Cluster Algebras, Steinmann Relations, and the Lie Cobracket in planar $\mathcal{N} = 4$ sYM

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Outline

- Planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (sYM) theory
  - Symmetries and Simplifications
  - Infrared and Helicity Structure
- Polylogarithms and Cluster-Algebraic Structure
  - Polylogarithms, the Coaction, and the Lie Cobracket
  - Cluster-Algebraic Structure in $\mathcal{N} = 4$ sYM
- Subalgebra Constructibility
  - Decomposing the Remainder Function

Amplitudes in $\mathcal{N} = 4$ sYM

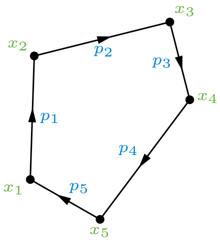
SUSY Ward identities	$\Rightarrow$	many relations among amplitudes with different helicity structure
Conformal theory	$\Rightarrow$	no running of the coupling or UV divergences
$AdS_5 \times S^5$ dual theory	$\Rightarrow$	multiple ways to calculate quantities of interest
Supersymmetric version of QCD	$\Rightarrow$	the types of functions that show up here also appear in QCD

Planar Limit and Dual Conformal Symmetry

We work with in the $N_c \to \infty$ limit with fixed $g^2 = g^2_{\text{YM}} N_c/(16\pi^2)$

- All non-planar graphs are suppressed in this limit, giving rise to a natural ordering of external particles
- This ordering can be used to define a set of dual coordinates

$$p_i^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}$$



- The coordinates x_i^μ label the cusps of a light-like polygonal Wilson loop in the dual theory, which respects a superconformal symmetry in this dual space

  [Alday, Maldacena; Drummond, Henn, Korchemsky, Sokatchev]

Helicity and Infrared Structure

Since we are working with all massless particles, our amplitudes $\mathcal{A}_n$ must be renormalized in the infrared

- Infrared divergences are universal and entirely accounted for by the 'BDS Ansatz' [Bern, Dixon, Smirnov]
- In the dual theory, the BDS Ansatz constitutes a particular solution to an anomalous conformal Ward identity that determines the Wilson loop up to a function of dual conformal invariants [Drummond, Henn, Korchemsky, Sokatchev]

$$\mathcal{A}_n = \underbrace{\mathcal{A}_n^{\text{BDS}}}_{\text{IR structure}} \times \overbrace{\exp(R_n) \times \underbrace{\left(1 + \mathcal{P}_n^{\text{NMHV}} + \mathcal{P}_n^{\text{N}^2\text{MHV}} + \cdots + \mathcal{P}_n^{\overline{\text{MHV}}}\right)}_{\text{helicity structure}}}^{\text{finite function of dual conformal invariants}}$$

Dual Conformal Invariants

- We can construct dual conformally invariant cross ratios out of combinations of Mandelstam invariants

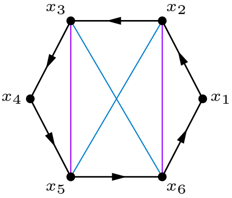
$$x_{ij}^2 = (x_i - x_j)^2 = (p_i + p_{i+1} + \cdots + p_{j-1})^2$$

that remain invariant under the dual inversion generator

$$I(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2} \qquad \Rightarrow \qquad I(x_{ij}^2) = \frac{x_{ij}^2}{x_i^2 x_j^2}$$

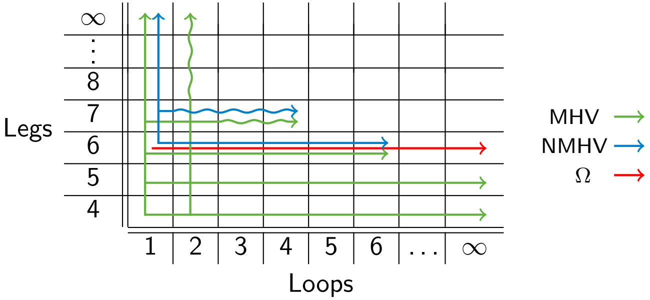
- These can first be constructed for $n = 6$ since $x_{i,i+1}^2 = p_i^2 = 0$

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \quad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}$$



- In general, we can form $3n - 15$ independent ratios

Loops and Legs in Planar $\mathcal{N}=4$



[Bern, Caron-Huot, Dixon, Drummond, Duhr, Foster, Gürdoğan, He, Henn, von Hippel, Golden, Kosower, AJM, Papathanasiou, Pennington, Roiban, Smirnov, Spradlin, Vergu, Volovich, . . .]

