Conformal anomalies of Feynman integrals

Emery Sokatchev

LAPTh, Annecy, France

10th May 2018, GGI, Florence

Based on arXiv:1709.03511, 1804.03571
in collaboration with D. Chicherin & J. Henn

Dedicated to the memory of my colleague and friend

Yassen Stanev

Outline

- Conformal anomaly of generalized form factors (vertex functions)
- Proof of the anomalous conformal Ward identity
- Application to one- and two-loop finite Feynman integrals for amplitudes
- Superconformal anomaly of vertex functions in $\mathcal{N} = 1$ matter theory
- Application to a nonplanar two-loop integral
- Conclusions

Generalized form-factors

- Our subject is CFT in D-dimensions. Only massless particles in the spectrum. Minkowski space-time metric $(+, -, \ldots, -)$

- Generalized form-factors

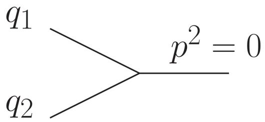
$$F(x_1, \ldots, x_n | p_1, \ldots, p_m) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_n)|p_1, p_2, \ldots, p_m\rangle$$

$\mathcal{O}(x)$ – operators, $|p_i\rangle$ – on-shell state, $p_i^2 = 0$, $i = 1, \ldots, m$.

- Example: ϕ^3-theory in $D = 6$ dimensions. $\mathcal{O} = \phi(x)$. Fourier transform $x \to q$:

$$\tilde{F}(q_1, q_2|p) = \langle \mathcal{O}(q_1)\mathcal{O}(q_2)|\phi(p)\rangle_{\mathrm{Born}}$$

$$= \frac{g}{(q_1^2 + i0)(q_2^2 + i0)}\delta^{(6)}(q_1 + q_2 + p) =$$



Is $\tilde{F}(q_1, q_2|p)$ conformally invariant?

Conformal algebra in D-dimensions

Conformal group of D-dimensional Minkowski space $SO(2, D)$

$$\text{generators} \qquad P_\mu \qquad D \qquad L_{\mu\nu} \qquad K_\mu$$

Off-shell conformal boost K_μ

coordinate space $$K^{(x)}_{\mu;\Delta} = i\left(x^2 \partial_{x^\mu} - 2x_\mu x^\nu \partial_{x^\nu} - 2\Delta x_\mu\right)$$

momentum space $$K^{(q)}_{\mu;\Delta} = -q_\mu \Box_q + 2q^\nu \partial_{q^\nu} \partial_{q_\mu} + 2(D - \Delta)\partial_{q^\mu}$$

Δ – conformal weight

Conformal algebra in D-dimensions

On-shell conformal boost $\mathbb{K}_\mu$

- $D = 4$ realization is well known: [Witten '03]
  Spinor-helicity parametrization of light-like momenta by $SL(2, C)$ spinors

$$\sigma^\mu_{\alpha\dot\alpha} p_\mu = \lambda_\alpha \tilde\lambda_{\dot\alpha} \qquad , \qquad \mathbb{K}_\mu = 2\,\tilde\sigma^{\dot\alpha\alpha}_\mu \frac{\partial^2}{\partial\lambda^\alpha \partial\tilde\lambda^{\dot\alpha}}$$

  $\lambda_\alpha\,,\;\tilde\lambda_{\dot\alpha}$ defined up to phase (helicity).

$$\mathbb{K}_\mu = K^{(p)}_{\mu;\Delta=3} \quad \text{on the space} \quad \varphi = \varphi(\lambda \otimes \tilde\lambda)$$

- In $D = 6$ we use chiral $SL(4)$ spinors λ^{Aa}, $A = 1, \ldots, 4$; [Cheung, O'Connell '09]
  helicity (little group) index $a = 1, 2$

$$p^\mu \tilde\sigma^{AB}_\mu = \lambda^{Aa}\lambda^B_a \qquad , \qquad \mathbb{K}_\mu = -\tilde\sigma^{AB}_\mu \frac{\partial^2}{\partial\lambda^{Aa}\partial\lambda^B_a}$$

$$\mathbb{K}_\mu = K^{(p)}_{\mu;\Delta=4} \quad \text{on the space} \quad \varphi = \varphi(\lambda^a \otimes \lambda_a)$$

