Geometric IR subtraction for real radiation

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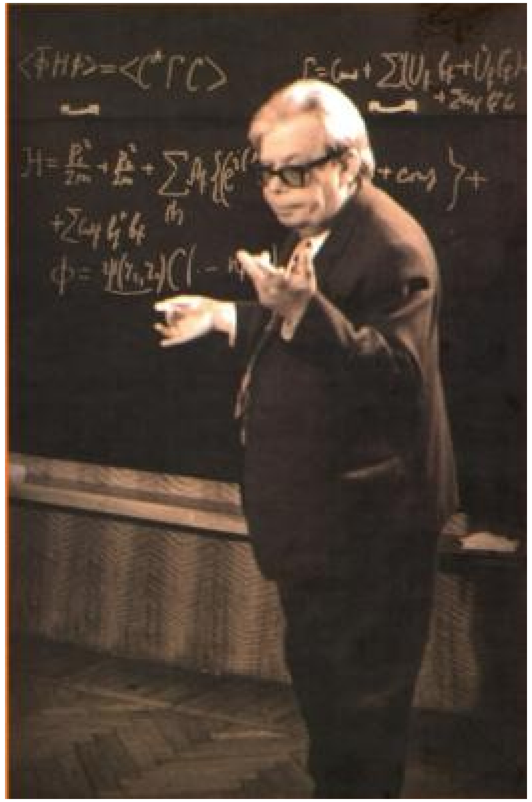
Amplitudes in the LHC Era

GGI Florence 19.10.2018

The Forest Formula

BPHZ (Bogoliubov, Parasiuk 1955; Hepp, Zimmermann)



$$R(\Gamma) = \sum_{U \in \mathcal{U}_r(\Gamma)} \prod_{\gamma \in U} (-K_\gamma)\,\Gamma$$

Ingredients

- Scheme dependent counter-term operation K
- Notion of sets of divergent subgraphs/regions U

What is a forest?

A forest is a set of subgraphs $\{\gamma_1, .., \gamma_n\}$ which are either **nested** $\gamma_i \subset \gamma_j$ or **disjoint** $\gamma_i \cap \gamma_j = 0$.

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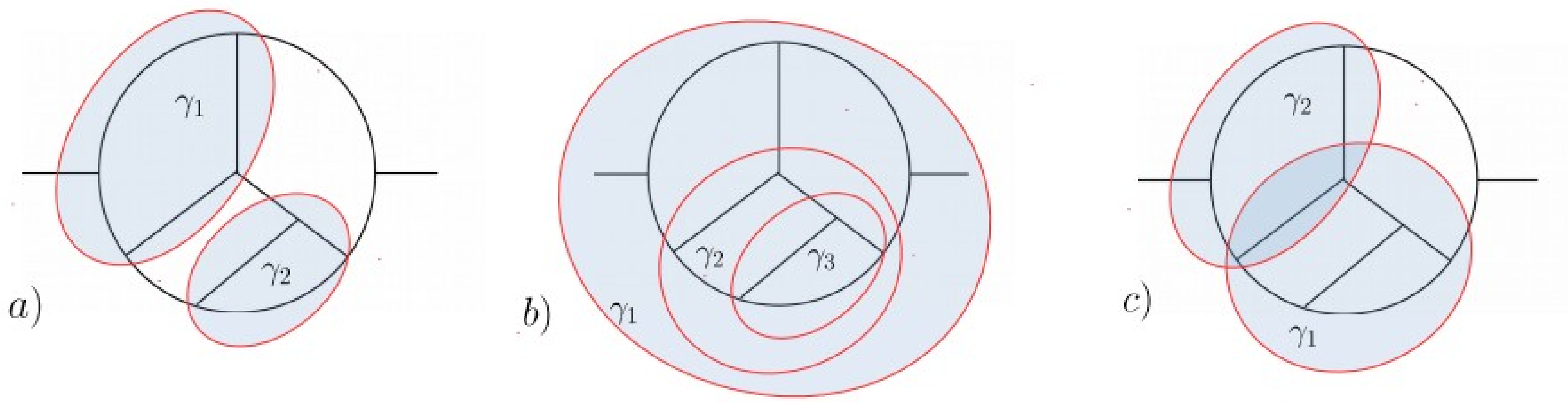

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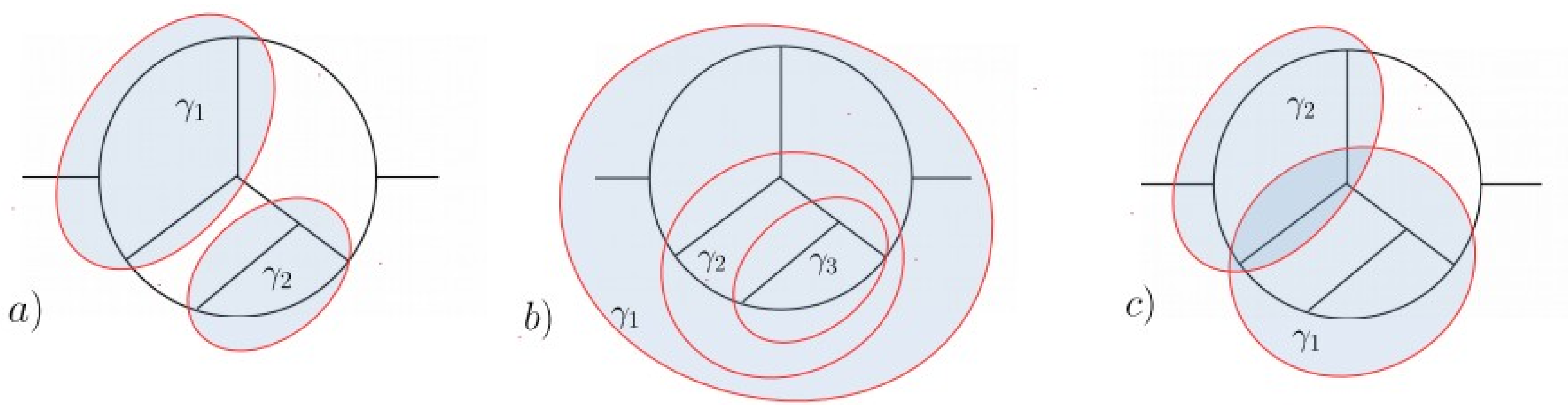

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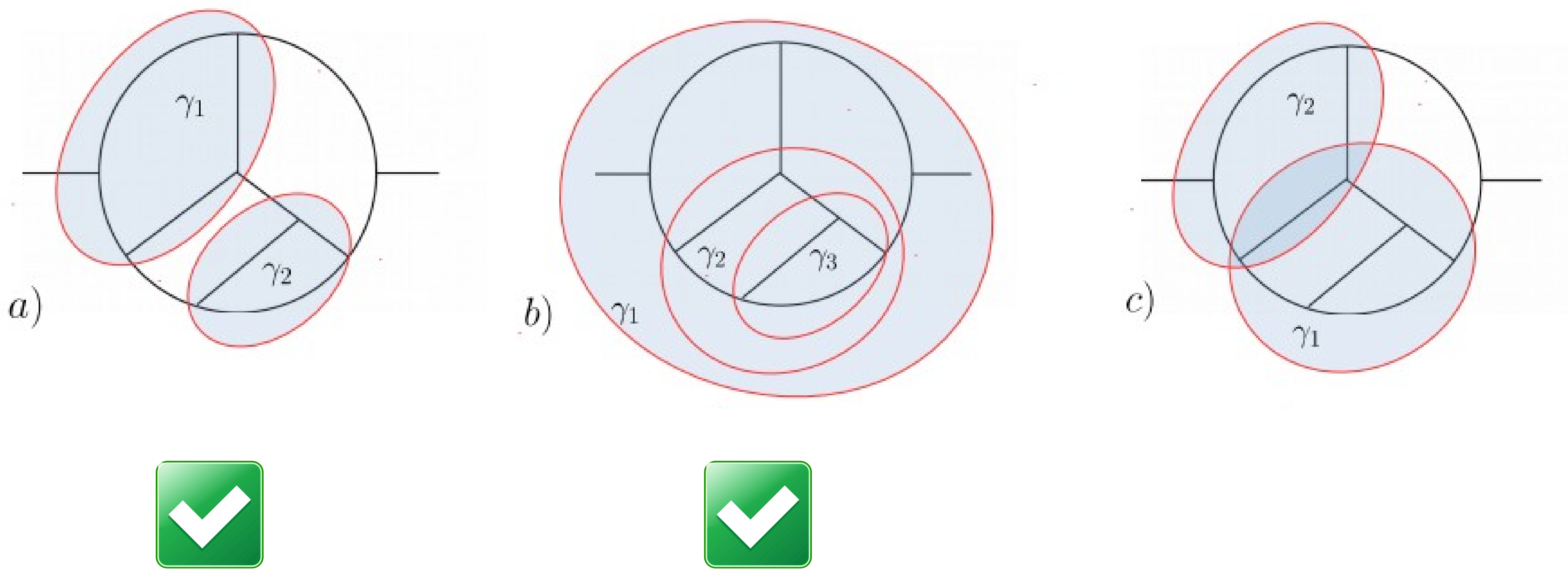

