

Conformal anomalies of Feynman integrals

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Dedicated to the memory of my colleague and friend

Yassen Stanev



Outline

- Conformal anomaly of generalized form factors (vertex functions)
- Proof of the anomalous conformal Ward identity
- Application to one- and two-loop finite Feynman integrals for amplitudes
- Superconformal anomaly of vertex functions in $\mathcal{N} = 1$ matter theory
- Application to a nonplanar two-loop integral
- Conclusions

Generalized form-factors

- Our subject is CFT in D-dimensions. Only massless particles in the spectrum.
Minkowski space-time metric $(+, -, \dots, -)$
- Generalized form-factors

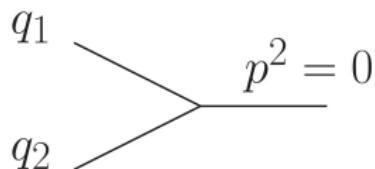
$$F(x_1, \dots, x_n | p_1, \dots, p_m) = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \dots \mathcal{O}(x_n) | p_1, p_2, \dots, p_m \rangle$$

$\mathcal{O}(x)$ – operators, $|p_i\rangle$ – on-shell state, $p_i^2 = 0$, $i = 1, \dots, m$.

- Example: ϕ^3 -theory in $D = 6$ dimensions. $\mathcal{O} = \phi(x)$. Fourier transform $x \rightarrow q$:

$$\tilde{F}(q_1, q_2 | p) = \langle \mathcal{O}(q_1) \mathcal{O}(q_2) | \phi(p) \rangle_{\text{Born}}$$

$$= \frac{g}{(q_1^2 + i0)(q_2^2 + i0)} \delta^{(6)}(q_1 + q_2 + p) =$$



Is $\tilde{F}(q_1, q_2 | p)$ conformally invariant?

Conformal algebra in D-dimensions

Conformal group of D -dimensional Minkowski space $SO(2, D)$

generators P_μ D $L_{\mu\nu}$ K_μ

Off-shell conformal boost K_μ

coordinate space $K_{\mu;\Delta}^{(x)} = i \left(x^2 \partial_{x^\mu} - 2x_\mu x^\nu \partial_{x^\nu} - 2\Delta x_\mu \right)$

momentum space $K_{\mu;\Delta}^{(q)} = -q_\mu \square_q + 2q^\nu \partial_{q^\nu} \partial_{q_\mu} + 2(D - \Delta) \partial_{q^\mu}$

Δ – conformal weight

Conformal algebra in D-dimensions

On-shell conformal boost \mathbb{K}_μ

- $D = 4$ realization is well known:

[Witten '03]

Spinor-helicity parametrization of light-like momenta by $SL(2, C)$ spinors

$$\sigma_{\alpha\dot{\alpha}}^\mu p_\mu = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \quad , \quad \mathbb{K}_\mu = 2 \tilde{\sigma}_\mu^{\dot{\alpha}\alpha} \frac{\partial^2}{\partial \lambda^\alpha \partial \tilde{\lambda}^{\dot{\alpha}}}$$

$\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$ defined up to phase (helicity).

$$\mathbb{K}_\mu = K_{\mu; \Delta=3}^{(\rho)} \quad \text{on the space} \quad \varphi = \varphi(\lambda \otimes \tilde{\lambda})$$

- In $D = 6$ we use chiral $SL(4)$ spinors $\lambda^{Aa}, A = 1, \dots, 4$; [Cheung, O'Connell '09] helicity (little group) index $a = 1, 2$

$$p^\mu \tilde{\sigma}_\mu^{AB} = \lambda^{Aa} \lambda_a^B \quad , \quad \mathbb{K}_\mu = -\tilde{\sigma}_\mu^{AB} \frac{\partial^2}{\partial \lambda^{Aa} \partial \lambda_a^B}$$

$$\mathbb{K}_\mu = K_{\mu; \Delta=4}^{(\rho)} \quad \text{on the space} \quad \varphi = \varphi(\lambda^a \otimes \lambda_a)$$

Reminder: 4D Holomorphic anomaly

MHV tree-level color ordered 4D amplitude for n gluons

$$\mathcal{A}_n^{-+-\dots+} = \frac{\langle 12 \rangle^3 \delta^{(4)} \left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i \right)}{\langle 23 \rangle \langle 34 \rangle \dots \langle n1 \rangle}, \quad \langle ij \rangle \equiv \lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta$$