- Unexpected and striking structure exists in the the direction of both higher loops and legs
  - Galois Coaction Principle
  - Cluster-Algebraic Structure
- This talk will focus on using these polylogarithmic amplitudes (especially the two-loop MHV ones) as a data mine

Polylogarithms

- Loop-level contributions to MHV (and NMHV) amplitudes are expected to be multiple polylogarithms of uniform transcendental weight $2L$, meaning that the derivatives of these functions satisfy

$$dF = \sum_i F^{s_i} d\log s_i$$

for some set of 'symbol letters' $\{s_i\}$, where F^{s_i} is a multiple polylogarithm of weight $2L-1$

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

- The symbol letters $\{s_i\}$ can in general be algebraic functions of dual conformal invariants
- Examples of such functions (and their special values) include $\log(z)$, $i\pi$, $\text{Li}_m(z)$, and ζ_m. The classical polylogarithms $\text{Li}_m(z)$ involve only the symbol letters $\{z, 1-z\}$

$$\text{Li}_1(z) = -\log(1-z), \quad \text{Li}_m(z) = \int_0^z \frac{\text{Li}_{m-1}(t)}{t} dt$$

The Coaction

- Multiple polylogarithms are endowed with a coaction that maps functions to a tensor space of lower-weight functions [Goncharov; Brown]

$$\mathcal{H}_w \xrightarrow{\Delta} \bigoplus_{p+q=w} \mathcal{H}_p \otimes \mathcal{H}_q^{\mathfrak{dr}}$$

- If we iterate this map $w-1$ times we will arrive at a function's 'symbol', in terms of which all identities reduce to familiar logarithmic identities
- The location of branch cuts is determined by the $\Delta_{1,w-1}$ component of the coproduct, up to terms involving transcendental constants
- The derivatives of a function are encoded in the $\Delta_{w-1,1}$ component of its coproduct

$$\Delta_{1,\ldots,1}\mathrm{Li}_m(z) = -\log(1-z) \otimes \log z \otimes \cdots \otimes \log z$$

Symbol Alphabets and Discontinuities

The symbol exposes the discontinuity structure of polylogarithms

- In six-particle kinematics there are only 9 symbol letters:

$$\mathcal{S}_6 = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$s_{i\dots k} = (p_i + \cdots + p_k)^2, \quad u = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

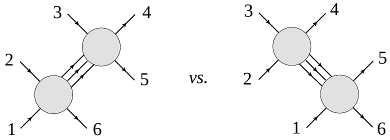
$$y_u = \frac{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}}$$

- Only letters whose vanishing locus coincides with internal propagators going on shell can appear in the first symbol entry
- In seven-particle kinematics there are 42 analogous symbol letters, 14 of which are parity odd
- For more than seven particles, symbol alphabets not as well understood
  - algebraic roots appear in symbol letters even at one loop in N^2MHV amplitudes [Prlina, Spradlin, Stankowicz, Stanojevic, Volovich]

The Steinmann Relations

- The Steinmann relations tell us that amplitudes cannot have double discontinuities in partially overlapping channels

  [Steinmann; Cahill, Stapp]



$$\text{Disc}_{s_{234}}(\text{Disc}_{s_{345}}(\mathcal{A}_n)) = 0$$

$$\cancel{\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{w}{uv}\right)} \otimes \ldots \qquad \cancel{\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{v}{uw}\right)} \otimes \ldots$$

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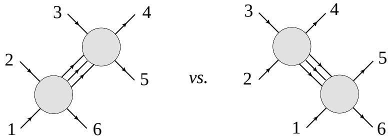

$$\text{Disc}_{s_{234}}(\text{Disc}_{s_{345}}(\mathcal{A}_n)) = 0$$

$$\cdots \otimes \cancel{\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{w}{uv}\right)} \otimes \dots \qquad \cdots \otimes \cancel{\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{v}{uw}\right)} \otimes \dots$$

- ...in fact, the Steinmann relations constrain not just double discontinuities, but all iterated discontinuities
  [Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou]
- For six and seven particles, this appears to be equivalent to requiring 'cluster adjacency' [Drummond, Foster, Gürdoğan]

Infrared Normalization

- Steinmann functions don't form a ring

$$\text{Disc}_{s_{i-1,i,i+1}}\left[\mathcal{A}_n^{(1)}\right] \neq 0$$

$$\Downarrow$$

$$\text{Disc}_{s_{234}}\left[\text{Disc}_{s_{345}}\left[\left(\mathcal{A}_n^{(1)}\right)^2\right]\right] \neq 0$$

- The BDS ansatz exponentiates the one-loop amplitude, leading to products of amplitudes starting at two loops (and obscuring the Steinmann relations)

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- This is fixed by the BDS-like ansatz, which only depends on two-particle invariants

$$\mathcal{A}_n^{\text{BDS}} \times \exp(R_n) \to \mathcal{A}_n^{\text{BDS-like}} \times \mathcal{E}_n^{\text{MHV}}$$

- The BDS-like ansatz only scrambles Steinmann relations involving two-particle invariants, which are obfuscated in massless kinematics anyways

