Tree Amplitudes for Some Supersymmetric Theories in Six Dimensions

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GGI April 16, 2019

Introduction

My most recent work that is relevant to "string theory from a worldsheet viewpoint" is arXiv:1506.07706, which is entitled

New Formulation of the Type IIB Superstring Action in $AdS_5 \times S^5$.

Even though I think that this paper is interesting, it has not had much impact. In any case, I prefer to speak about more recent work on a different topic, and the organizers said that this would be okay. So here goes.

The techniques that have been developed in recent years for constructing tree amplitudes of 4D supersymmetric theories can be extended to 6D. The results are contained in three articles:

- M. Heydeman, JHS, C. Wen, arXiv:1710.02170
- F. Cachazo, A. Guevara, M. Heydeman, S. Mizera, JHS, C. Wen, arXiv:1805.11111
- M. Heydeman, JHS, C. Wen, S-Q Zhang, arXiv:1812.06111

Additional references can be found in these articles.

Theories Considered

- (2,0) world-volume theory of flat M5-brane in $\mathbb{R}^{10,1}$
- (1,1) world-volume theory of flat D5-brane in $\mathbb{R}^{9,1}$
- (1,1) $SU(N)$ Yang–Mills amplitudes in $\mathbb{R}^{5,1}$
- (2,2) supergravity in $\mathbb{R}^{5,1}$
- (2,0) supergravity coupled to tensor multiplets in $\mathbb{R}^{5,1}$

The number of particles, n, is even for nonzero M5 and D5 amplitudes. It can be odd for the other theories, though the number of tensor particles must be even in the last case.

Spinor–helicity formalism in six dimensions

We will only consider massless theories in flat $\mathbb{R}^{5,1}$. The massless little group in 6D is

$$Spin(4) = SU(2) \times SU(2).$$

The on-shell six-momentum of a massless particle may be written in the form

$$p^{AB} = \varepsilon^{ab}\lambda_a^A\lambda_b^B = \langle\lambda^A\lambda^B\rangle, \quad p^2 \sim \mathrm{Pf}(p^{AB}) = 0,$$

where $A = 1, 2, 3, 4$ is a Lorentz spinor index and $a = \pm$ labels a doublet of the first little-group $SU(2)$ factor.

One can also define

$$p_{AB} = \varepsilon^{\hat{a}\hat{b}} \tilde{\lambda}_{A\hat{a}} \tilde{\lambda}_{B\hat{b}} = [\tilde{\lambda}_A \tilde{\lambda}_B],$$

where the subscript A labels the opposite chirality Lorentz spinor and $\hat{a}$ labels a doublet of the second $SU(2)$ factor in the little group.

$\tilde{\lambda}_{A\hat{a}}$ is determined by λ_a^A via the condition

$$p^{AB} = \frac{1}{2} \varepsilon^{ABCD} p_{CD}$$

up to $SU(2)$ transformations. In particular,

$$\lambda_a^A \tilde{\lambda}_{A\hat{b}} = 0.$$

M5-brane supermultiplet

The M5-brane theory is a 6D theory of a self-interacting tensor supermultiplet with $(2,0)$ supersymmetry. It has $Spin(5) = USp(4)$ R symmetry, which arises from rotational symmetry of the five dimensions transverse to the M5-brane in $\mathbb{R}^{10,1}$ (11D Minkowski spacetime).

The 16 on-shell degrees of freedom are

$$B^{ab} : (3,1;1) \qquad \psi^a_I : (2,1;4) \qquad \phi_{IJ} : (1,1;5).$$

The interacting theory is of Born–Infeld type, with the B field replacing the usual $U(1)$ gauge field.

The M5-brane theory has 16 supercharges

$$q^{AI} = \varepsilon^{ab}\lambda_a^A\eta_b^I = \langle\lambda^A\eta^I\rangle, \text{ where } \quad \{\eta_a^I, \eta_b^J\} = \varepsilon_{ab}\Omega^{IJ}.$$

Ω^{IJ} is the $USp(4)$ metric. The ten R charges are

$$R^{IJ} = \frac{1}{2}(\langle\eta^I\eta^J\rangle + \langle\eta^J\eta^I\rangle).$$

One can treat four of the eight η_a^I's that anticommute as Grassmann numbers. This choice can either keep the R symmetry or the little-group symmetry manifest, but not both. The latter choice leads to more concise formulas for scattering amplitudes. The two choices are related by a Grassmann Fourier transform, which can be implemented to verify the non-manifest symmetries.