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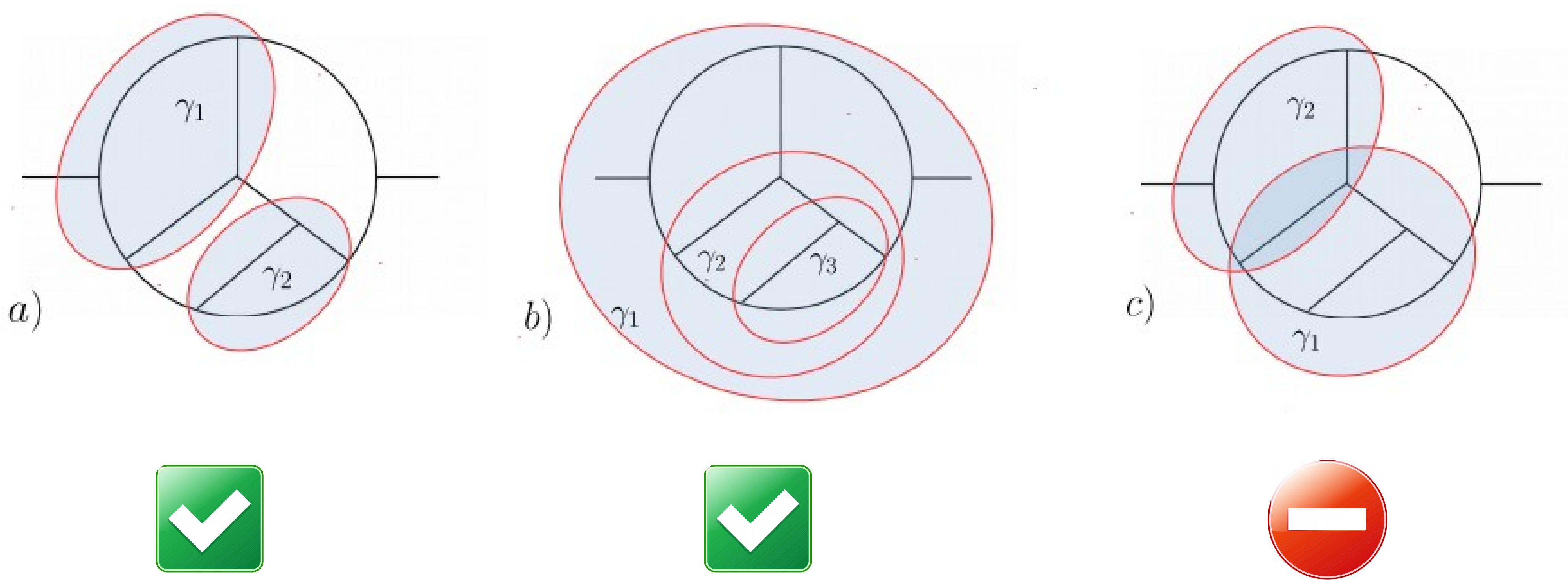

Forest formula generalisations

- Hopf algebraic formulation
  [Kreimer,Connes; Bloch; Brown]
- Generalisation to euclidean IR
  [Chetyrkin, Tkachov; Smirnov; Brown]
- Generalisation to collinear
  [van Neerven]
- Towards soft+collinear forest formula for Real radiation [Collins, Soper, Sterman; Caola, Raul, Roentsch; Somogyi, Trocszanyi, DelDuca; Magnea, Maina, Torielli, Uccirati; FH]
  - On-shell delta functions
  - Overlapping divergences now appear: soft-collinear
  - [FH]: Use a slicing/blow-up scheme to classify and treat overlap

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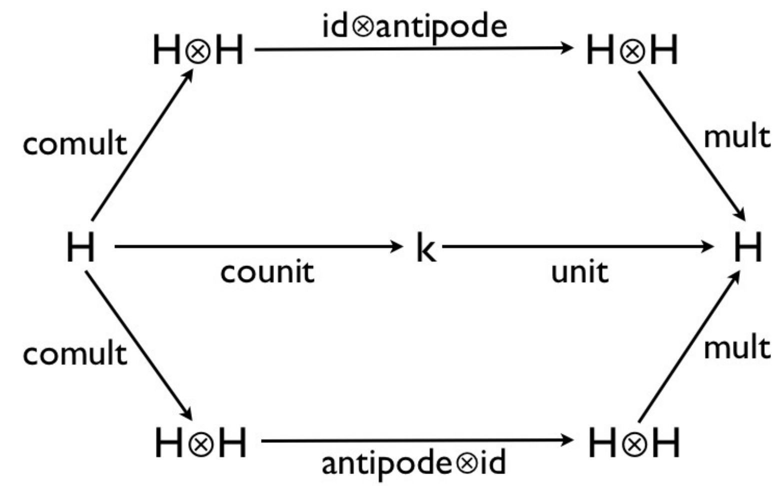

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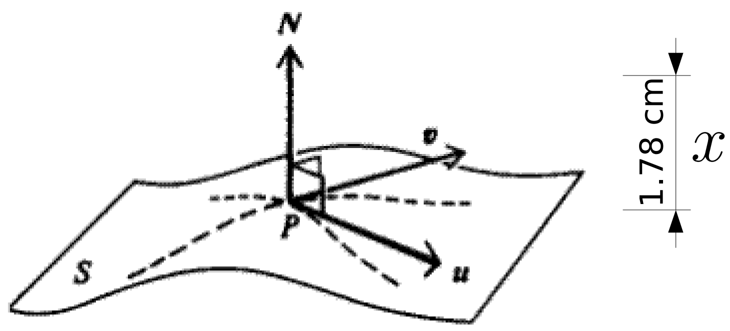
Conjecture:

$$\mathcal{U}^{(l)} = \mathcal{U}_S^{(l)} \times \mathcal{U}_C^{(l)} \mod \mathcal{J}^{(l)}$$

Normal coordinates/slicing parameters



Singular surface S

Normal coordinates measure distance along the surface "normal"

Counter-term prescription:

$$\int_0^{x^+} \frac{dx}{x} f(x) \to \lim_{\epsilon \to 0} \int_\epsilon^{x^+} \frac{dx}{x} f(x)$$

$$= \int_0^{x^+} \frac{dx}{x} f(x) - \int_0^\epsilon \frac{dx}{x} f(0)$$

Normal coordinates

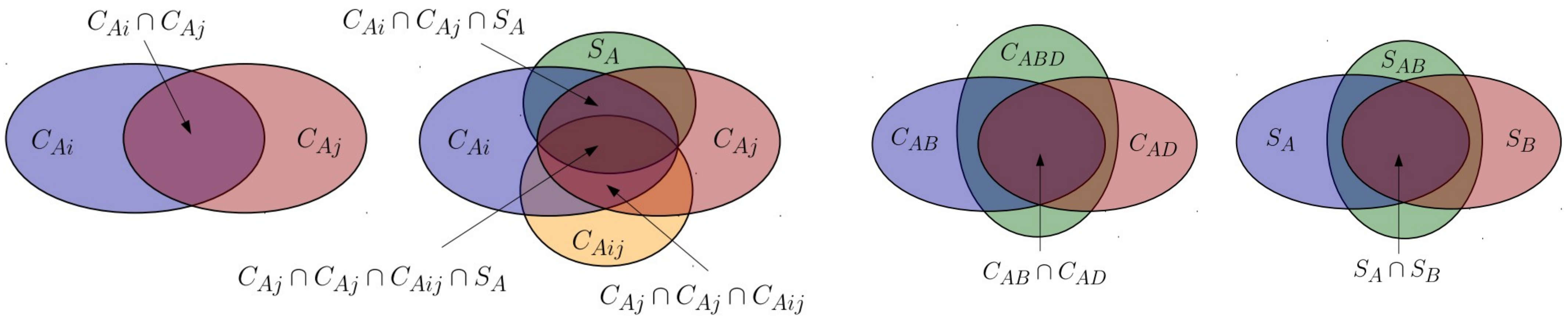
	Region	Normal coordinate	Upper bound
Collinear	$1\|\|2\|\|..\|\|n$	$\frac{s_{12...n}}{Q^2}$	$\leq b_{12..n}$
Soft	$12..n \to 0$	$\frac{2p_{kl}\cdot p_{12...n}}{s_{kl}}$	$\leq a_{12..n}$