Reminder: 4D Holomorphic anomaly

MHV tree-level color ordered 4D amplitude for n gluons

$$\mathcal{A}_n^{--+\ldots+} = \frac{\langle 12\rangle^3\,\delta^{(4)}\left(\sum_{i=1}^n \lambda_i\,\tilde\lambda_i\right)}{\langle 23\rangle\langle 34\rangle\ldots\langle n1\rangle}\,, \qquad \langle ij\rangle \equiv \lambda_i^\alpha \epsilon_{\alpha\beta}\lambda_j^\beta$$

Singular denominators generate holomorphic anomaly [Cachazo,Svrcek,Witten '04]

$$\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}\frac{1}{\langle\lambda\chi\rangle} = \pi\tilde\chi_{\dot\alpha}\delta^2(\langle\lambda\chi\rangle) \qquad \Longleftarrow \qquad \frac{\partial}{\partial\bar z}\frac{1}{z} = \pi\delta^2(z)$$

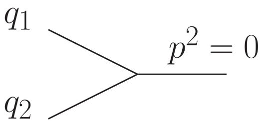
The anomaly is localized on collinear configurations

$$p_2 \sim p_3 \quad , \quad p_3 \sim p_4 \quad , \quad \ldots \quad , \quad p_{n-1}\sim p_n \quad , \quad p_n \sim p_1$$

Anomalous conformal Ward identity [Korchemsky, ES '09; Bargheer, Beisert, Loebbert, McLoughlin, Galleas '09]

$$\mathbb{K}_\mu\,\mathcal{A}_n(1,\ldots,n) \sim \text{“}\sum_{i=1}^n \delta^2(\langle i,i+1\rangle)\,\mathcal{A}_{n-1}(1,\ldots,\check{i},\ldots,n)\text{”}$$

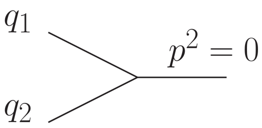
6D vertex function



$$\left(K^{(q_1)}_{\mu;\Delta=2} + K^{(q_2)}_{\mu;\Delta=2} + \mathbb{K}^{(p)}_{\mu}\right) \frac{\delta^{(6)}(q_1+q_2+p)}{(q_1^2+i0)(q_2^2+i0)}$$

$$= \quad ???$$

6D vertex function



$$\left(K^{(q_1)}_{\mu;\Delta=2} + K^{(q_2)}_{\mu;\Delta=2} + \mathbb{K}^{(p)}_{\mu}\right) \frac{\delta^{(6)}(q_1+q_2+p)}{(q_1^2+i0)(q_2^2+i0)}$$

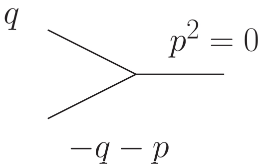
$$= 4i\pi^3\, p_\mu \int_0^1 d\xi\, \xi(1-\xi)\, \delta^{(6)}(q_1+\xi p)\, \delta^{(6)}(q_2+(1-\xi)p)$$

Anomaly is contact and it lives on collinear configurations of momenta

$$q_1 \sim q_2 \sim p$$

6D vertex function

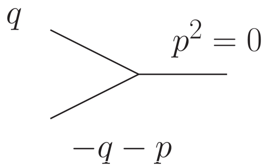
Truncated vertex function:



$$\left(K^{(q)}_{\mu;\Delta=2} + \mathbb{K}^{(p)}_{\mu}\right) \frac{1}{(q^2 + i0)((q+p)^2 + i0)} = 4i\pi^3 \, p_\mu \int_0^1 d\xi \, \xi(1-\xi) \, \delta^{(6)}(q + \xi p)$$

6D vertex function

Truncated vertex function:



$$\left(K^{(q)}_{\mu;\Delta=2} + \mathbb{K}^{(p)}_{\mu}\right) \frac{1}{(q^2 + i0)((q+p)^2 + i0)} = 4i\pi^3\, p_\mu \int_0^1 d\xi\, \xi(1-\xi)\, \delta^{(6)}(q + \xi p)$$

How to prove this distribution relation?