Infrared Normalization

- However, the BDS-like ansatz only exists for particle multiplicities that are not a multiple of four [Alday, Maldacena, Sever, Vieira; Yang; Dixon, Drummond, Harrington, AJM, Papathanasiou, Spradlin]

Infrared Normalization

- However, the BDS-like ansatz only exists for particle multiplicities that are not a multiple of four [Alday, Maldacena, Sever, Vieira; Yang; Dixon, Drummond, Harrington, AJM, Papathanasiou, Spradlin]
- To unpack this statement: there exists a unique decomposition of the one-loop MHV amplitude taking the form

$$\mathcal{A}^{(1)}_{\text{MHV},n} = \underbrace{X_n(\epsilon, \{s_{i,i+1}\})}_{\text{IR structure}} + \underbrace{Y_n(\{s_{i,\dots,i+j}\})}_{\text{dual conformal invariant}}$$

  for all particle multiplicities n that are not a multiple of four
- Exponentiating the function X_n rather than the full one-loop amplitude accounts for the full infrared structure of this theory, yet is invisible to the operation of taking discontinuities in three- and higher-particle channels

Infrared Normalization

- This is problematic if we want to test the equivalence of the Steinmann relations and cluster adjacency in eight-particle kinematics
- However, if this is test is our only objective the last slide makes clear there is a way out: normalize the amplitude by a 'minimal BDS ansatz' only consisting of the infrared-divergent part of the one-loop amplitude
- It can be explicitly checked that this restores not only all (higher-particle) Steinmann relations, but also all cluster adjacency relations
    - this provides further evidence that the these conditions are equivalent (when cluster adjacency can be unambiguously applied)

[Golden, AJM (to appear)]

Lie Cobracket

Polylogarithms also come equipped with a Lie cobracket structure

$$\delta(F) \equiv \sum_{i=1}^{k-1} (\rho_i \wedge \rho_{k-i}) \rho(F)$$

$$\rho(s_1 \otimes \cdots \otimes s_k) = \frac{k-1}{k} \Big(\rho(s_1 \otimes \cdots \otimes s_{k-1}) \otimes s_k - \rho(s_2 \otimes \cdots \otimes s_k) \otimes s_1 \Big)$$

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- The cobracket of classical polylogarithms is especially simple:

$$\delta\big(\text{Li}_k(-z)\big) = -\{z\}_{k-1} \wedge \{z\}_1, \quad k > 2$$
$$\delta\big(\text{Li}_2(-z)\big) = -\{1+z\}_1 \wedge \{z\}_1$$

where

$$\{z\}_1 = \rho(\log(z)), \quad \{z\}_k = \rho(-\text{Li}_k(-z))$$

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- In fact, any weight four function that has no $\delta_{2,2}$ component can be written in terms of classical polylogarithms

  [Dan; Gangl; Goncharov, Rudenko]