For the choice $\Omega^{13} = \Omega^{24} = 1$, we have $\{\eta_a^I, \eta_b^J\} = 0$ where $I, J = 1, 2$. Then the 16 supercharges are

$$q^{AI} = \langle \lambda^A \eta^I \rangle \quad \text{and} \quad \tilde{q}_I^A = \lambda_a^A \frac{\partial}{\partial \eta_a^I}.$$

In this basis an on-shell supermultiplet is described by

$$\tilde{\Phi}(\eta) = \phi + \eta_a^I \psi_I^a + \frac{1}{2}\varepsilon_{IJ}\eta_a^I \eta_b^J B^{ab} + \frac{1}{2}\langle \eta^I \eta^J \rangle \phi_{IJ} + \ldots + \eta^4 \phi'.$$

As we have explained, only an $SU(2)$ subgroup of the R symmetry is manifest in these formulas.

D5-brane and Yang–Mills supermultiplets

They are both $(1,1)$ vector supermultiplets that have $Spin(4) = SU(2) \times SU(2)$ R symmetry. The on-shell content is

$$A^{a\hat{a}} : (2,2;1,1) \qquad \phi^{I\hat{I}} : (1,1;2,2)$$
$$\psi^{a\hat{I}} : (2,1;1,2) \qquad \psi^{\hat{a}I} : (1,2;2,1)$$

where I and $\hat{I}$ label R-symmetry doublets.

The 16 supercharges are given by

$$q^{AI} = \langle\lambda^A \eta^I\rangle, \quad \tilde{q}_A^{\hat{I}} = [\tilde{\lambda}_A \tilde{\eta}^{\hat{I}}],$$

where

$$\{\eta_a^I, \eta_b^J\} = \varepsilon_{ab}\,\varepsilon^{IJ}, \ \{\tilde{\eta}_{\hat{a}}^{\hat{I}}, \tilde{\eta}_{\hat{b}}^{\hat{J}}\} = \varepsilon_{\hat{a}\hat{b}}\,\varepsilon^{\hat{I}\hat{J}}, \ \{\eta_a^I, \tilde{\eta}_{\hat{a}}^{\hat{I}}\} = 0.$$

The representation with manifest little-group symmetry utilizes the $I = 1$ components of η_a^I, now denoted η_a, and the $\hat{I} = 1$ components of $\tilde{\eta}_{\hat{a}}^{\hat{I}}$, now denoted $\tilde{\eta}_{\hat{a}}$. The on-shell superfield in this representation is

$$\begin{gathered}\tilde{\Phi}(\eta) = \phi^{1\hat{1}} + \eta_a \psi^{a\hat{1}} + \tilde{\eta}_{\hat{a}} \psi^{\hat{a}1} + \eta_a \tilde{\eta}_{\hat{a}} A^{a\hat{a}} \\ + \eta^2 \phi^{2\hat{1}} + \tilde{\eta}^2 \phi^{1\hat{2}} + \ldots + \eta^2 \tilde{\eta}^2 \phi^{2\hat{2}},\end{gathered}$$

where

$$\eta^2 = \frac{1}{2} \varepsilon^{ab} \eta_a \eta_b \quad \text{and} \quad \tilde{\eta}^2 = \frac{1}{2} \varepsilon^{\hat{a}\hat{b}} \tilde{\eta}_{\hat{a}} \tilde{\eta}_{\hat{b}}.$$

Maximal supergravity supermultiplet

The $(2,2)$ supergravity multiplet can be constructed by tensoring $(2,0)\otimes(0,2)$ or $(1,1)\otimes(1,1)$. The latter choice is essential for n odd. The theory has $USp(4)\times USp(4)$ R symmetry.

The on-shell bosons are

$$(3,3;1,1)+(3,1;1,5)+(1,3;5,1)+(2,2;4,4)+(1,1;5,5),$$

and the on-shell fermions are

$$(3,2;1,4)+(2,3;4,1)+(2,1;4,5)+(1,2;5,4).$$

This supermultiplet, $\Phi(\eta,\tilde{\eta})$, is a function of four η's and four $\tilde{\eta}$'s.