Soft variable requires choosing suitable momentum p_{kl}

Hierarchy of regions

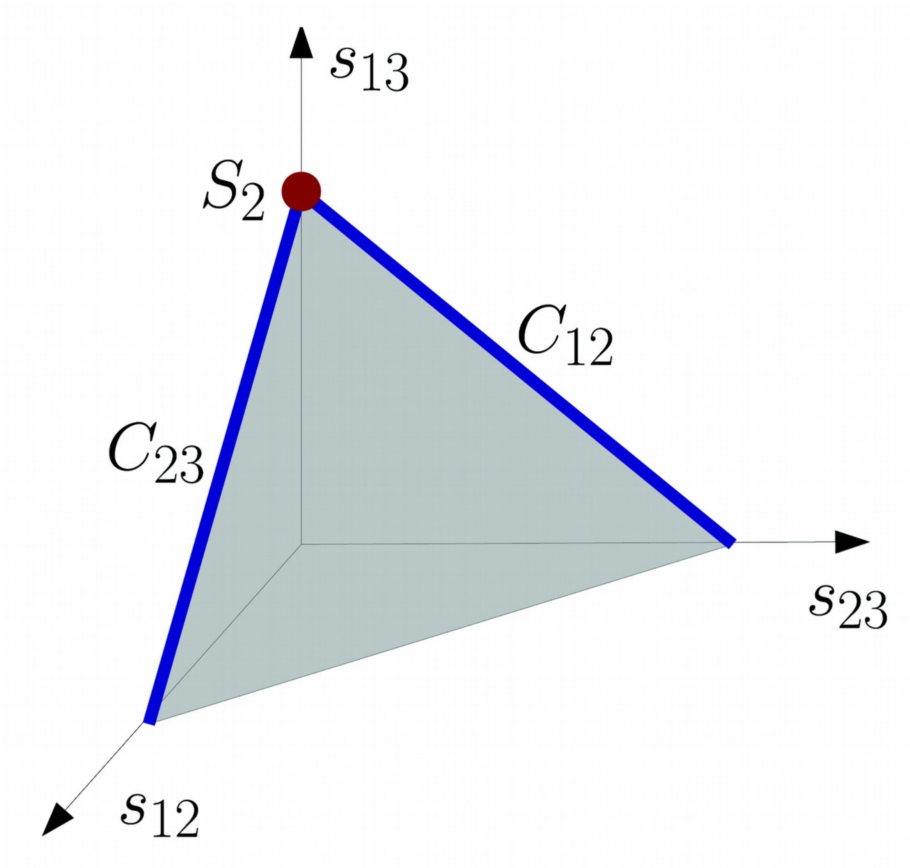
- A forest formula emerges (at least conjecturally) from region cancellations with the hierarchy:

$$a_{i_1 \ldots i_l} \gg a_{i_1 .. i_{l-1}} \gg .. \gg b_{i_1 .. i_{l+1}} \gg .. \gg b_{i_1 i_2}$$



Simple Example

$$\int d\Phi_{123} \frac{s_{13}}{s_{12} s_{23}}$$

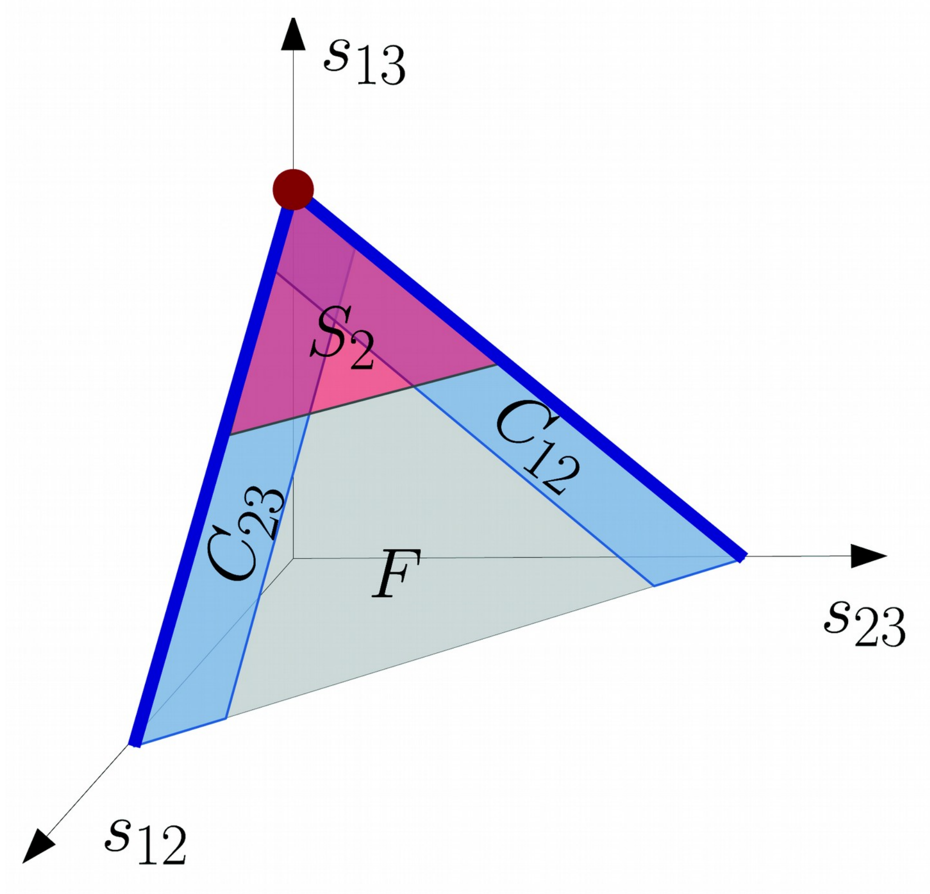


Locations of singularities in Mandelstam space

Simple Example cont.

$$1 = \Theta(F) + \Theta(S_2) + \Theta(C_{12}) + \Theta(C_{23})$$

$$- \Theta(C_{12} \cap S_2) - \Theta(C_{23} \cap S_2)$$



Note: Hierarchy implies that the soft region contains the overlap of the collinear regions .

Simple Example cont.

C_{12}

$$\int \mathrm{d}\Phi_{C_{12}}\Theta(C_{12}) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)}\int_0^{b_{12}Q^2} \mathrm{d}s_{12} s_{12}^{-\epsilon}\int_0^1 \mathrm{d}z_1\,\mathrm{d}z_2\,\delta(1-z_1-z_2)\,(z_1 z_2)^{-\epsilon}$$

$$\int \mathrm{d}\Phi_{C_{12}}\frac{\Theta(C_{12})}{s_{12}}\frac{z_1}{z_2} = (4\pi)^{-2+\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\frac{(b_{12}Q^2)^{-\epsilon}}{\epsilon^2}$$

S_2

$$\int \mathrm{d}\Phi_{S_2}^{(1,3)}\Theta(S_2) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} s_{13}^{-1-\epsilon}\int_0^{\infty} \mathrm{d}s_{12}\,\mathrm{d}s_{23}\,(s_{12}s_{23})^{-\epsilon}\,\Theta(s_{12}+s_{23}<a_2 s_{13})$$

$$\int \mathrm{d}\Phi_{S_2}^{(1,3)}\frac{\Theta(S_2)s_{13}}{s_{12}s_{23}} = (4\pi)^{-2+\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{s_{13}^{-\epsilon}a_2^{-2\epsilon}}{\epsilon^2}$$

$S_2 \cap C_{12}$

$$\int \mathrm{d}\Phi_{C_{12}S_2}\Theta(C_{12}\cap S_2) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)}\int_0^{b_{12}Q^2} \mathrm{d}s_{12} s_{12}^{-\epsilon}\int_0^{a_2} \mathrm{d}z_2\, z_2^{-\epsilon}$$

$$\int \mathrm{d}\Phi_{C_{12}S_2}\frac{\Theta(C_{12}\cap S_2)}{s_{12}z_2} = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)}\frac{(a_2 b_{12}Q^2)^{-\epsilon}}{\epsilon^2}$$