Direct proof in momentum space

- Introduce Feynman parameter ξ and regulator ϵ

$$\frac{1}{q^2(q+p)^2} = \int_0^1 \frac{d\xi}{[(1-\xi)q^2+\xi(q+p)^2]^2}$$
$$= \int_0^1 \frac{d\xi}{[(q+\xi p)^2]^2} = \lim_{\epsilon\to 0}\int_0^1 \frac{d\xi}{[(q+\xi p)^2]^{2-\epsilon}}$$

- Act with $K_\mu = -q_\mu \Box_q - p_\mu \Box_p + \ldots$

$$\left(K^{(q)}_{\mu;\Delta=2} + \mathbb{K}^{(p)}_\mu\right) \int_0^1 \frac{d\xi}{[(q+\xi p)^2]^{2-\epsilon}} \sim p_\mu \int_0^1 \frac{\epsilon\, d\xi}{[(q+\xi p)^2]^{3-\epsilon}}$$

- This distribution has a pole with residue

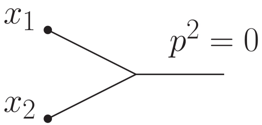
$$[(q+\xi p)^2]^{-3+\epsilon} = \frac{i\pi^3}{2\epsilon}\delta^{(6)}(q+\xi p) + O(\epsilon^0)$$

- The limit $\epsilon \to 0$ produces the contact anomaly

Proof by Lagrangian insertion

- Mixed x/p representation

$$F(x_1, x_2|p) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)|\phi(p)\rangle_{\rm Born} =$$

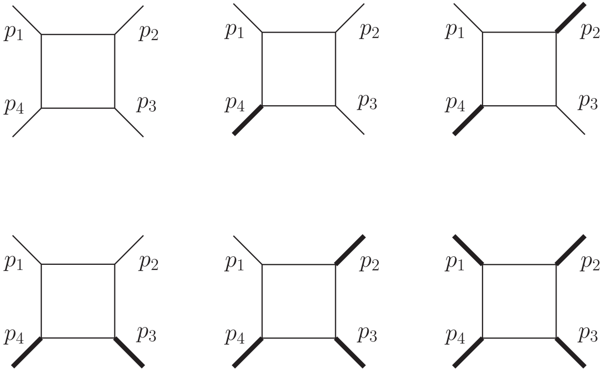


- Conformal variation as Lagrangian insertion

$$\left(K^{(x_1)}_{\mu;\Delta=2} + K^{(x_2)}_{\mu;\Delta=2} + \mathbb{K}^{(p)}_{\mu}\right) \; F(x_1, x_2|p)$$
$$\sim \lim_{\epsilon\to 0} \epsilon \int d^{6-2\epsilon}x \, \langle \phi(x_1)\phi(x_2)\, (x_\mu L(x)) \, |\phi(p)\rangle$$
$$= p_\mu \int_0^1 d\xi \, \xi(1-\xi)\, e^{i(px_1)\xi + i(px_2)(1-\xi)}$$

- Anomaly is not contact in the x/p-representation, so easier to detect.

6D Box integrals



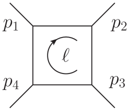
- No UV or IR/collinear divergences
- 6D boxes = Finite parts of 4D boxes + 4D three-mass triangles

  [Bern, Dixon, Kosower '93]
- Are they conformal?