Singular denominators generate holomorphic anomaly [Cachazo,Svrcek,Witten '04]

$$\frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} \frac{1}{\langle \lambda \chi \rangle} = \pi \tilde{\chi}_{\dot{\alpha}} \delta^2(\langle \lambda \chi \rangle) \iff \frac{\partial}{\partial \bar{z}} \frac{1}{z} = \pi \delta^2(z)$$

The anomaly is localized on collinear configurations

$$p_2 \sim p_3 , \quad p_3 \sim p_4 , \quad \dots , \quad p_{n-1} \sim p_n , \quad p_n \sim p_1$$

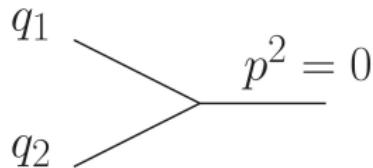
Anomalous conformal Ward identity

[Korchemsky, ES '09;

Bargheer, Beisert, Loebbert, McLoughlin, Galleas '09]

$$\mathbb{K}_\mu \mathcal{A}_n(1, \dots, n) \sim " \sum_{i=1}^n \delta^2(\langle i, i+1 \rangle) \mathcal{A}_{n-1}(1, \dots, \check{i}, \dots, n) "$$

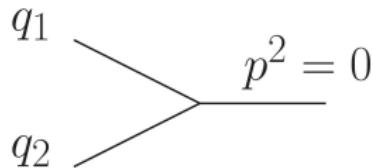
6D vertex function



$$\left(K_{\mu; \Delta=2}^{(q_1)} + K_{\mu; \Delta=2}^{(q_2)} + \mathbb{K}_{\mu}^{(p)} \right) \frac{\delta^{(6)}(q_1 + q_2 + p)}{(q_1^2 + i0)(q_2^2 + i0)}$$

= ???

6D vertex function



$$\left(K_{\mu; \Delta=2}^{(q_1)} + K_{\mu; \Delta=2}^{(q_2)} + \mathbb{K}_{\mu}^{(p)} \right) \frac{\delta^{(6)}(q_1 + q_2 + p)}{(q_1^2 + i0)(q_2^2 + i0)}$$

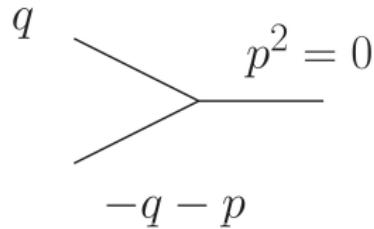
$$= 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1-\xi) \delta^{(6)}(q_1 + \xi p) \delta^{(6)}(q_2 + (1-\xi)p)$$

Anomaly is contact and it lives on collinear configurations of momenta

$$q_1 \sim q_2 \sim p$$

6D vertex function

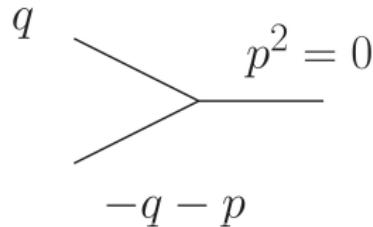
Truncated vertex function:



$$\left(K_{\mu; \Delta=2}^{(q)} + \mathbb{K}_{\mu}^{(p)} \right) \frac{1}{(q^2 + i0)((q + p)^2 + i0)} = 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1 - \xi) \delta^{(6)}(q + \xi p)$$

6D vertex function

Truncated vertex function:



$$\left(K_{\mu; \Delta=2}^{(q)} + \mathbb{K}_{\mu}^{(p)} \right) \frac{1}{(q^2 + i0)((q + p)^2 + i0)} = 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1 - \xi) \delta^{(6)}(q + \xi p)$$

How to prove this distribution relation?

