Calabi-Yaus and Other Animals

[arXiv:1805.09326] with J. Bourjaily, Y.-H. He, A. Mcleod, and M. Wilhelm
[arXiv:1810.07689] with J. Bourjaily, A. Mcleod, and M. Wilhelm

Matt von Hippel (Niels Bohr International Academy)

Multiple Polylogarithms

Integrals over rational factors:

$$G(w_1, w_2, \ldots; z) = \int_0^z \frac{1}{x - w_1} G(w_2, \ldots; x) dx$$

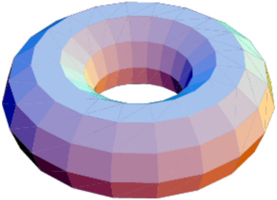
Elliptic Multiple Polylogarithms

Integrals over an elliptic curve:

$$\mathrm{E}\left(\begin{smallmatrix} 0 & n_2 & \cdots \\ 0 & c_2 & \cdots \end{smallmatrix}; z\right) = \int_0^z \frac{1}{y(x)} \mathrm{E}\left(\begin{smallmatrix} n_2 & \cdots \\ c_2 & \cdots \end{smallmatrix}; x\right) dx$$

where

$$y^2 \sim (x^4) + x^3 + \ldots$$



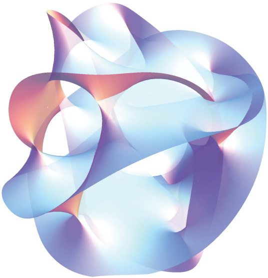
??? Multiple Polylogarithms

Integrals over a higher-dimensional manifold:

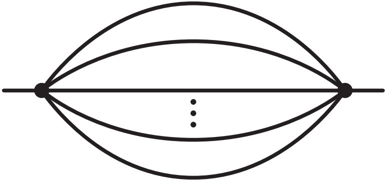
$$F(???) = \int \frac{1}{y(x_1, x_2, \ldots)} F(???; x_1, x_2, \ldots) dx_1 dx_2 \ldots$$

where

$$y^2 \sim P(x_1, x_2, \ldots)$$



Known Examples are Calabi-Yau



- Known to be CY_{L-1} at L loops [Bloch, Kerr, Vanhove; Broadhurst]

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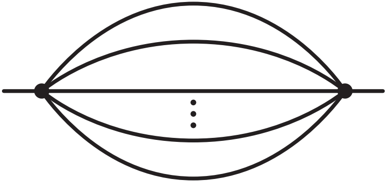

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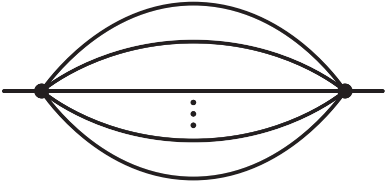

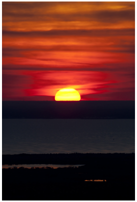

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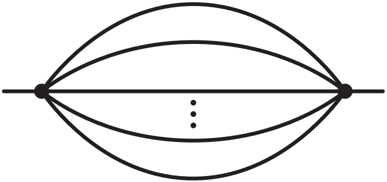

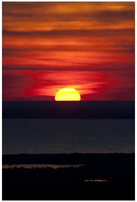

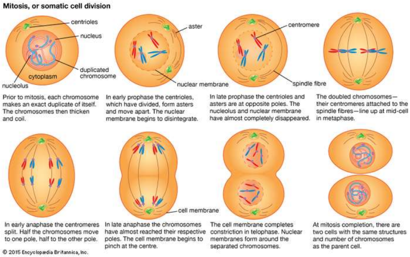

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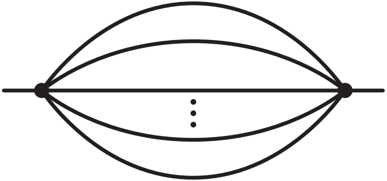

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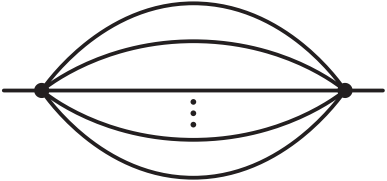
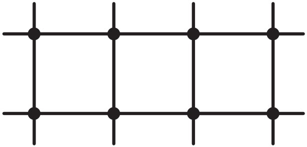