Different Levels of Polylogarithmic Structure

Full Function

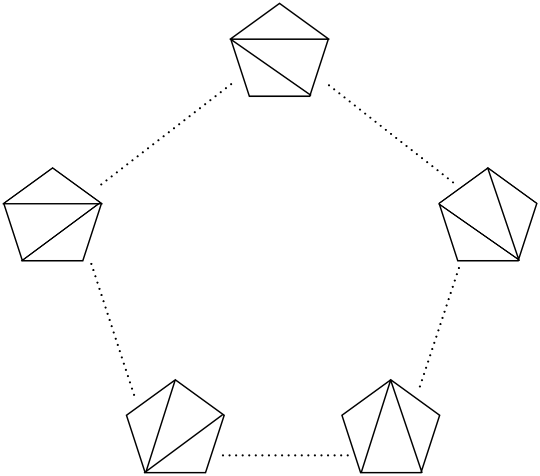
$\downarrow \mathcal{S}$

Analytic Structure

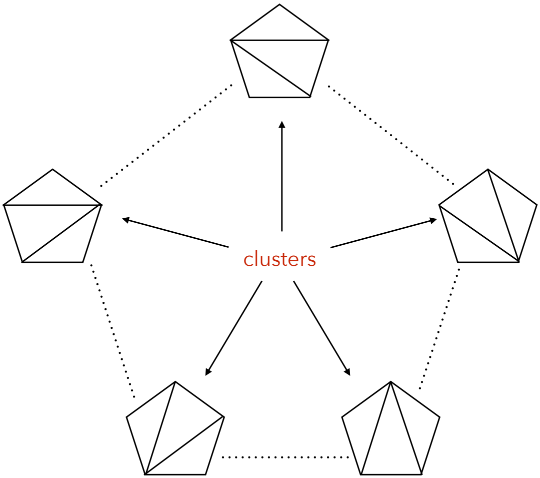
$\downarrow \delta$

Nonclassical Structure

Cluster Algebras



Cluster Algebras



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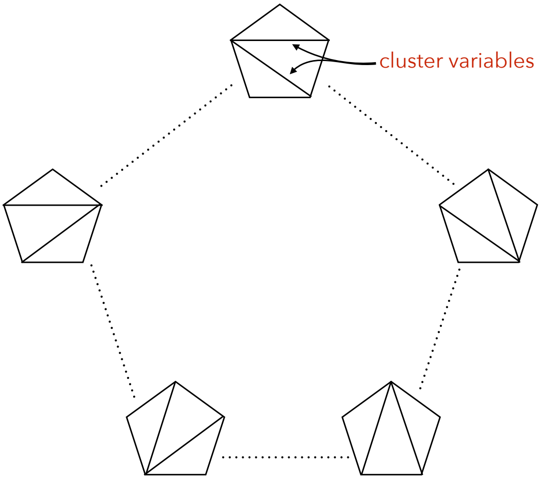

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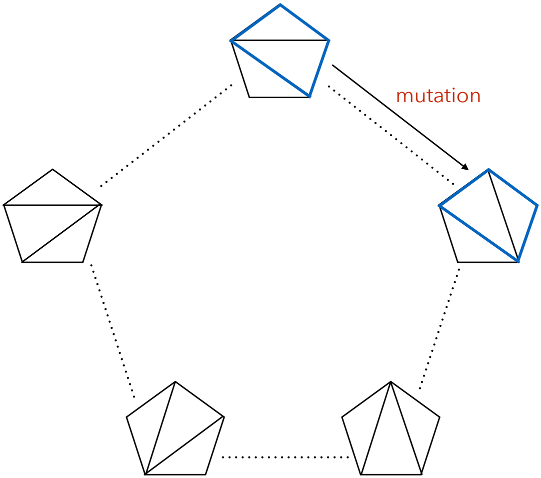

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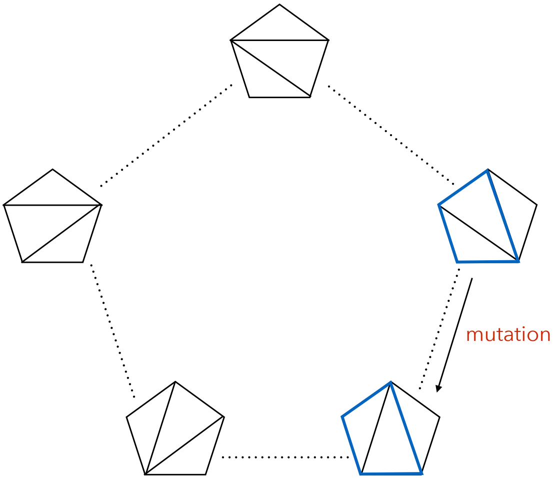

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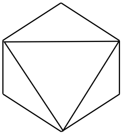

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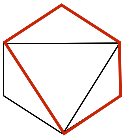
Cluster Algebras



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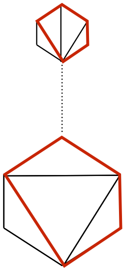

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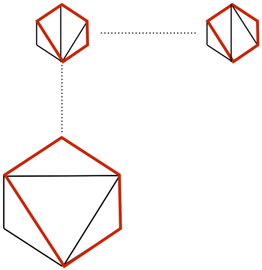

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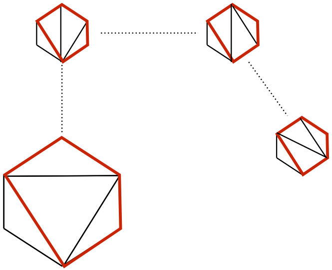

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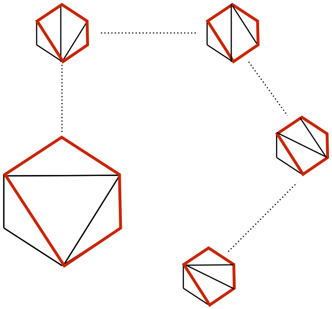

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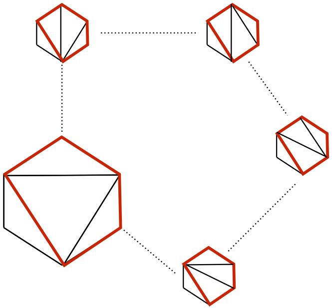

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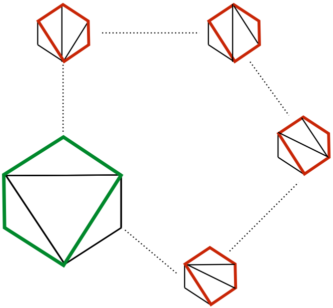