Four-particle amplitudes

Up to constant coefficients and momentum-conservation delta functions, the four-particle ($n = 4$) amplitudes are:

$$A_4^{\mathrm{M5}} = \delta^8\Big(\sum_i q_i^{AI}\Big)$$

$$A_4^{\mathrm{D5}} = \delta^4\Big(\sum_i q_i^{A}\Big)\,\delta^4\Big(\sum_i \tilde{q}_{iA}\Big)$$

$$A_4^{(1,1)\ \mathrm{SYM}} = \delta^4\Big(\sum_i q_i^{A}\Big)\,\delta^4\Big(\sum_i \tilde{q}_{iA}\Big)/st$$

$$A_4^{(2,2)\ \mathrm{Sugra}} = \delta^8\Big(\sum_i q_i^{AI}\Big)\,\delta^8\Big(\sum_i \tilde{q}_{iA}^{\hat{I}}\Big)/stu$$

These formulas are extremely concise!

The SYM formula has cyclic symmetry and should be multiplied by a color factor. Then one should sum over cyclically inequivalent permutations. The other three formulas have total symmetry. These features hold for all n.

As mentioned earlier, a suitable Grassmann Fourier transform makes the full R symmetry manifest for each theory. This symmetry, together with the supersymmetries made manifest by the Grassmann delta functions, implies that the other half of the supercharges are also conserved.

M5-brane amplitudes

In the CHY formalism, a coordinate $\sigma_i \in CP^1$ is associated to the ith particle. These are defined up to a common $SL(2, \mathbb{C})$ transformation.

The n-particle tree-level scattering amplitude for the M5 theory has the structure

$$\mathcal{A}_n^{\mathrm{M5}} = \int d\mu_n^{6D}\, \mathcal{I}_{\mathrm{DBI}}\, \mathcal{I}_{(2,0)},$$

where $d\mu_n^{6D}$ is the n-particle CHY measure in 6D. The number of particles, n, is necessarily even for all DBI theories, including the M5-brane and D5-brane theories.

Each of the $\mathcal{I}$ factors is required to transform with weight -2 under $SL(2,\mathbb{C})$, since the CHY measure has weight 4. This means that if

$$\sigma_i \to (a\sigma_i + b)/(c\sigma_i + d)$$

for all i, with $ad - bc = 1$, then

$$\mathcal{I} \to \prod_{i=1}^{n} (c\sigma_i + d)^2 \times \mathcal{I}.$$

Our construction utilizes bosonic and fermionic polynomials of degree $m = \frac{n}{2} - 1$:

$$\rho_a^A(\sigma) = \sum_{k=0}^{m} \rho_{k,a}^A \sigma^k \,, \quad \chi_a^I(\sigma) = \sum_{k=0}^{m} \chi_{k,a}^I \sigma^k \,.$$

The CHY measure is given by

$$d\mu_{n\ \mathrm{even}}^{\mathrm{6D}} = \frac{\prod_{i=1}^{n} d\sigma_i\ \prod_{k=0}^{m} d^8\rho_{k,a}^{A}}{\mathrm{vol}(SL(2,\mathbb{C})_\sigma \times SL(2,\mathbb{C})_\rho)} \frac{\Delta_B(\rho)}{V_n^2},$$

where

$$\Delta_B(\rho) = \prod_{i=1}^{n} \delta^6 \left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i) \rangle}{\prod_{j\neq i} \sigma_{ij}} \right),$$

$$\sigma_{ij} = \sigma_i - \sigma_j$$

and

$$V_n = \prod_{1\leq i<j\leq n} \sigma_{ij}.$$

The CHY *scattering equations*

$$E_i = \sum_j (A_n)_{ij} = 0 \quad i = 1, 2, \ldots, n$$

where

$$(A_n)_{ij} = \frac{p_i \cdot p_j}{\sigma_{ij}}, \quad i, j = 1, 2, \ldots, n,$$

are a consequence of the delta functions in the measure.

The scattering equations are $n-3$ linearly independent equations for the σ_i's – defined up to an overall $SL(2,\mathbb{C})$ – that have $(n-3)!$ solutions.