Simple Example cont.

$$I_{\text{Singular}}(Q; a_1, b_{12}, b_{23}) = \tag{3.25}$$
$$\frac{\Phi_2}{Q^2}\Big[+I_{S1}(a_2, Q^2) + I_{C_{12}}(b_{12}Q^2) + I_{C_{12}}(b_{23}Q^2) - I_{C_{12}S_1}(b_{23}Q^2, a_2) - I_{C_{12}S_1}(b_{12}Q^2, a_2)\Big]$$

$$\begin{aligned}
= \frac{\Phi_3}{(Q^2)^2}\Bigg[&+ \left(\frac{2}{\epsilon^2} + \frac{-9 - 4\ln a_2}{\epsilon} + \left(9 + 4\zeta_2 + 18\ln a_2 + 4\ln^2 a_2\right) + \mathcal{O}(\epsilon)\right) \\
&+ \left(\frac{2}{\epsilon^2} + \frac{-7 - 2\ln b_{12}}{\epsilon} + \left(4 + 4\zeta_2 + 7\ln b_{12} + \ln^2 b_{12}\right) + \mathcal{O}(\epsilon)\right) \\
&+ \left(\frac{2}{\epsilon^2} + \frac{-7 - 2\ln b_{23}}{\epsilon} + \left(4 + 4\zeta_2 + 7\ln b_{23} + \ln^2 b_{23}\right) + \mathcal{O}(\epsilon)\right) \\
&- \Bigg(\frac{2}{\epsilon^2} + \frac{-9 - 2\ln a_2 - 2\ln b_{12}}{\epsilon} + (9 + 6\zeta_2 + 9\ln a_2 + 9\ln b_{12} \\
&\qquad + 2\ln a_2 \ln b_{12} + \ln^2 a_2 + \ln^2 b_{12}) + \mathcal{O}(\epsilon)\Bigg) \\
&- \Bigg(\frac{2}{\epsilon^2} + \frac{-9 - 2\ln a_2 - 2\ln b_{23}}{\epsilon} + (9 + 6\zeta_2 + 9\ln a_2 + 9\ln b_{23} \\
&\qquad + 2\ln a_2 \ln b_{23} + \ln^2 a_2 + \ln^2 b_{23}) + \mathcal{O}(\epsilon)\Bigg)\Bigg]
\end{aligned} \tag{3.26}$$

$$= \frac{\Phi_3}{(Q^2)^2}\left[\frac{2}{\epsilon^2} + \frac{-5}{\epsilon} + \left(-1 - 2\ln b_{12} - 2\ln b_{23} - 2\ln a_2 \ln b_{12} - 2\ln a_2 \ln b_{23} + 2\ln^2 a_2\right) + \mathcal{O}(\epsilon)\right].$$

The finite part

- a) Slicing:

$$I_F(Q;a_1,b_{12},b_{23}) = \int d\Phi_{123}\,\Theta(F)\,\frac{s_{13}}{s_{12}\,s_{23}}$$

$$\Theta(F) = \Theta(s_{12} > b_{12}Q^2)\Theta(s_{23} > b_{23}Q^2)\Theta(s_{2(13)} > a_2 s_{13})$$

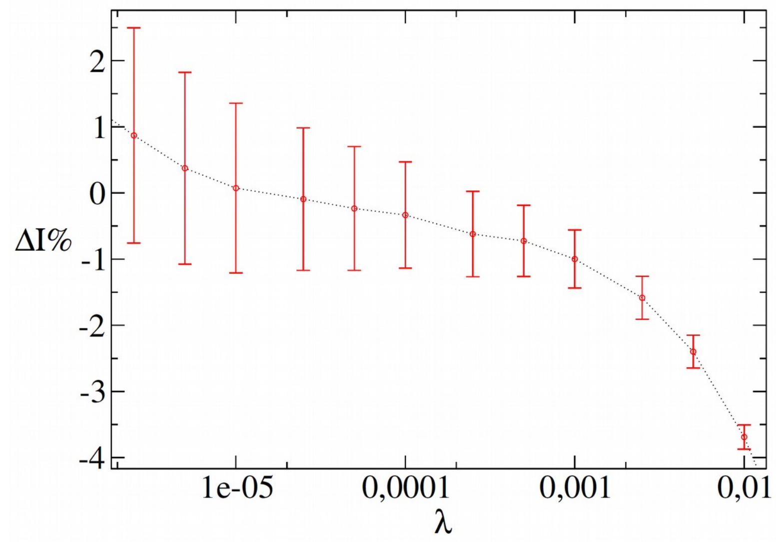
- b) Subtraction:

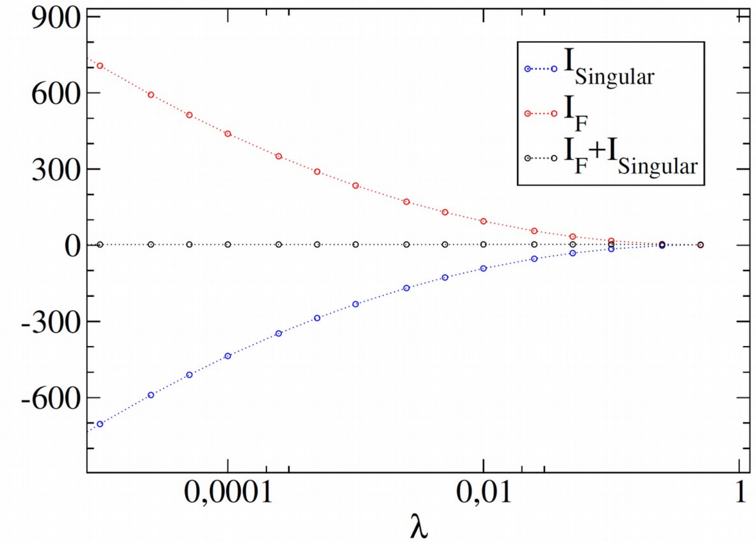
$$\begin{aligned} I_F(Q;a_1,b_{12},b_{23}) = \int d\Phi_{123}\Big[& \frac{s_{13}}{s_{12}\,s_{23}} - \frac{Q^2}{s_{12}\,s_{23}}\Theta(s_{2(13)} < a_2 Q^2) \\ & - \frac{\big(z_{12} - \Theta(z_{21} < a_2)\big)}{s_{12}\,z_{21}(1 - s_{12}/Q^2)}\Theta(s_{12} < b_{12}Q^2) \\ & - \frac{\big(z_{32} - \Theta(z_{23} < a_2)\big)}{s_{23}\,z_{23}(1 - s_{23}/Q^2)}\Theta(s_{23} < b_{23}Q^2)\Big] \end{aligned}$$