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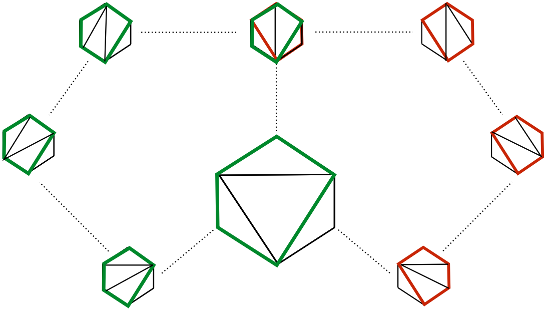

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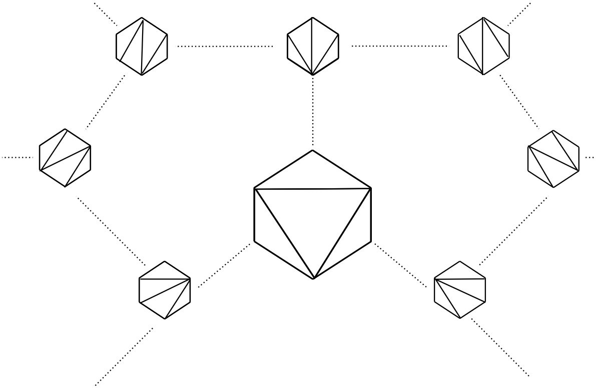
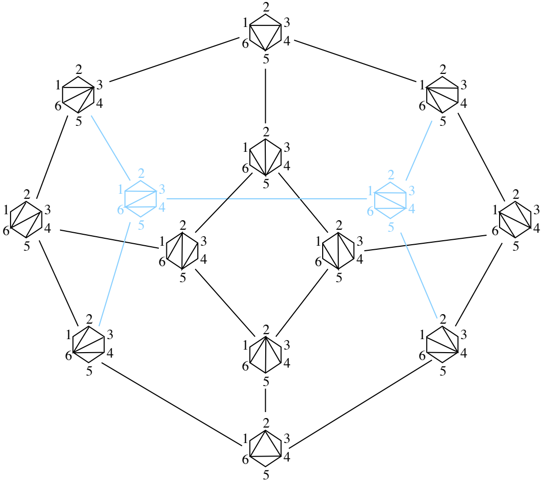


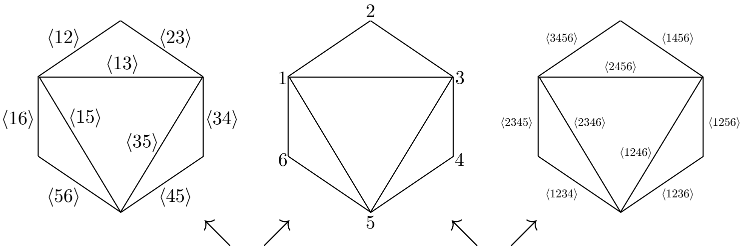
Cluster Algebras

$$\text{Gr}(4,6) \sim A_3$$



[Williams]

Cluster Coordinates



Gr$(2,6)$ Gr$(4,6)$

$\mathcal{A}$-coordinates

$$\langle 2456 \rangle$$

$$\langle 2346 \rangle$$

$$\langle 1246 \rangle$$

$\mathcal{X}$-coordinates

$$\frac{\langle 1246\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2346\rangle} = \sqrt{\frac{u(1-v)}{v(1-u)y_u y_v}}$$

$$\frac{\langle 1234\rangle\langle 2456\rangle}{\langle 1246\rangle\langle 2345\rangle} = \sqrt{\frac{v(1-w)}{w(1-v)y_v y_w}}$$

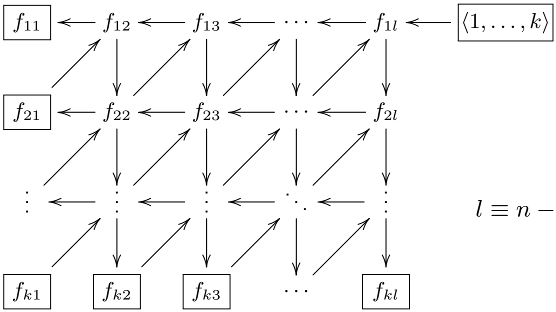
$$\frac{\langle 1256\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 2456\rangle} = \sqrt{\frac{w(1-u)}{u(1-w)y_u y_w}}$$

$$Z_i^{R=\alpha,\dot\alpha} = (\lambda_i^\alpha, x_i^{\beta\dot\alpha}\lambda_{i\beta}), \quad \langle abcd\rangle = \epsilon_{RSTU} Z_a^R Z_b^S Z_c^T Z_d^U$$

Cluster Algebras

- More generally, clusters can be defined to be quiver diagrams that have a cluster coordinate associated with every node