- Known to be CY_{L-1} at L loops [Bloch, Kerr, Vanhove; Broadhurst]

- Eight-loop ϕ^4 vacuum graph with a K3 (CY_2) [Brown, Schnetz]
- L-loop "traintracks" appear to be CY_{L-1} [Bourjaily, He, Mcleod, MvH, Wilhelm]



Questions:

- Why are these examples Calabi-Yau?
- Are more Feynman integrals Calabi-Yau? (All?)
- How bad can it get? (Dimensions vs. loop order)

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My Goals Today:

- Make what definite statements I can
- Inspire further investigation!

Symanzik Form

- Introduce "alpha parameters" for each propagator:

$$\frac{1}{p^2 - m^2} = \int_0^\infty e^{i(p^2-m^2)\alpha} d\alpha$$

- Get well-known form, projective integral over one variable per edge:

$$\Gamma(E - LD/2) \int_{x_i \geq 0} [d^{E-1} x_i] \frac{\mathfrak{U}^{E-(L+1)D/2}}{\mathfrak{F}^{E-LD/2}}$$

- Graph polynomials $\mathfrak{U}$ and $\mathfrak{F}$ defined by:

$$\mathfrak{U} \equiv \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i, \quad \mathfrak{F} \equiv \left[\sum_{\{T_1, T_2\} \in \mathfrak{T}_2} s_{T_1} \Bigg(\prod_{e_i \notin T_1 \cup T_2} x_i \Bigg) \right] + \mathfrak{U} \sum_{e_i} x_i m_i^2$$

- (Neglecting numerators, higher propagator powers)

Symanzik Form: Special Cases

Two cases where things simplify, both for even dimensions:

- $E = LD/2$: Explored by mathematicians. Superficial divergence from gamma function, if there are no subdivergences can strip this off, no need for dim reg. Only $\mathfrak{U}$ contributes.

$$\int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{U}^{D/2}}$$

- $E = (L+1)D/2$: **Marginal**. If finite, can again avoid dim reg. Only $\mathfrak{F}$ contributes.

$$\int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{F}^{D/2}}$$

  - In $D = 2$, these are the sunrise/banana graphs!
  - Many more cases in $D = 4$

Direct Integration

We can attempt to integrate these with direct integration:

- Start with a rational function. Can partial-fraction in some variable x, getting
$$\int_{x\geq 0} \frac{P(z)}{x-Q(z)} + \frac{R(z)}{(x-S(z))^2} + \ldots$$
where z represents the other variables.
- Linear denominators integrate to logarithms, double poles and higher stay rational
- If $P, Q, \ldots$ rational in another variable, repeat: get polylogarithms

Rigidity

- What if some of $P, Q, \ldots$ aren't rational?
  - Square root of a quadratic: this is expected to still be polylogarithmic. Sometimes possible to manifestly rationalize with a change of variables, see e.g. [Besier, Van Straten, Weinzierl]
  - Square root of cubic or higher: in general, cannot be rationalized, sign of non-polylogarithmicity
- Try all possible integration orders. We define the **rigidity** of an integral as the minimum number of variables left in the root.
- N.B.: This does not rule out more unusual changes of variables/re-parametrizations! To do that, would need a "more invariant" picture (differential equations?)

What is a Calabi-Yau?

- Compact Kähler manifold with vanishing first Chern class
- Ricci-flat
- Preserves N=1 supersymmetry of compactifications

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- Compact Kähler manifold with vanishing first Chern class
- Ricci-flat
- Preserves N=1 supersymmetry of compactifications
- **. . . not helpful!**

How do you **diagnose** a Calabi-Yau?

Embed the patient in a weighted projective space!

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$$(x_1, x_2, \ldots) \sim (\lambda x_1, \lambda x_2, \ldots)$$

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- Curve should scale uniformly in λ (homogeneous polynomial)
- If the sum of the coordinate weights equals the overall scaling (degree), your curve is Calabi-Yau!

Did you check the patient for singularities?

