functional by $\frac{1}{24c}\mathcal{A}_\mu$.

The normalization follows from comparing the following two terms in the anomaly polynomial

$$\mathcal{A} \supset \int \Big(-2\,c\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{6}F_{\mu\nu}\,\mathcal{F}^{\mu\nu}\Big)$$

This is the general formulation of factorization that we are using.

This raises the following question:

To what extend does factorization determine the form of the anomaly polynomial or, more generally, the effective action?

If it were given completely by factorization, the two $U(1)$ gauge fields would only appear in the combination $A_\mu + \frac{1}{24c}\mathcal{A}_\mu$. This is clearly not the case.

For instance, while there is a term $\alpha\, F_{\mu\nu}\tilde{F}^{\mu\nu}$, there is no term $\alpha\, \mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}$, the corresponding terms constructed from $\mathcal{A}_\mu$.

This seems to be dictated by supersymmetry, because there is no way to supersymmetrize $\alpha\, \mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}$, at least not within the setup used here (e.g. moduli in chiral multiplets)

This being said, we will now show explicitly that factorization is required by supersymmetry, but it is 'contaminated' by 'ordinary' terms which contribute to moduli correlators.

We will do this by looking at a simply computable example ...

$\mathcal{N} = 2$ Super-Maxwell theory

It contains of a single vector multiplet (A_μ, λ^i, ϕ) where

λ^i	$SU(2)_\mathcal{R}$ doublet of Weyl fermions
ϕ	complex boson

The action is (we ignore the $SU(2)_\mathcal{R}$ triplet of auxiliary fields as it plays no role)

$$S = -\frac{1}{g^2}\int d^4x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2}{32\,\pi^2}\theta\,F_{\mu\nu}\,\tilde{F}^{\mu\nu} + i\,\bar{\lambda}_i\,\bar{\sigma}^\mu\partial_\mu\lambda^i + \partial_\mu\phi\,\partial^\mu\bar{\phi}\right)$$

The theory has the $U(1)_\mathcal{R}$ current

$$j_\mu = -\bar{\lambda}_i\,\bar{\sigma}_\mu\lambda^i + 2\,i(\phi\,\partial_\mu\bar{\phi} - \bar{\phi}\,\partial_\mu\phi)$$

and one complex modulus with source $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

$$M = \frac{i\,\pi}{2}\left(\frac{1}{8}F^+_{\mu\nu}F^{+\mu\nu} + i\,\bar{\lambda}_i\,\bar{\sigma}^\mu\partial_\mu\lambda^i - \bar{\phi}\,\Box\phi\right) \qquad\qquad F^\pm = F \pm i\,\tilde{F}$$

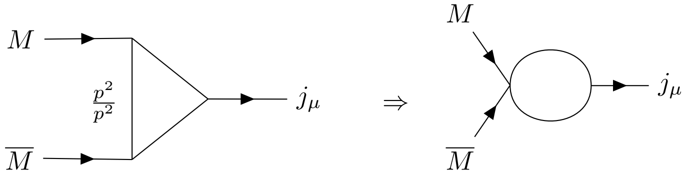
Note that the last two terms in M are 'redundant' operators, i.e. they vanish on-shell.

One might be tempted to drop them, but as we will see, SUSY forbids this. While they do not contribute to the Zamolodchikov metric

$$\langle M(x)\,\overline{M}(y)\rangle$$

which receives only contributions from 'ordinary' $F^{\pm}F^{\pm}$ terms,

they contribute to higher point amplitudes via the 'cancelled propagator mechanism', e.g.



They are responsible for the local term in the $M\overline{M}$ OPE and they are the only parts in M which couple to the $U(1)_{\mathcal{R}}$ current.

Explicit calculation of the one-loop triangle diagram gives

$$\langle M(k_1)\,\overline{M}(k_2)\,j_\mu(-k_1-k_2)\rangle = -\frac{1}{64}\big(q^2\,r_\mu - q\cdot r\,q_\mu\big)\log\frac{\Lambda^2}{q^2} + \text{local} \qquad (1)$$

as expected from the anomaly polynomial.

Even more interesting is the four-point function:

we know that the Osborn anomaly is the signal of a log-divergent counterterm in the four-point function and that $\mathcal{N} = 2$ SUSY dictates that it is of the form

$$\log \Lambda^2 \int \mathcal{R}_{i\bar{k}j\bar{l}}\, \partial^\mu J^i \partial_\mu J^j\, \partial^\nu \bar{J}^k \partial_\nu \bar{J}^l$$

where $\mathcal{R}_{i\bar{k}j\bar{l}}$ is the Riemann tensor on $\mathcal{M}_{conf}$ which is $\mathbb{H}_+$ with metric

$$G_{\tau\bar{\tau}} = \frac{1}{2\,\tau_2}$$

For pure Maxwell (no SUSY) where the modulus only contains the non-redundant part, Osborn has computed the four-point function. It cannot be expressed in terms of the Riemann tensor and is therefore not consistent with $\mathcal{N} = 2$ SUSY. The difference

$$2\,(\nabla^\mu \tau\, \nabla_\mu \bar{\tau})^2 - 5\,|\nabla^\mu \tau\, \nabla_\mu \tau|^2$$

can only be accounted for by the redundant part of M and the local part in the $M\bar{M}$ OPE.

In fact, if one computes the contribution from the fermions and scalars in M to the four-point function

$$\langle M(k_1)\, M(k_2)\, \overline{M}(k_3)\, \overline{M}(k_4)\rangle$$

and adds them up, one precisely recovers the mismatch between Osborn's result and that required by supersymmetry.

This proves that the local-terms in the OPE are necessary to reproduce the result consistent with SUSY.

But this also shows that the factorized contribution gets mixed with the 'ordinary' ones, here those due to the non-redundant part of M.

While the fact that the factorized contribution is due to the redundant part of M is a peculiarity of this simple free model, the general message is not.

Remark: In this simple model we can also explicitly determine the local terms in the $M\bar{M}$ OPE. Besides the $U(1)_{\mathcal{R}}$ current two other operators appear: a second 'accidental' $U(1)$ current and a scalar operator.

Further comments, etc.

- A similar analysis can be performed for $\mathcal{N} = (2, 2)$ in two dimensions, using their anomaly polynomial. While there are differences in exactly how it is used, the main conclusion is once more, that factorized contributions to the OPE of moduli are universal and indispensible.

  The result in $d = 2$ is even stronger: the complete anomalous part of the (non-local) effective action is determined by factorization.

  This can be verified on a simple example with a free (twisted-chiral) superfield coupled to a chiral source.

- The existence of local contributions in certain OPs as required by SUSY should have relevance for the bootstrap of these theories.
- It is an interesting question whether there is any relation to the issue of Kähler shift anomalies which was discussed in recent papers by (Seiberg)-Tachikawa-Yonekura. The factorization assumption might suggest that there is.

Thank you !