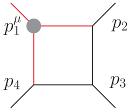
Conformal Ward identities for finite loop integrals

Zero-mass box

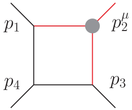
$$\left(\sum_{i=1}^{4}\mathbb{K}_i^{\mu}\right)$$



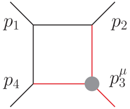
$=$



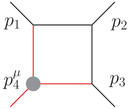
$+$



$+$



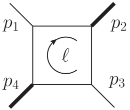
$+$



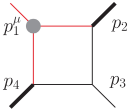
$$\left(\sum_{i=1}^{4}\mathbb{K}_i^{\mu}\right)\delta^{(6)}(P)I_{\text{zero mass}} = \delta^{(6)}(P)\sum_{i=1}^{4}\frac{4p_i^{\mu}}{p_{i-1i}^2 p_{ii+1}^2} = 0$$

Two-mass-easy box

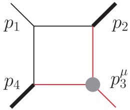
$$(\mathbb{K}_1^{\mu} + K_2^{\mu} + \mathbb{K}_3^{\mu} + K_4^{\mu})$$



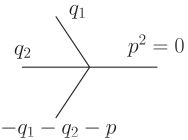
$=$



$+$



Conformal anomalies of 4D ϕ^4 theory



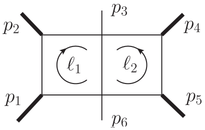
$$= \langle \phi(q_1)\,\phi(q_2)\,\phi(q_3)|\phi(p)\rangle$$

$$\left(\sum_{i=1}^{3} K^{(q_i)}_{\mu;\Delta=1} + \mathbb{K}^{(p)}_{\mu}\right) \frac{\delta^{(4)}\left(\sum_{i=1}^{3} q_i + p\right)}{(q_1^2 + i0)(q_2^2 + i0)(q_3^2 + i0)}$$

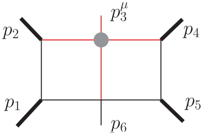
$$= 4\pi^4 p_\mu \int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(\alpha+\beta+\gamma-1)\, \delta^{(4)}(q_1 + \alpha p)\, \delta^{(4)}(q_2 + \beta p)\, \delta^{(4)}(q_3 + \gamma p)$$

Application to 10-leg double box (elliptic integral)

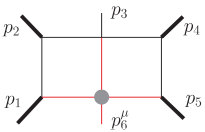
$$\left(\sum_{i=1}^{6} K_i^{\mu}\right)$$



$$=$$



$$+$$



Superconformal anomaly

- Conformal anomalous Ward identity is a 2nd-order differential equation
- Superconformal algebra

$$\{S_\alpha, \bar{S}_{\dot\alpha}\} = \sigma^\mu_{\alpha\dot\alpha} K_\mu$$

  Superconformal generators are 1st-order $\to$ more powerful anomalous WIs
- $\mathcal{N} = 1$ supersymmetric Wess-Zumino model:
  - chiral superfield and action

$$\Phi(x, \theta) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x)$$

$$S_{WZ} = \int d^4x d^2\theta d^2\bar\theta \, \bar\Phi \Phi + \frac{g}{3!} \int d^4x d^2\theta \, \Phi^3 + \frac{g}{3!} \int d^4x d^2\bar\theta \, \bar\Phi^3$$

  - on-shel states (with $p^{\alpha\dot\alpha} = \lambda^\alpha \tilde\lambda^{\dot\alpha}$, $\eta = \lambda^\alpha \theta_\alpha$)

$$\bar\Phi(p, \eta) = \bar\phi(p) + \eta\psi_-(p) \, , \quad \Psi(p, \eta) = \psi_+(p) + \eta\phi(p)$$

Superconformal anomaly

- Vertex function (super form factor)

$$\mathcal{F} \equiv \langle \bar\Phi(q_1,\bar\theta_1)\bar\Phi(q_2,\bar\theta_2)|\bar\Phi(p,\eta)\rangle_{\rm Born} = \delta^{(4)}(P)\,\delta^{(2)}(Q)\,\frac{g}{q_1^2 q_2^2}$$