$\mathsf{Gr}(k,n)$



$l \equiv n - k$

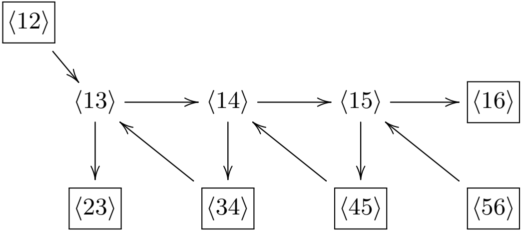
$$f_{ij} = \begin{cases} \langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle, & i \le l-j+1, \\ \langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle, & i > l-j+1. \end{cases}$$

Cluster Algebras

- We can translate between clusters in $\mathcal{A}$-coordinates and $\mathcal{X}$-coordinates using

$$x_i = \prod_j a_j^{b_{ji}}$$

$$b_{ij} = (\#\text{ of arrows } i \to j) - (\#\text{ of arrows } j \to i)$$



$$\Updownarrow$$

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle} \longrightarrow \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle} \longrightarrow \frac{\langle 14\rangle\langle 56\rangle}{\langle 16\rangle\langle 45\rangle}$$

Cluster Algebras

- A cluster algebra is the closure of a given quiver under cluster mutation

$$a_k a_k' = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}$$

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i,j\}, \\ b_{ij}, & \text{if } b_{ik} b_{kj} \leq 0, \\ b_{ij} + b_{ik} b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik} b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases}$$

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i \left(1 + x_k^{\mathrm{sgn}(b_{ik})}\right)^{b_{ik}}, & i \neq k \end{cases}$$

Cluster-Algebraic Structure in Planar $\mathcal{N} = 4$

Cluster algebras appear in planar $\mathcal{N} = 4$ sYM in a number of striking ways

$$\delta(F)$$

- [Building Blocks] The cobracket of all two-loop MHV amplitudes can be expressed in terms of Bloch group elements evaluated on cluster $\mathcal{X}$-coordinates, $\{\mathcal{X}\}_k$ [Golden, Paulos, Spradlin, Volovich]
- [Cluster Adjacency] The cobracket of all two-loop MHV can be expressed as a linear combination of terms $\{\mathcal{X}_i\}_2 \wedge \{\mathcal{X}_j\}_2$ and $\{\mathcal{X}_k\}_3 \wedge \{\mathcal{X}_l\}_1$ where each pair of $\mathcal{X}$-coordinates appears together in a cluster of Gr$(4, n)$ [Golden, Spradlin]
- [Subalgebra Constructibility] The nonclassical part of all two-loop MHV amplitudes can be decomposed into functions defined on their A_2 and A_3 subalgebras [Golden, Paulos, Spradlin, Volovich]

Cluster-Algebraic Structure in Planar $\mathcal{N} = 4$

Cluster algebras appear in planar $\mathcal{N} = 4$ sYM in a number of striking ways

$$\mathcal{S}(F)$$

- [Building Blocks] The symbol alphabets for $n \in \{6, 7\}$ are precisely cluster coordinates on the Grassmannian Gr$(4, n)$, and all symbol letters in the two-loop MHV amplitudes are also cluster coordinates on Gr$(4, n)$

  [Golden, Goncharov, Spradlin, Vergu, Volovich; Drummond, Papathanasiou, Spradlin]

- [Cluster Adjacency] In the symbol of (appropriately normalized) amplitudes in which no algebraic roots arise, each pair of adjacent $\mathcal{A}$-coordinates appears together in a cluster of Gr$(4, n)$

  [Drummond, Foster, Gürdoğan]

Cluster-Algebraic Structure in Planar $\mathcal{N} = 4$

Cluster algebras appear in planar $\mathcal{N} = 4$ sYM in a number of striking ways

$$F$$

- [Building Blocks] The two-loop MHV amplitudes are expressible as (products of) functions taking only negative cluster $\mathcal{X}$-coordinate coordinates, $\text{Li}_{n_1,\ldots,n_d}(-\mathcal{X}_i, \ldots, -\mathcal{X}_j)$ [Golden, Spradlin]

Cluster-Algebraic Structure in Planar $\mathcal{N} = 4$

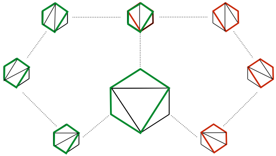
Cluster algebras appear in planar $\mathcal{N} = 4$ sYM in a number of striking ways

$$\int \mathcal{I}$$

- [Building Blocks] The integrands in this theory are encoded by plabic graphs, which are dual to cluster algebras
  [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]
- [Cluster Adjacency] Cluster adjacency translates to the statement that cluster coordinates only appear in adjacent entries of the symbol or cobracket when the boundaries corresponding to their zero-loci are simultaneously accessible

Cluster Polylogarithms

- Using cluster $\mathcal{A}$- or $\mathcal{X}$-coordinates, we can define polylogarithms on any cluster algebra that can be represented as a quiver
- In particular, we can consider functions that live on the cluster subalgebras of $\text{Gr}(4, n)$



- The union of all $\mathcal{A}$- or $\mathcal{X}$-coordinates on the clusters in a (sub)algebra provide a symbol alphabet