Numerics

- Slicing

$$\lambda = a_i, \quad \lambda^2 = b_{ij}$$





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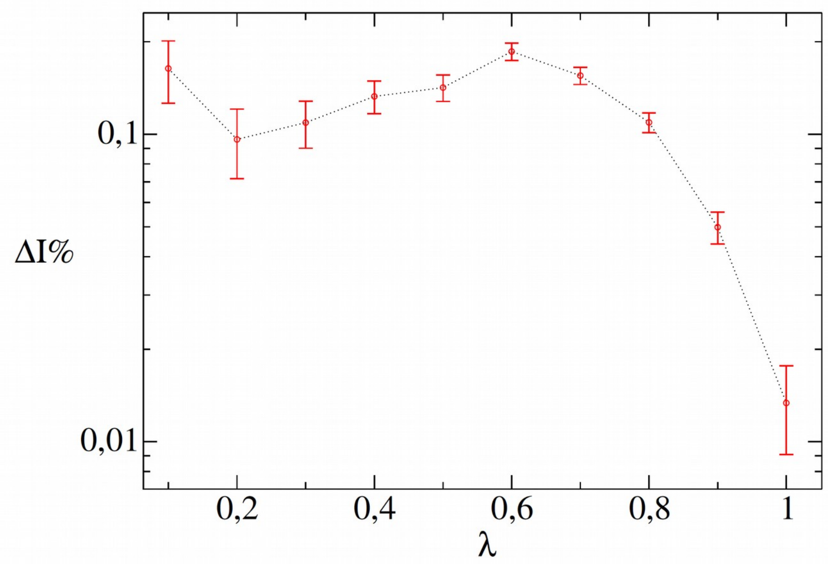
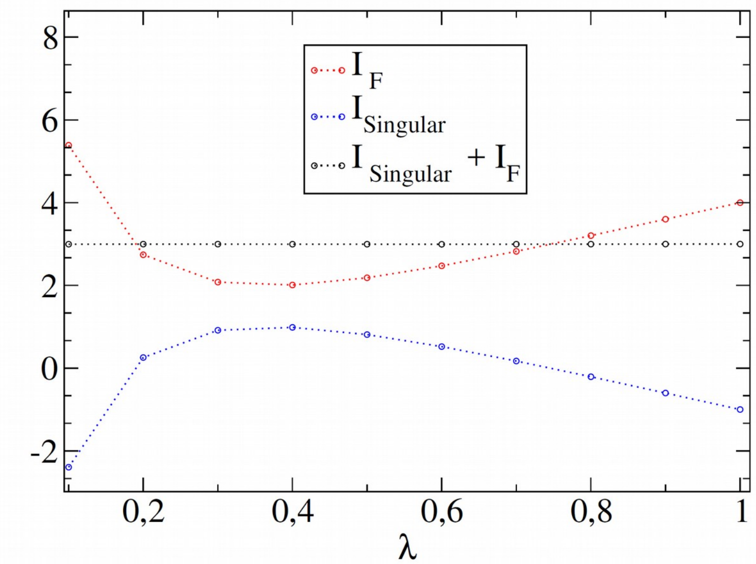




General framework

Start with:

$$\Theta(\text{Singular}) + \Theta(F) = 1 \qquad \Theta(F) = \prod_{r \in R} (1 - \Theta(r))$$

with R the set of all singular regions; to get

$$\Theta(\text{Singular}) = -\sum_{U \subset R} (-1)^{|U|} \prod_{r \in U} \Theta(r)$$

where U is any non-empty subset of R.

Final step: argue that non-desired regions cancel using hierarchy.

A subtraction scheme for QCD

Soft integrals simplify by choosing different soft reference vectors p_{kl} for different diagrams contributing to different eikonal factors!

$$\boldsymbol{\Theta}(\text{Singular}) * |\mathcal{M}_{1..n+l}|^2 = \sum_{k,m} (\mathcal{M}_k^*)_{1..n+l} \, (\mathcal{M}_m)_{1..n+l} \, \Theta(\text{Singular}(k,m))$$

NLO singular part

$$\mathcal{O}^{\text{Singular}}_{1;1..n+1} = -\lim_{a_i \to 0} \lim_{b_{ij} \to 0}$$

$$\cdot \sum_{U \in \mathcal{U}^{(1)}} (-1)^{|U|} \int \mathrm{d}\Phi_{1..n+1}\, \mathcal{J}^{(1)}_{1..n+1} \prod_{r \in U} \mathbf{\Theta}(r) * |\mathcal{M}_{1..n+1}|^2$$

$$\mathcal{U}^{(1)} = \{\{C_{ij}\}, \{S_i\}, \{C_{ij}, S_i\}\}$$

Soft and collinear limits

- Soft:

$$\lim_{a_k \to 0} \mathbf{\Theta}(S_k) * |\mathcal{M}_{1..n+1}|^2 = \sum_{ij} |\mathcal{M}^{(i,j)}_{1..\not{k}..n+1}|^2 \, \mathcal{S}^{(i,j)}_k \, \Theta(a_k s_{ij} - s_{k(ij)})$$

- Collinear:

$$\lim_{b_{ij} \to 0} \mathbf{\Theta}(C_{ij}) * |\mathcal{M}_{..i..j..}|^2 = \frac{2}{s_{ij}} (P_{ij})_{\mu_1 \mu_2} \, |\mathcal{M}^{\boldsymbol{\mu_1 \mu_2}}{}_{..\widehat{ij}..}|^2 \, \Theta(b_{ij} Q^2 - s_{ij})$$

Integrated counterterms NLO for final state gluonic radiation

$$\mathcal{O}^{\text{Singular}}_{1;1..n+1} = \sum_{i>j} \mathcal{I}^{\widehat{C}}_{ij}(Q^2 b_{ij}, a_i, a_j)\, \mathcal{O}_{0;1..\widehat{ij}..n+1} + \sum_i \sum_{k,l\neq i} \int \mathrm{d}\mathcal{O}^{(k,l)}_{0;1..\not{i}..n+1} \mathcal{I}^S_{g_i}(s_{kl}, a_i)$$

$$\mathrm{d}\mathcal{O}^{(i,j)}_{l;1..n+l} = \mathrm{d}\Phi_{1..n+l}\, |\mathcal{M}^{(i,j)}_{1..n+l}| \mathcal{J}^{(l)}_{1..n+l}\,.$$

$$\begin{aligned}\mathcal{I}^S_g(s_{kl}, a_i) &= \int \mathrm{d}\Phi^{(k,l)}_{S_i}(s_{kl}, a_i)\, \mathcal{S}^{(k,l)}_i \\ &= 2c_\Gamma \frac{(a_i^2 s_{kl})^{-\epsilon}}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}\end{aligned}$$

$$\begin{aligned}\mathcal{I}^C_{gg}(Q^2, b_{ij}) &= \int \mathrm{d}\Phi_{C_{ij}}(Q^2 b_{ij})\, \frac{2}{s_{ij}} \langle P_{gg}(z_i)\rangle \\ &= 6C_A c_\Gamma \frac{(Q^2 b_{ij})^{-\epsilon}}{\epsilon^2} \frac{(1-\epsilon)(4-3\epsilon)}{(3-2\epsilon)} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}\end{aligned}$$

$$\begin{aligned}\mathcal{I}^{SC}_{gg}(Q^2, b_{ij}, a_i) &= \int \mathrm{d}\Phi_{C_{ij}S_i}(Q^2 b_{ij}, a_i)\, \frac{2}{s_{ij}} \langle P_{gg}(z_i)\rangle \Big|_{z_i\to 0} \\ &= 4C_A c_\Gamma \frac{(Q^2 b_{ij} a_i)^{-\epsilon}}{\epsilon^2}\end{aligned}$$

$$\mathcal{I}^{\widehat{C}}_{ab}(Q^2, b_{ij}, a_i, a_j) = \mathcal{I}^C_{ab}(Q^2, b_{ij}) - \mathcal{I}^{SC}_{ab}(Q^2, b_{ij}, a_i) - \mathcal{I}^{SC}_{ab}(Q^2, b_{ij}, a_j)$$

NNLO singular part

$$\mathcal{O}^{\text{Singular}}_{2;1..n+2} = - \lim_{a_{ij}\to 0} \lim_{a_i \to 0} \lim_{b_{ijk}\to 0} \lim_{b_{ij}\to 0}$$

$$\cdot \sum_{U \in \mathcal{U}^{(2)}} (-1)^{|U|} \int d\Phi_{1..n+2}\, \mathcal{J}^{(2)}_{1..n+2} \prod_{r\in U} \mathbf{\Theta}(r) * |\mathcal{M}_{1..n+2}|^2$$

$$\begin{aligned}\mathcal{U}^{(2)} = \Big\{ & \{S_i\}, \{S_{ij}\}, \{C_{ij}\}, \{C_{ijk}\}, \{C_{ijk}, C_{ij}\}, \{C_{ijk}, S_{ij}\}, \{C_{ijk}, S_i\}, \{C_{ij}, C_{kl}\}, \\ & \{C_{ij}, S_{ij}\}, \{C_{ij}, S_i\}, \{C_{ij}, S_k\}, \{S_{ij}, S_i\}, \{S_i, S_j\}, \{S_i, S_j, S_{ij}\}, \{C_{ijk}, C_{ij}, S_{ij}\}, \\ & \{C_{ijk}, C_{ij}, S_i\}, \{C_{ijk}, C_{ij}, S_k\}, \{C_{ijk}, S_{ij}, S_i\}, \{C_{ijk}, S_i, S_j\}, \{C_{ijk}, S_i, S_j, S_{ij}\}, \\ & \{C_{ij}, C_{kl}, S_i\}, \{C_{ij}, S_{ij}, S_i\}, \{C_{ij}, S_i, S_k\}, \{C_{ij}, S_i, S_k, S_{ik}\}, \{C_{jk}, S_{ij}, S_i\}, \\ & \{C_{ijk}, C_{ij}, S_{ij}, S_i\}, \{C_{ijk}, C_{ij}, S_{ik}, S_k\}, \{C_{ijk}, C_{ij}, S_i, S_k\}, \{C_{ijk}, C_{ij}, S_i, S_k, S_{ik}\}, \\ & \{C_{ij}, C_{kl}, S_i, S_k\}, \{C_{ij}, C_{kl}, S_i, S_k, S_{ik}\} \Big\}\end{aligned}$$

Double soft limit

Soft momenta factorised but color kinematic correlations
with up to 4 Wilson lines

$$\lim_{k,l\to 0} |\mathcal{M}_{1..n+2}|^2 = \frac{1}{2} \sum_{i,j,r,t=0}^{n} |\mathcal{M}^{(i,j)(r,t)}_{1..\not{k}..\not{l}..n}|^2 \, \mathcal{S}_k^{(i,j)} \, \mathcal{S}_l^{(r,t)}$$

$$- \frac{1}{2} C_A \sum_{i>j=1}^{n} |\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n}|^2 \left(2\, \mathcal{S}_{kl}^{(i,j)} - \mathcal{S}_{kl}^{(i,i)} - \mathcal{S}_{kl}^{(j,j)} \right)$$

Double soft momenta correlated, but only 2 Wilson lines

Double soft limit cont.