Strictly, this only works if the Calabi-Yau is not singular

- F is singular $\equiv$ points where $\nabla F = 0$
- Generically, our manifolds **are** singular!
- Can blow up to smooth singularities – we usually skip this part

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Excuses:

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Excuses:

- All cases we've checked in detail work [ongoing with Candelas, Elmi, Schafer-Nameki, Wang]
- Even mathematicians assume this will work [Brown 0910.0114]
- Charles Doran: "A Calabi-Yau is whatever you want it to be"

Marginal Integrals are Calabi-Yau

Let's look at our "special cases".

[Brown 0910.0114] explored the $E = LD/2$ case, argument for marginal integrals ($E = (L+1)D/2$) similar:

- $\mathfrak{F}$ is homogenous, degree $L+1$, so $\mathfrak{F}^{D/2}$ has degree $(L+1)D/2 = E$ in E variables
- Direct integration preserves this: each integration removes one variable, and decreases the degree of the denominator by one.
- Suppose we encounter a square root. For rigidity m, root $\sqrt{Q(x_i)}$ will contain a degree $2m$ polynomial in m variables.
- Curve $y^2 = Q(x_i)$. Give the x_i weight 1, y weight m. Then sum of the weights is equal to degree $\rightarrow$ diagnosed Calabi-Yau!

Example: Massless $D = 4$

- Specialize to $D = 4$, massless propagators:

$$\int_{x_i \geq 0} [d^{2L+1} x_i] \frac{1}{\mathfrak{F}^2}$$

- $\mathfrak{F}$ is linear in every variable (x_i^2 only shows up in the mass term). We may integrate out any one parameter x_j. Writing $\mathfrak{F} \equiv \mathfrak{F}_0^{(j)} + x_j \mathfrak{F}_1^{(j)}$:

$$\int_{x_i \geq 0} [d^{2L} x_i] \frac{1}{\mathfrak{F}_0^{(j)} \mathfrak{F}_1^{(j)}}$$

- Each factor is still linear, so we can integrate in another variable x_k. Writing $\mathfrak{F}_i^{(j)} \equiv \mathfrak{F}_{i,0}^{(j,k)} + x_k \mathfrak{F}_{i,1}^{(j,k)}$:

$$\int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log\left(\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)}\right) - \log\left(\mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}\right)}{\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}}$$

Example: Massless $D = 4$

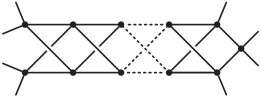
$$\int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log\left(\mathfrak{F}^{(j,k)}_{0,0} \mathfrak{F}^{(j,k)}_{1,1}\right) - \log\left(\mathfrak{F}^{(j,k)}_{0,1} \mathfrak{F}^{(j,k)}_{1,0}\right)}{\mathfrak{F}^{(j,k)}_{0,0} \mathfrak{F}^{(j,k)}_{1,1} - \mathfrak{F}^{(j,k)}_{0,1} \mathfrak{F}^{(j,k)}_{1,0}}$$

- Denominator is at most quadratic in each remaining variable.
- If irreducibly quadratic in all variables (and discriminants irreducibly cubic or quartic in all other variables), then Calabi-Yau with rigidity $2L - 2$.
- Thus for massless marginal integrals in $4D$, rigidity is **bounded**.

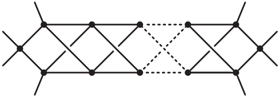
Is this bound saturated?

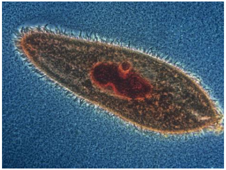
Yes!

Even $L \geq 2$
Tardigrades

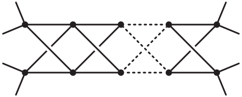


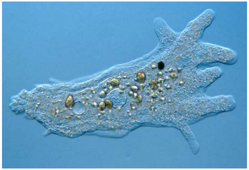
Odd $L \geq 1$
Paramecia





Odd $L \geq 5$
Amoebas

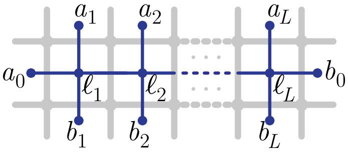




Observations:

- The $L = 2$ tardigrade is a two-loop, five-point (three external masses) K3!
- We've looked at other marginal integrals with box power counting through seven loops, the majority are maximally rigid.
- The $L = 3$ amoeba is oddly enough *not* maximally rigid.