Direct proof in momentum space

- Introduce Feynman parameter ξ and regulator ϵ

$$\begin{aligned}\frac{1}{q^2(q+p)^2} &= \int_0^1 \frac{d\xi}{[(1-\xi)q^2 + \xi(q+p)^2]^2} \\ &= \int_0^1 \frac{d\xi}{[(q+\xi p)^2]^2} = \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{d\xi}{[(q+\xi p)^2]^{2-\epsilon}}\end{aligned}$$

- Act with $K_\mu = -q_\mu \square_q - p_\mu \square_p + \dots$

$$\left(K_{\mu; \Delta=2}^{(q)} + \mathbb{K}_\mu^{(p)} \right) \int_0^1 \frac{d\xi}{[(q+\xi p)^2]^{2-\epsilon}} \sim p_\mu \int_0^1 \frac{\epsilon \, d\xi}{[(q+\xi p)^2]^{3-\epsilon}}$$

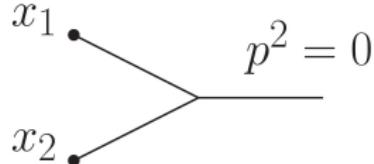
- This distribution has a pole with residue

$$[(q+\xi p)^2]^{-3+\epsilon} = \frac{i\pi^3}{2\epsilon} \delta^{(6)}(q+\xi p) + O(\epsilon^0)$$

- The limit $\epsilon \rightarrow 0$ produces the contact anomaly

Proof by Lagrangian insertion

- Mixed x/p representation

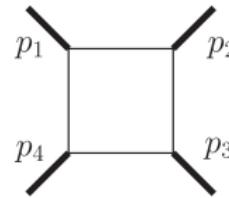
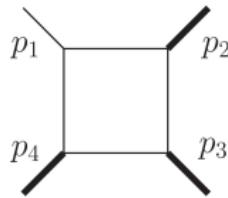
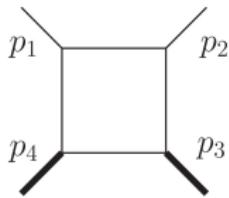
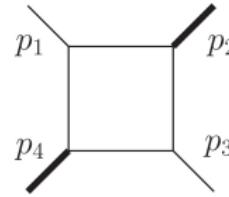
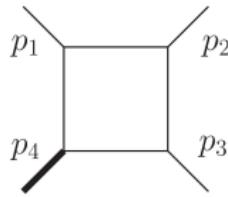
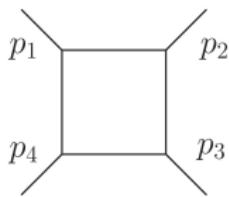
$$F(x_1, x_2 | p) = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) | \phi(p) \rangle_{\text{Born}} =$$


- Conformal variation as Lagrangian insertion

$$\begin{aligned} & \left(K_{\mu; \Delta=2}^{(x_1)} + K_{\mu; \Delta=2}^{(x_2)} + \mathbb{K}_{\mu}^{(p)} \right) F(x_1, x_2 | p) \\ & \sim \lim_{\epsilon \rightarrow 0} \epsilon \int d^{6-2\epsilon} x \langle \phi(x_1) \phi(x_2) (\mathbf{x}_\mu L(\mathbf{x})) | \phi(p) \rangle \\ & = p_\mu \int_0^1 d\xi \xi(1-\xi) e^{i(px_1)\xi + i(px_2)(1-\xi)} \end{aligned}$$

- Anomaly is not contact in the x/p -representation, so easier to detect.

6D Box integrals



- No UV or IR/collinear divergences
- 6D boxes = Finite parts of 4D boxes + 4D three-mass triangles
- Are they conformal?

[Bern, Dixon, Kosower '93]

Conformal Ward identities for finite loop integrals

Zero-mass box

$$\left(\sum_{i=1}^4 \mathbb{K}_i^\mu \right) \text{Diagram} = \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 + \text{Diagram}_4$$

The diagram consists of a square with four external legs labeled p_1, p_2, p_3, p_4 . A circular arrow labeled ℓ indicates a clockwise direction around the square. The first term on the right is a square with a red vertical line connecting the top-left vertex to the top-right vertex, labeled p_1^μ at the top-left and p_2 at the top-right. The second term is a square with a red vertical line connecting the top-right vertex to the bottom-right vertex, labeled p_2^μ at the top-right and p_3 at the bottom-right. The third term is a square with a red vertical line connecting the bottom-right vertex to the bottom-left vertex, labeled p_3^μ at the bottom-right and p_4 at the bottom-left. The fourth term is a square with a red vertical line connecting the bottom-left vertex to the top-left vertex, labeled p_4^μ at the bottom-left and p_1 at the top-left.

$$\left(\sum_{i=1}^4 \mathbb{K}_i^\mu \right) \delta^{(6)}(P) I_{\text{zero mass}} = \delta^{(6)}(P) \sum_{i=1}^4 \frac{4p_i^\mu}{p_{i-1i}^2 p_{ii+1}^2} = 0$$