- Let the kinematics follow the color!

$$\lim_{a_{kl}\to 0} \mathbf{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 =$$

$$-\frac{1}{2} C_A \sum_{i,j=1\neq k,l}^{n+2} |\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n+2}|^2 \,(2\mathcal{S}^{(i,j)}_{kl} - \mathcal{S}^{(i,i)}_{kl} - \mathcal{S}^{(j,j)}_{kl})\,\Theta(a_{kl}s_{ij} - s_{(kl)(ij)})$$

$$\lim_{a_{kl}\to 0}\lim_{(a_k,a_l)\to 0}(1-\mathbf{\Theta}(S_{kl}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_l) * |\mathcal{M}_{1..n+2}|^2 =$$

$$+\frac{1}{2} \sum_{i,j,r,t\neq k,l} |\mathcal{M}^{(i,j)(r,t)}_{1..\not{k}..\not{l}..n+2}|^2 \,\mathcal{S}^{(i,j)}_{k}\, \mathcal{S}^{(r,t)}_{l}\; \Theta(a_k s_{rt} - s_{k(rt)})\,\Theta(a_l s_{ij} - s_{l(ij)})$$

Master Integrals and reverse unitarity

- 2 double soft integrals appear in higgs threshold production [Anastasiou, Buehler, Duhr, FH]
- 4 triple collinear integrals are identical from the n-jettines jet and beam function [Waalewijn, Ritzmann]
- Large number of overlap contibutions but integrals are “trivial”

$$
\begin{aligned}
\mathbf{M}_S^{(2;1)}(s_{12}, a_{34}) &= \int \mathrm{d}\Phi_{S_{34}}^{(1,2)}(s_{12}, a_{34}) \frac{(s_{12})^2}{(s_{(12)(34)})^4} \\
&= -c_\Gamma^2 \frac{(s_{12})^{-2\epsilon}(a_{34})^{-4\epsilon}}{4\epsilon} \frac{\Gamma^4(1-\epsilon)}{\Gamma(4-4\epsilon)}\,, \\
\mathbf{M}_S^{(2;2)}(s_{12}, a_{34}) &= \int \mathrm{d}\Phi_{S_{34}}^{(1,2)}(s_{12}, a_{34}) \frac{s_{12}}{s_{34}s_{13}s_{24}} \\
&= \mathbf{M}_S^{(2;1)}(s_{12}, a_{34})\, {}_3F_2(1, 1, -\epsilon; 1-\epsilon, 1-2\epsilon; 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_C^{(2;1)}(Q^2 b_{123}) &= \int \mathrm{d}\Phi_{C_{123}}(Q^2 b_{123}) \frac{1}{s_{123}^2} \\
&= -c_\Gamma^2 \frac{(Q^2 b_{123})^{-2\epsilon}}{2\epsilon} \frac{\Gamma^5(1-\epsilon)}{\Gamma(2-2\epsilon)\Gamma(3-3\epsilon)}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_C^{(2;2)}(Q^2 b_{123}) &= \int \mathrm{d}\Phi_{C_{123}}(Q^2 b_{123}) \frac{1}{s_{123}s_{12}z_{23}} \\
&= -\frac{2-3\epsilon}{\epsilon} \mathbf{M}_C^{(2;1)}(Q^2 b_{123})\, {}_3F_2(1, 1-2\epsilon, 1-\epsilon; 2-3\epsilon, 2-2\epsilon; 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_C^{(2;3)}(Q^2 b_{123}) &= \int \mathrm{d}\Phi_{C_{123}}(Q^2 b_{123}) \frac{1}{s_{12}s_{13}z_{13}z_{12}} \\
&= c_\Gamma^2 \frac{(Q^2 b_{123})^{-2\epsilon}}{2\epsilon} \frac{\Gamma^4(1-\epsilon)}{\Gamma(1-4\epsilon)}\, {}_4F_3(1-\epsilon, -2\epsilon, -2\epsilon, -2\epsilon; 1-2\epsilon, 1-2\epsilon, -4\epsilon; 1)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_C^{(2;4)}(Q^2 b_{123}) &= \int \mathrm{d}\Phi_{C_{123}}(Q^2 b_{123}) \frac{1}{s_{12}s_{13}z_2 z_3} \\
&= c_\Gamma^2 (Q^2 b_{123})^{-2\epsilon} \Bigg[3 \frac{\Gamma(1-\epsilon)^5}{\epsilon^4 \Gamma(1-2\epsilon)\Gamma(1-3\epsilon)} \\
&\quad - \frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)}{2\epsilon^4\Gamma(1-4\epsilon)} {}_3F_2(-2\epsilon, -2\epsilon, -2\epsilon; 1-2\epsilon, -4\epsilon; 1) \\
&\quad + \frac{\Gamma(1-\epsilon)^5}{\epsilon^2(1-\epsilon)(1+\epsilon)\Gamma(1-3\epsilon)\Gamma(1-2\epsilon)} {}_4F_3(1, 1-\epsilon, 1-\epsilon, 1-\epsilon; 1-3\epsilon, 2-\epsilon, 2+\epsilon; 1) \Bigg]
\end{aligned}
$$

NNLO integrated counter-terms for final state gluonic radiation

$$
\begin{aligned}
\mathcal{O}^{\text{Singular}}_{2;1..n+2} = & \sum_{i>j} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j)\, \mathcal{O}_{1;1..\widehat{ij}..n+2} \\
& - \sum_{k} \sum_{i,j \neq k} \int \mathrm{d}\mathcal{O}^{(i,j)}_{1;1..\not{k}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{ij}, a_k) \\
& - \sum_{i>j>k>l} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j)\, \mathcal{I}^{\bar{C}}_{g_k g_l}(t_{kl}, a_k, a_l)\, \mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2} \\
& + \sum_{i>j>k} \mathcal{I}^{\bar{C}}_{g_i g_j g_k}(t_{ijk}, t_{ij}, t_{ik}, t_{jk}, a_{ij}, a_{ik}, a_{jk}, a_i, a_j, a_k)\, \mathcal{O}_{0;1..\widehat{ijk}..n+2} \\
& + \sum_{i>j} \sum_{k \neq i,j} \sum_{l,m \in \{1,..,\widehat{ij},..,\not{k},..n+2\}} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j) \int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{lm}, a_k) \\
& + \sum_{k,l} \sum_{i,j,m,n \neq k,l} \int \mathrm{d}\mathcal{O}^{(i,j)(m,n)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{ij}, a_k)\, \mathcal{I}^{S}_{g_l}(s_{mn}, a_l) \\
& - \frac{C_A}{2} \sum_{k,l} \sum_{i,j \neq k,l} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{\hat{S}}_{g_k g_l}(s_{ij}, a_{kl}, a_k, a_l, t_{kl}, t_{ik}, t_{jk}, t_{il}, t_{jl})
\end{aligned}
$$

H→gggg phase space integral check

$$
\begin{aligned}
\mathcal{O}_{H\to g_1g_2g_3g_4} &= 120(c_\Gamma)^2(C_A)^2\mathcal{O}_{H\to g_1g_2} \\
&\cdot\Bigg\{-\frac{1}{\epsilon^4}-\frac{1}{\epsilon^3}\frac{121}{30}+\frac{1}{\epsilon^2}\Big[\frac{39}{5}\zeta_2-\frac{872}{45}\Big]+\frac{1}{\epsilon}\Big[\frac{123}{5}\zeta_3+\frac{473}{15}\zeta_2-\frac{4691}{54}\Big] \\
&\quad+\Big[-\frac{37}{10}\zeta_4-\frac{304951}{810}+99\zeta_3+\frac{2303}{15}\zeta_2\Big]+\mathcal{O}(\epsilon)\Bigg\}
\end{aligned}
$$

Poles check out!