- Poincaré and conformal supersymmetry generators

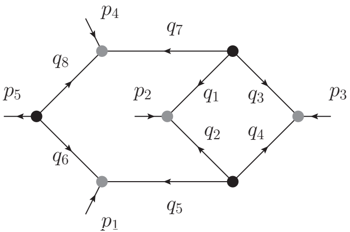
$$Q^\alpha = \sum_i q_i^{\alpha\dot\alpha}(\bar\theta_i)_{\dot\alpha} + \lambda^\alpha\eta\,, \qquad S_\alpha = \frac{1}{2}\sum_i \frac{\partial^2}{\partial q_i^{\alpha\dot\alpha}\partial\bar\theta_{i\,\dot\alpha}} + \frac{\partial^2}{\partial\lambda^\alpha\partial\eta}$$

- Superconformal anomaly from Lagrangian insertion $\int d^4x d^2\theta\,\theta^\alpha L_{int}(x,\theta)$

$$S^\alpha\mathcal{F} = \lambda^\alpha\int_0^1 d\xi\,\left(\eta + \tilde\lambda_{\dot\alpha}\bar\theta_1^{\dot\alpha}\xi + \tilde\lambda_{\dot\alpha}\bar\theta_2^{\dot\alpha}(1-\xi)\right)\delta^{(4)}(q_1+\xi p)\,\delta^{(4)}(q_2+(1-\xi)p)$$

$$S^\alpha\overline{\mathcal{F}} = 0$$

Computation of two-loop nonplanar integral



- Superamplitude (with $[ij] = \langle ij \rangle^*$)

$$\langle \bar{\Phi}(p_1,\eta_1)\bar{\Phi}(p_2,\eta_2)\bar{\Phi}(p_3,\eta_3)\bar{\Phi}(p_4,\eta_4)\Psi(p_5,\eta_5)\rangle$$
$$= \delta^{(4)}(P)\,\delta^{(2)}(Q)\,(\eta_1[23] + \eta_2[31] + \eta_3[12])\, f(p)$$

- Two-loop Feynman integral

$$f(p) = \frac{1}{\langle 45 \rangle} \int \frac{d^4\ell_1 d^4\ell_2}{q_1^2 \dots q_8^2} \langle 2|q_2\tilde{q}_4|3\rangle\langle 1|q_5\tilde{q}_7|4\rangle$$

Computation of two-loop nonplanar integral

- Superconformal anomalous Ward identity $\Rightarrow$ system of 1st-order PDEs

$$F^{\alpha}_{123} f(p) = \sum_{i=1,2,3,4} \lambda^{\alpha}_{i} A_i(p)\,.$$

  with $A_i(p)$ given by simple one-loop integrals
- Differential operator = twistor collinearity operator [Witten'03]

$$F^{\alpha}_{123} = [23]\frac{\partial}{\partial \lambda_{1\,\alpha}} + [31]\frac{\partial}{\partial \lambda_{2\,\alpha}} + [12]\frac{\partial}{\partial \lambda_{3\,\alpha}}$$

- Unique solution of 1st-order PDEs:
  - natural boundary conditions
  - predicts the 31-letter symbol alphabet of [ChicherinHennMitev'17]
  - gives the complete weight-4 HPL function

Conclusions

- Conformal anomaly of generalized form factors
  - Anomaly lives on collinear configurations of momenta
  - Examples: 6D ϕ^3, 4D ϕ^4, 4D Yukawa, 4D gauge theory
  - Conformal Ward identities for finite loop integrals (6D boxes, 6D hexagon with massless legs, 4D double box, 4D Yukawa couplings)

$$\left(\sum_{i=1}^{n-1} \mathbb{K}_i^{\mu}\right) I_n^{(L)}(p) = \sum_{i=1}^{n} p_i^{\mu} A_i^{(L-1)}(p)$$

  - Develop methods for solving the 2nd-order PDEs

- Superconformal anomaly $\to$ 1st-order PDEs
  - $\mathcal{N} = 1$ matter (scalars and fermions)
  - to do: $\mathcal{N} = 1$ vector multiplets (gluons and gluinos)
  - Extend to larger classes of integral topologies
  - Relationship to dual $\bar{Q}$ anomaly for Wilson loops? [Caron-Huot, BullimorSkinner'11]