Cluster Polylogarithms

- Cluster polylogarithms on the subalgebras of $\mathrm{Gr}(4,n)$ efficiently capture the nonclassical structure of $R_n^{(2)}$ (or equivalently $\mathcal{E}_n^{(2)}$)
- There exists only a single nonclassical polylogarithm defined on A_2, and only two on A_3, but they have special properties
  - Physically:

$$\delta_{2,2}(R_n^{(2)}) = \sum_{A_3 \subset \mathrm{Gr}(4,n)} d_i \; \delta_{2,2}(f_{A_3^{(i)}}) = \sum_{A_2 \subset \mathrm{Gr}(4,n)} c_i \; \delta_{2,2}(f_{A_2^{(i)}})$$

  - Mathematically:
    - f_{A_2} act as a basis for all nonclassical polylogarithms, while
    - f_{A_3} acts as a basis for all nonclassical cluster polylogarithms whose cobracket satisfies cluster adjacency

[Golden, Paulos, Spradlin, Volovich]

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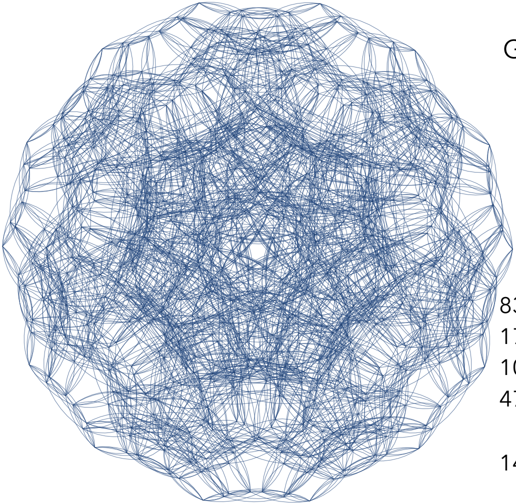
- This basis of f_{A_2} and f_{A_3} functions is massively overcomplete... what about larger subalgebras of Gr(4,7)?

Cluster Polytopes



Gr(4,7) = E_6

833 vertices

1785 $A_1 \times A_1$

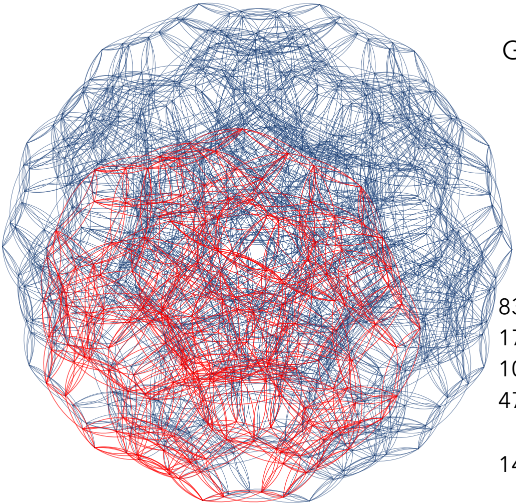
1071 A_2

476 A_3

⋮

14 D_5

Cluster Polytopes



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Cluster Polylogarithms

Type	Nonclassical Cobrackets	Automorphism Signature			
		$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	1 (0)	0	1 (0)	0	0
A_3	6 (1)	0	1 (0)	0	1 (1)
A_4	21 (6)	0	3 (0)	0	0
D_4	34 (9)				
A_5	56 (21)	2 (1)	5 (1)	2 (0)	5 (3)
D_5	116 (42)				
E_6	448 (195)				

Nonclassical D_5 Polylogarithms

$\sigma^+_{\mathfrak{D}_5}\tau^+_{\mathfrak{D}_5}$	
$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
5 (2)	0

$\sigma^+_{\mathfrak{D}_5}\tau^-_{\mathfrak{D}_5}$	
$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
9 (2)	0

$\sigma^-_{\mathfrak{D}_5}\tau^+_{\mathfrak{D}_5}$	
$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
0	3 (1)

$\sigma^-_{\mathfrak{D}_5}\tau^-_{\mathfrak{D}_5}$	
$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
0	7 (5)

Subalgebra Constructibility

- D_5 and A_5 are special in E_6, as only a single orbit of each type exists under the E_6 automorphism group
  - It follows that all D_5- and A_5-constructible polylogarithms in E_6 necessarily take the form:

$$\sum_{D_5 \subset E_6} f_{D_5}(x_i \to \dots) = \sum_{i=0}^{6} \sum_{j=0}^{1} (\pm 1)^i (\pm 1)^j \, \mathbb{Z}_{2,E_6}^j \circ \sigma_{E_6}^i \Big(f_{D_5}(x_i \to \dots) \Big)$$

$$\sum_{A_5 \subset E_6} f_{A_5}(x_i \to \dots) = \sum_{i=0}^{6} (\pm 1)^i \, \sigma_{E_6}^i \Big(f_{A_5}(x_i \to \dots) \Big)$$