Finite terms remain to be checked!

$$
\begin{aligned}
\mathcal{O}^{\text{Singular}}_{H\to g_1g_2g_3g_4} &= 120(c_\Gamma)^2(C_A)^2\mathcal{O}_{H\to g_1g_2} \\
&\cdot\Bigg\{-\frac{1}{\epsilon^4}-\frac{1}{\epsilon^3}\frac{121}{30}+\frac{1}{\epsilon^2}\Big[\frac{39}{5}\zeta_2-\frac{872}{45}\Big]+\frac{1}{\epsilon}\Big[\frac{123}{5}\zeta_3+\frac{473}{15}\zeta_2-\frac{4691}{54}\Big] \\
&\quad+\Big[-\frac{586351}{1620}+\frac{6788}{45}\zeta_2+\frac{1496}{15}\zeta_3-\frac{8}{5}\zeta_4-\frac{1}{5}L^4_{\alpha_2}-\frac{17}{3}L^2_{\alpha_1}-\frac{89}{135}L_{\beta_2} \\
&\quad-\frac{6}{5}L^2_{\beta_2}-\frac{22}{15}L_{\beta_2}L^2_{\alpha_2}-\frac{22}{15}L^2_{\beta_2}L_{\alpha_2}-\frac{2}{5}L^2_{\beta_2}L^2_{\alpha_2}-\frac{8}{5}L^2_{\alpha_1}L^2_{\beta_2}+\frac{4}{5}L^4_{\alpha_1} \\
&\quad-\frac{44}{15}L^2_{\alpha_1}L_{\beta_1}-\frac{22}{15}L^2_{\alpha_2}L_{\beta_1}-\frac{16}{5}L_{\beta_1}L^3_{\alpha_1}-\frac{22}{15}L_{\beta_2}L^2_{\alpha_1}-\frac{22}{5}L^2_{\beta_2}L_{\alpha_1} \\
&\quad-\frac{4}{5}L^2_{\beta_2}\zeta_2-\frac{16}{5}L_{\alpha_1}\zeta_3-\frac{8}{5}L_{\alpha_2}\zeta_3-\frac{44}{15}L_{\alpha_2}\zeta_2+\frac{22}{15}L^3_{\alpha_2}+\frac{503}{27}L_{\alpha_1} \\
&\quad+\frac{187}{18}L_{\beta_1}+\frac{121}{90}L^2_{\beta_1}-\frac{44}{15}L_{\alpha_1}\zeta_2+4\zeta_3L_{\beta_2}+\frac{8}{5}L_{\beta_2}L_{\beta_1}\zeta_2 \\
&\quad+\frac{16}{5}L_{\beta_1}L^2_{\alpha_1}L_{\beta_2}+\frac{44}{15}L_{\beta_1}L_{\beta_2}L_{\alpha_2}+\frac{4}{5}L^2_{\alpha_2}L_{\beta_1}L_{\beta_2}+\frac{44}{5}L_{\beta_1}L_{\alpha_1}L_{\beta_2} \\
&\quad-\frac{8}{5}L_{\alpha_1}L_{\beta_1}\zeta_2+\frac{8}{5}L^2_{\alpha_1}\zeta_2-\frac{16}{5}L_{\alpha_2}L_{\alpha_1}\zeta_2-\frac{8}{5}L_{\beta_2}L_{\alpha_1}\zeta_2+\frac{8}{5}L^2_{\alpha_2}\zeta_2 \\
&\quad+\frac{4}{5}L^2_{\alpha_2}L^2_{\alpha_1}+\frac{134}{45}L_{\beta_2}L_{\alpha_1}+\frac{12}{5}L_{\beta_2}L_{\beta_1}+\frac{8}{5}L_{\alpha_1}L^3_{\alpha_2}+\frac{644}{45}L_{\alpha_1}L_{\beta_1} \\
&\quad+\frac{44}{15}L^2_{\beta_1}L_{\alpha_1}+\frac{8}{5}L^2_{\beta_1}L^2_{\alpha_1}-\frac{12}{5}L_{\beta_1}L_{\alpha_1}L^2_{\alpha_2}-\frac{8}{5}L^2_{\alpha_1}L_{\alpha_2}L_{\beta_2} \qquad \text{(5} \\
&\quad-\frac{8}{5}L_{\alpha_1}L^2_{\beta_2}L_{\alpha_2}-\frac{4}{5}L_{\alpha_1}L^2_{\alpha_2}L_{\beta_2}+\frac{16}{5}L_{\beta_1}L_{\alpha_1}L_{\beta_2}L_{\alpha_2}\Big]+\mathcal{O}(\epsilon)\Bigg\}
\end{aligned}
$$

Beyond?

Maximal forest partition

A maximal forest U_m is a forest which has maximal size in $\mathcal{U}$.

$$1 = \sum_{U_m \in \mathcal{U}} \rho(U_m)$$

In each sector the only singularities which can occur are those contained
In the particular maximal forest.

[FKS at NLO, Stripper at NNLO, sector-improved subtraction...]

EXAMPLE

$$I_1 = \int \frac{\mathrm{d}\Phi_{1234}}{s_{12} s_{123} s_{124}}$$

Maximal forests:

$$U_m^1 = \{C_{12}, C_{123}, S_{12}\}\,, \qquad U_m^2 = \{C_{12}, C_{124}, S_{12}\}$$

Partitions:

$$\rho(U_m^1) = \frac{s_{124}}{s_{123} + s_{124}}\,, \qquad \rho(U_m^2) = \frac{s_{123}}{s_{123} + s_{124}}\,.$$