Two-mass-easy box

$$(\mathbb{K}_1^\mu + K_2^\mu + \mathbb{K}_3^\mu + K_4^\mu) \text{Diagram} = \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3$$

The diagram consists of a square with four external legs labeled p_1, p_2, p_3, p_4 . A circular arrow labeled ℓ indicates a clockwise direction around the square. The first term on the right is a square with a red vertical line connecting the top-left vertex to the top-right vertex, labeled p_1^μ at the top-left and p_2 at the top-right. The second term is a square with a red vertical line connecting the top-right vertex to the bottom-right vertex, labeled p_2^μ at the top-right and p_3 at the bottom-right. The third term is a square with a red vertical line connecting the bottom-right vertex to the bottom-left vertex, labeled p_3^μ at the bottom-right and p_4 at the bottom-left.

Conformal anomalies of 4D ϕ^4 theory

$$= \langle \phi(q_1) \phi(q_2) \phi(q_3) | \phi(p) \rangle$$

$$\begin{aligned} & \left(\sum_{i=1}^3 K_{\mu; \Delta=1}^{(q_i)} + \mathbb{K}_{\mu}^{(p)} \right) \frac{\delta^{(4)} \left(\sum_{i=1}^3 q_i + p \right)}{(q_1^2 + i0)(q_2^2 + i0)(q_3^2 + i0)} \\ &= 4\pi^4 p_\mu \int_0^1 d\alpha \, d\beta \, d\gamma \, \delta(\alpha + \beta + \gamma - 1) \delta^{(4)}(q_1 + \alpha p) \delta^{(4)}(q_2 + \beta p) \delta^{(4)}(q_3 + \gamma p) \end{aligned}$$

Application to 10-leg double box (elliptic integral)

$$\left(\sum_{i=1}^6 K_i^\mu \right) \text{Diagram A} = \text{Diagram B} + \text{Diagram C}$$

Diagram A: A 10-leg double box Feynman diagram. It consists of two vertical columns of five legs each. The left column has legs p_1, p_2, p_3, p_4, p_5 and the right column has legs $p_6, p_7, p_8, p_9, p_{10}$. There are two internal horizontal lines connecting the columns. The top internal line has a clockwise arrow labeled ℓ_1 and the bottom internal line has a clockwise arrow labeled ℓ_2 .

Diagram B: A 10-leg double box Feynman diagram similar to A, but with a red horizontal line connecting the two columns. A grey dot is located at the intersection of this red line and the bottom internal line.

Diagram C: A 10-leg double box Feynman diagram similar to A, but with a red horizontal line connecting the two columns. A grey dot is located at the intersection of this red line and the top internal line.

Superconformal anomaly

- Conformal anomalous Ward identity is a 2nd-order differential equation
- Superconformal algebra

$$\{S_\alpha, \bar{S}_{\dot{\alpha}}\} = \sigma_{\alpha\dot{\alpha}}^\mu K_\mu$$

Superconformal generators are 1st-order \rightarrow more powerful anomalous WIs

- $\mathcal{N} = 1$ supersymmetric Wess-Zumino model:
 - chiral superfield and action

$$\Phi(x, \theta) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x)$$

$$S_{WZ} = \int d^4x d^2\theta d^2\bar{\theta} \bar{\Phi}\Phi + \frac{g}{3!} \int d^4x d^2\theta \Phi^3 + \frac{g}{3!} \int d^4x d^2\bar{\theta} \bar{\Phi}^3$$

- on-shell states (with $p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$, $\eta = \lambda^\alpha \theta_\alpha$)

$$\bar{\Phi}(p, \eta) = \bar{\phi}(p) + \eta \psi_-(p), \quad \Psi(p, \eta) = \psi_+(p) + \eta \phi(p)$$

Superconformal anomaly

- Vertex function (super form factor)

$$\mathcal{F} \equiv \langle \bar{\Phi}(q_1, \bar{\theta}_1) \bar{\Phi}(q_2, \bar{\theta}_2) | \bar{\Phi}(p, \eta) \rangle_{\text{Born}} = \delta^{(4)}(P) \delta^{(2)}(Q) \frac{g}{q_1^2 q_2^2}$$