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- Surprisingly, a D_5 and A_5 decomposition of $\delta_{2,2}(R_7^{(2)})$ both exist

$$\delta_{2,2}(R_n^{(2)}) = \sum_{D_5 \subset \mathrm{Gr}(4,7)} \delta_{2,2}(f_{D_5}^{---}) = \sum_{A_5 \subset \mathrm{Gr}(4,7)} \delta_{2,2}(f_{A_5}^{--})$$

Subalgebra Constructibility

- ...moreover, we can play the same game with the new f_{D_5} and f_{A_5} functions
  - there exists only a single orbit of A_4 subalgebras in each D_5 and A_5

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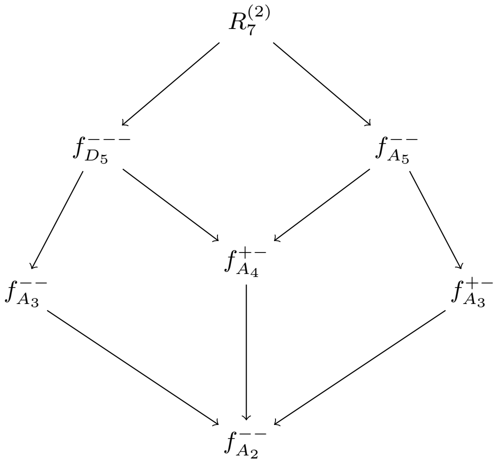
  - there exists only a single orbit of A_4 subalgebras in each D_5 and A_5
- Both f_{D_5} and f_{A_5} turn out to be decomposable into the *same* A_4 function:

$$R_7^{(2)} = \sum_{D_5 \subset \text{Gr}(4,7)} \sum_{A_4 \subset D_5} f_{A_4}^{+-}(x_1 \to x_2 \to x_3 \to x_4) + \dots$$
$$= \sum_{A_5 \subset \text{Gr}(4,7)} \sum_{A_4 \subset A_5} f_{A_4}^{+-}(x_1 \to x_2 \to x_3 \to x_4) + \dots$$

[Golden, AJM]

Subalgebra Constructibility

In fact, many nested decompositions are possible (although, none involving D_4), each making different properties of $\delta_{2,2}(R_7^{(2)})$ manifest



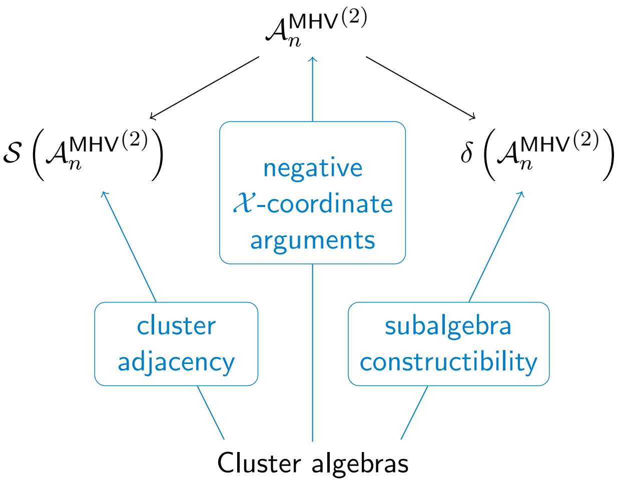
Subalgebra Constructibility

- The same game can be played in eight-particle kinematics, particularly using the new functions $f_{D_5}^{---}$, $f_{A_5}^{--}$, and $f_{A_4}^{+-}$ found in seven-particle kinematics [Golden, AJM (to appear)]
- It is completely systematic, starting from such a representation of their nonclassical component, to generate the full analytic expression for $R_8^{(2)}$ or $\mathcal{E}_8^{(2)}$ [Duhr, Gangl, Rhodes; Golden, Spradlin]
- Subalgebras of Gr(4,n) can also be associated with R-invariants, perhaps allowing a similar story to be developed in the NMHV sector [Drummond, Foster, Gürdoğan]

Conclusions

- A great deal of surprising structure remains to be explained in planar $\mathcal{N} = 4$
- In particular, the role of cluster algebras in this theory deserves to be better understood
  - The 'meaning' of these nonclassical decompositions remains obscure
- The big looming question is whether any similar types of structure can be found that extend beyond the polylogarithmic parts of this theory (or even to amplitudes involving algebraic roots)
  [Paulos, Spradlin, Volovich; Caron-Huot, Larsen; Bourjaily, Herrmann, Trnka]

Thanks!


$\mathcal{A}_n^{\mathsf{MHV}(2)}$
$\mathcal{S}\left(\mathcal{A}_n^{\mathsf{MHV}(2)}\right)$
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cluster adjacency
subalgebra constructibility
Cluster algebras