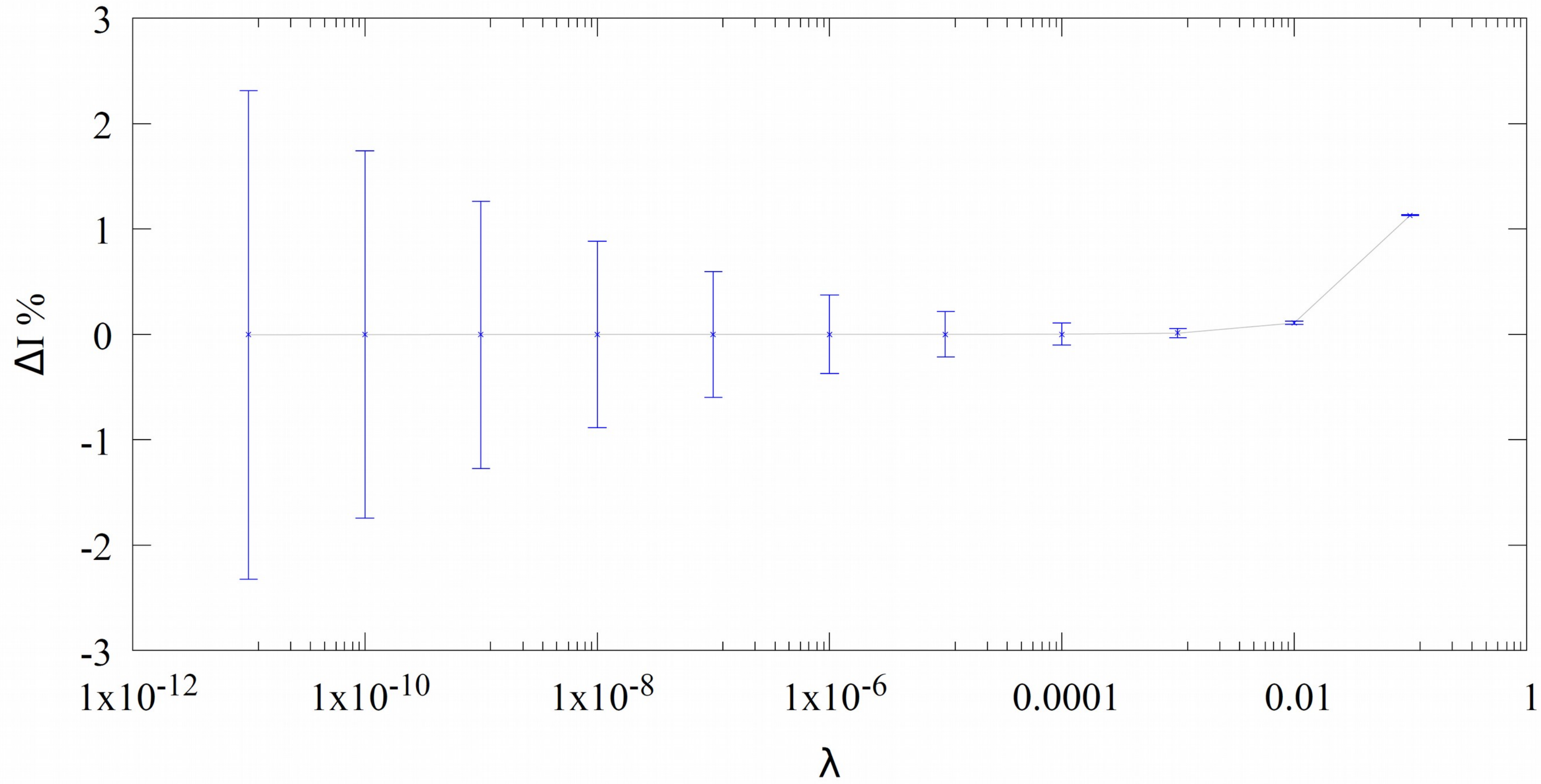
Maximal forest integral representation

$$\int \mathrm{d}\Phi_{1234}\Theta(S_{12})\Theta(C_{123})\Theta(C_{12})\frac{f(..)}{y_{12}y_{123}\bar{y}_{34}} =$$

$$\mathcal{N}\int_{a_{12}}^{1}\frac{\mathrm{d}\bar{y}_{34}}{\bar{y}_{34}}\int_{b_{123}}^{\bar{y}_{34}}\frac{\mathrm{d}y_{123}}{y_{123}}\int_{b_{12}}^{\frac{(\bar{y}_{34}-y_{123})y_{123}}{\bar{y}_{123}}}\frac{\mathrm{d}y_{12}}{y_{12}}\Delta_3^{-\epsilon}f(..)$$

There appears to exist a unique representation which allows to insert the cutoffs explicitely.

Numerically stable slicing



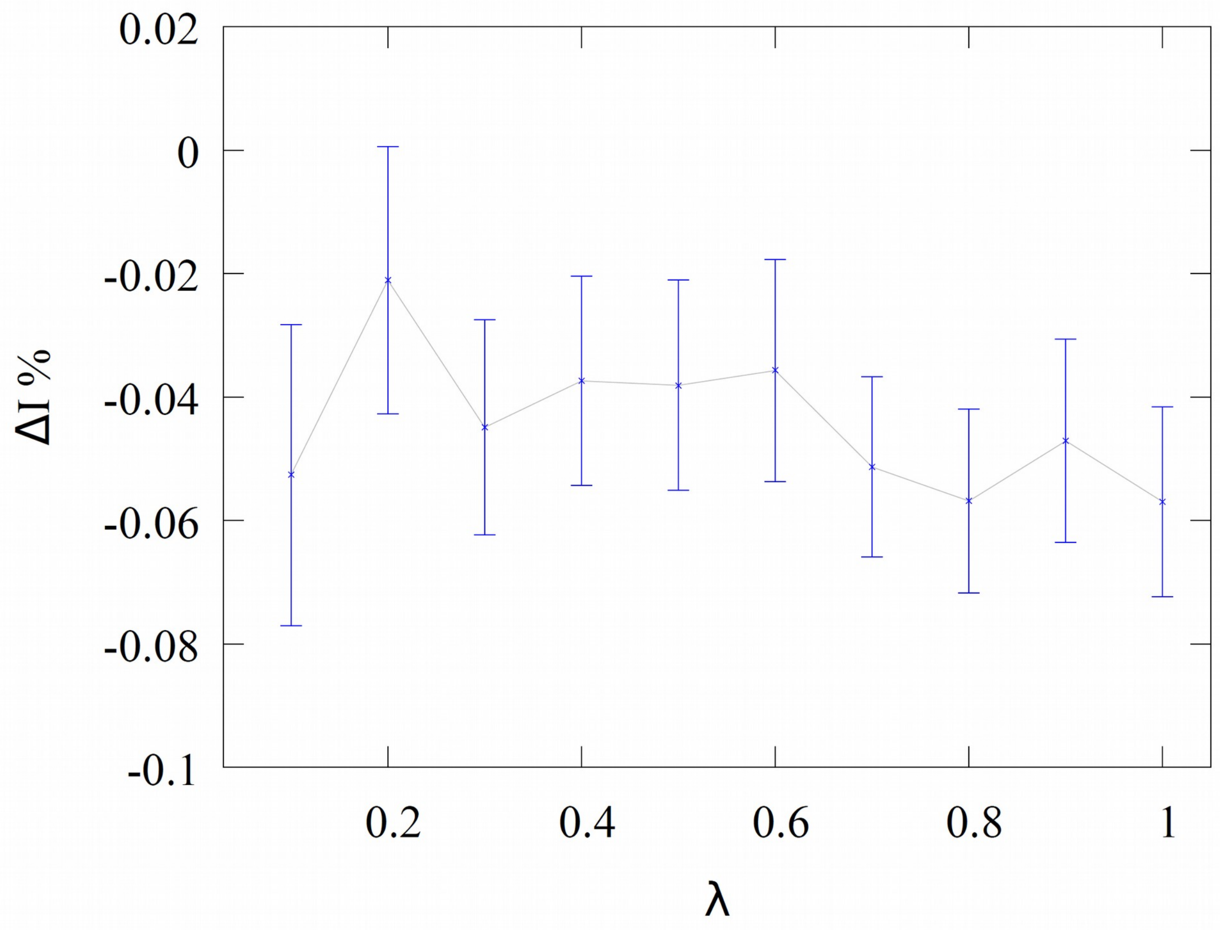
$$a_{12} = \lambda\,, \quad b_{123} = \lambda^2\,, \qquad b_{12} = \lambda^3\,.$$

Subtraction without mappings!

$$
\begin{aligned}
\int \mathrm{d}I_F\,\rho(U_m^1) = \quad & \mathcal{N}\Bigg[\int_0^1 \mathrm{d}\bar{y}_{34}\Big[\int_0^{\bar{y}_{34}} \mathrm{d}y_{123}\Big[\int_0^{\frac{(\bar{y}_{34}-y_{123})y_{123}}{\bar{y}_{123}}} \mathrm{d}y_{12}\,\mathcal{I}_F - \int_0^{b_{12}} \mathrm{d}y_{12}\,\mathcal{I}_S\Big] \\
& \qquad - \int_0^{b_{123}} \mathrm{d}y_{123}\Big[\int_0^{\bar{y}_{34}y_{123}} \mathrm{d}y_{12}\,\mathcal{I}_S - \int_0^{b_{12}} \mathrm{d}y_{12}\,\mathcal{I}_S\Big]\Big] \\
& - \int_0^{a_{12}} \mathrm{d}\bar{y}_{34}\Big[\int_0^{\bar{y}_{34}} \mathrm{d}y_{123}\Big[\int_0^{(\bar{y}_{34}-y_{123})y_{123}} \mathrm{d}y_{12}\,\mathcal{I}_S - \int_0^{b_{12}} \mathrm{d}y_{12}\,\mathcal{I}_S\Big] \\
& \qquad - \int_0^{b_{123}} \mathrm{d}y_{123}\Big[\int_0^{\bar{y}_{34}y_{123}} \mathrm{d}y_{12}\,\mathcal{I}_S - \int_0^{b_{12}} \mathrm{d}y_{12}\,\mathcal{I}_S\Big]\Big]\Bigg] \quad (21)
\end{aligned}
$$

Subtraction without mappings

$$a_{12} = b_{12} = b_{123} = \lambda$$



Conclusions

- Presented a new subtraction formalism based on Feynman diagram dependent slicing observable.
- Have analytically integrated all counter-terms for gluonic final state real radiation at NNLO.
- Integrated counter-terms are simple and can be recycled from Higgs soft expansion and n-jettiness beam and jet function.
- Scheme is not very (?) suitable as a slicing scheme.
- Outlook: Promote the integrated limits to local subtraction terms