- Poincaré and conformal supersymmetry generators

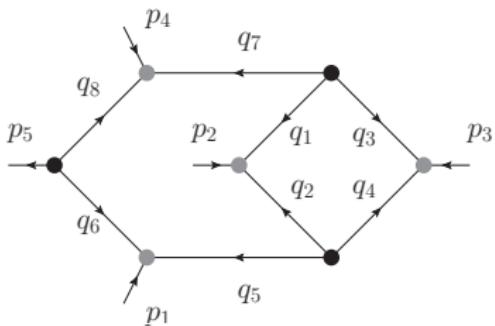
$$Q^\alpha = \sum_i q_i^{\alpha\dot{\alpha}} (\bar{\theta}_i)_{\dot{\alpha}} + \lambda^\alpha \eta, \quad S_\alpha = \frac{1}{2} \sum_i \frac{\partial^2}{\partial q_i^{\alpha\dot{\alpha}} \partial \bar{\theta}_i{}^{\dot{\alpha}}} + \frac{\partial^2}{\partial \lambda^\alpha \partial \eta}$$

- Superconformal anomaly from Lagrangian insertion $\int d^4x d^2\theta \theta^\alpha L_{int}(x, \theta)$

$$S^\alpha \mathcal{F} = \lambda^\alpha \int_0^1 d\xi \left(\eta + \tilde{\lambda}_{\dot{\alpha}} \bar{\theta}_1^{\dot{\alpha}} \xi + \tilde{\lambda}_{\dot{\alpha}} \bar{\theta}_2^{\dot{\alpha}} (1 - \xi) \right) \delta^{(4)}(q_1 + \xi p) \delta^{(4)}(q_2 + (1 - \xi)p)$$

$$S^\alpha \overline{\mathcal{F}} = 0$$

Computation of two-loop nonplanar integral



- Superamplitude (with $[ij] = \langle ij \rangle^*$)

$$\begin{aligned} & \langle \bar{\Phi}(p_1, \eta_1) \bar{\Phi}(p_2, \eta_2) \bar{\Phi}(p_3, \eta_3) \bar{\Phi}(p_4, \eta_4) \Psi(p_5, \eta_5) \rangle \\ &= \delta^{(4)}(P) \delta^{(2)}(Q) (\eta_1[23] + \eta_2[31] + \eta_3[12]) \textcolor{red}{f(p)} \end{aligned}$$

- Two-loop Feynman integral

$$f(p) = \frac{1}{\langle 45 \rangle} \int \frac{d^4 \ell_1 d^4 \ell_2}{q_1^2 \dots q_8^2} \langle 2 | q_2 \tilde{q}_4 | 3 \rangle \langle 1 | q_5 \tilde{q}_7 | 4 \rangle$$

Computation of two-loop nonplanar integral

- Superconformal anomalous Ward identity \Rightarrow system of 1st-order PDEs

$$F_{123}^\alpha f(p) = \sum_{i=1,2,3,4} \lambda_i^\alpha A_i(p).$$

with $A_i(p)$ given by simple one-loop integrals

- Differential operator = twistor collinearity operator

[Witten'03]

$$F_{123}^\alpha = [23] \frac{\partial}{\partial \lambda_{1\alpha}} + [31] \frac{\partial}{\partial \lambda_{2\alpha}} + [12] \frac{\partial}{\partial \lambda_{3\alpha}}$$

- Unique solution of 1st-order PDEs:

- natural boundary conditions
- predicts the 31-letter symbol alphabet of
- gives the complete weight-4 HPL function

[ChicherinHennMitev'17]

Conclusions

- Conformal anomaly of generalized form factors
 - Anomaly lives on collinear configurations of momenta
 - Examples: 6D ϕ^3 , 4D ϕ^4 , 4D Yukawa, 4D gauge theory
 - Conformal Ward identities for finite loop integrals (6D boxes, 6D hexagon with massless legs, 4D double box, 4D Yukawa couplings)

$$\left(\sum_{i=1}^{n-1} \mathbb{K}_i^\mu \right) I_n^{(L)}(p) = \sum_{i=1}^n p_i^\mu A_i^{(L-1)}(p)$$

- Develop methods for solving the 2nd-order PDEs
- Superconformal anomaly \rightarrow 1st-order PDEs
 - $\mathcal{N} = 1$ matter (scalars and fermions)
 - to do: $\mathcal{N} = 1$ vector multiplets (gluons and gluinos)
 - Extend to larger classes of integral topologies
 - Relationship to dual \bar{Q} anomaly for Wilson loops? [Caron-Huot, BullimoreSkinner'11]