What about the Traintracks?



- Not marginal: $E = 3L + 1 \neq (L+1)D/2$ for $L \neq 1$
- Not Symanzik:

$$\int_0^\infty [d^L\alpha] d^L\beta \frac{1}{(f_1 \cdots f_L) g_L}$$

$$f_k \equiv (a_0 a_{k-1}; a_k b_{k-1})(a_{k-1} b_k; b_{k-1} a_0)(a_k b_k; a_{k-1} b_{k-1}) f_{k-1} + \alpha_0(\alpha_k + \beta_k) + \alpha_k \beta_k$$
$$+ \sum_{j=1}^{k-1} \left[\alpha_j \alpha_k (b_j a_0; a_j a_k) + \alpha_j \beta_k (b_j a_0; a_j b_k) + \alpha_k \beta_j (a_0 a_j; a_k b_j) + \beta_j \beta_k (a_0 a_j; b_k b_j) \right]$$

$$g_L \equiv \alpha_0 + \sum_{j=1}^{L} \left[\alpha_j (b_j a_0; a_j b_0) + \beta_j (a_0 a_j; b_0 b_j) \right]; \quad (ab; cd) \equiv \frac{x_{a,b}\, x_{c,d}}{x_{a,c}\, x_{b,d}}$$

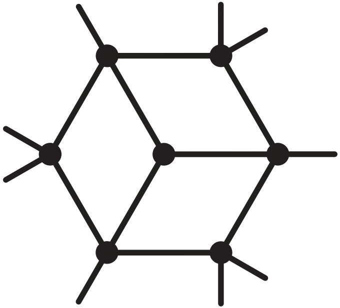
Three-Loop K3

- Take codimension $L+1$ residue, uncovering rigidity
- Get $\sqrt{Q}$, where Q is degree 4 in α_2 and degree 6 in α_1 and α_0
- Can transform to Weierstrass form, rational transformation $\alpha_2 \to x$ s.t. the curve becomes:

$$y^2 = 4x^3 - xg_2(\alpha_0, \alpha_1) - g_3(\alpha_0, \alpha_1)$$

  where g_2 has degree 8 and g_3 has degree 12
- Assign weight 6 to y, weight 4 to x, and weight 1 to α_0, α_1. $6+4+1+1=12$, satisfies Calabi-Yau condition.

Wheel/Coccolithophore

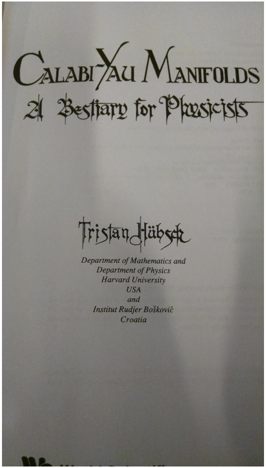


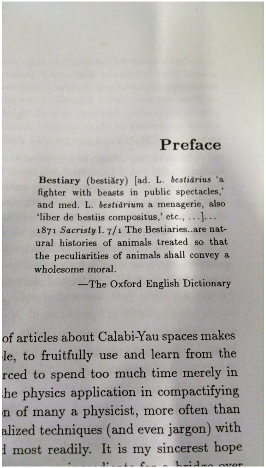
- Once again, not marginal, not Symanzik
- Planar, relevant to $\mathcal{N} = 4$ sYM
- For special kinematics, is CY_3
- We haven't found embedding for general kinematics though...maybe rigid, but not CY?

Further Questions

- Do different integration pathways give different Calabi-Yaus? Different parametrizations?
  - Unlike elliptic curves, no general way to determine if two Calabi-Yaus are the same
  - Could show two curves are *different* by checking geometric data
  - Is there an invariant notion of “the geometry”? Maybe from differential equations?
- Generalizations?
  - Traintracks are not marginal, but they are Calabi-Yau. How general is this?
  - Are *all* Feynman integrals Calabi-Yau? Currently looking at a potential counterexample.
  - If they are, does this rule out higher genus?

Thank You





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