The left factor in the integrand is

$$\mathcal{I}_{\text{DBI}} = \det{}' A_n = (\text{Pf}' A_n)^2 \,.$$

As before,

$$(A_n)_{ij} = \frac{p_i \cdot p_j}{\sigma_{ij}}, \quad i, j = 1, 2, \ldots, n.$$

Since A_n has co-rank 2 when n is even, one defines

$$\text{Pf}' A_n = \frac{(-1)^{p+q}}{\sigma_{pq}} \text{Pf} A_n^{[pq]},$$

where two rows and columns, p and q, are removed. $\text{Pf}' A_n$ has weight -1 and is independent of p, q, so $\mathcal{I}_{\text{DBI}}$ has weight -2.

The right factor in the integrand implements $(2,0)$ SUSY

$$\mathcal{I}_{(2,0)} = \mathrm{Pf}' A_n \int d\Omega_F^{(2,0)},$$

where

$$d\Omega_F^{(2,0)} = V_n \left(\prod_{k=0}^{m} d^4 \chi_{k,a}^I \right) \Delta_F^{(2,0)}(q, \rho, \chi),$$

and

$$\Delta_F^{(2,0)}(q, \rho, \chi) = \prod_{i=1}^{n} \delta^8 \left(q_i^{AI} - \frac{\langle \rho^A(\sigma_i) \chi^I(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}} \right).$$

$\mathcal{I}_{(2,0)}$ also has weight -2, as required.

The logic is as follows:

$$\mathcal{A}_n^{\mathrm{M5}} = \left(\prod_{i=1}^{n} \delta(p_i^2) \delta^4 \left(\tilde{\lambda}_{iA\hat{a}} q_i^{AI} \right) \right) A_n^{\mathrm{M5}} \,.$$

The factor $\delta(p_i^2)$ allows us to express p_i^{AB} as $\langle \lambda_i^A \lambda_i^B \rangle$, as described earlier. Similarly, $\delta^4 \left(\tilde{\lambda}_{iA\hat{a}} q_i^{AI} \right)$ allows us to express q_i^{AI} as $\langle \lambda_i^A \eta_i^I \rangle$. Then A_n^{M5} only depends on λ_{ia}^A and η_{ia}^I, and it is the desired amplitude.

There are analogous formulas for the other theories to be discussed. To save time, I will show less detail.

D5-brane amplitudes

These amplitudes are similar to the M5-brane ones with $\mathcal{I}_{(2,0)}$ replaced by $\mathcal{I}_{(1,1)} = \mathrm{Pf}' A_n \int d\Omega_F^{(1,1)}$, where

$$d\Omega_F^{(1,1)} = V_n \left(\prod_{k=0}^{m} d^2\chi_{k,a} d^2\tilde{\chi}_{k,\hat{a}} \right) \Delta_F^{(1,1)}(q,\rho,\chi),$$

and

$$\Delta_F^{(1,1)}(q,\rho,\chi) = \prod_{i=1}^{n} \delta^4 \left(q_i^A - \frac{\langle \rho^A(\sigma_i)\chi(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}} \right)$$

$$\times \prod_{i=1}^{n} \delta^4 \left(\tilde{q}_{iA} - \frac{[\tilde{\rho}_A(\sigma_i)\tilde{\chi}(\sigma_i)]}{\prod_{j\neq i}\sigma_{ij}} \right).$$

(1,1) Super Yang–Mills amplitudes

For even n, these amplitudes are obtained by the replacement $\mathcal{I}_{\mathrm{DBI}} \to \mathrm{PT}(\alpha)$ in the D5-brane formula. Thus, the color-stripped amplitudes are

$$\mathcal{A}_{n\ \mathrm{even}}^{(1,1)\ \mathrm{SYM}}(\alpha) = \int d\mu_{n\ \mathrm{even}}^{\mathrm{6D}}\, \mathrm{PT}(\alpha)\, \mathcal{I}_{(1,1)},$$

where $\alpha \in S_n/Z_n$ labels cyclically inequivalent permutations of the n external particles and

$$\mathrm{PT}(12\cdots n) = \frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}}.$$

The nontrivial challenge is to find the formula for the amplitudes when n is odd.

The case of odd multiplicity

The amplitudes for $n = 2m+1$ take the form

$$\mathcal{A}_{2m+1}^{(1,1)\,\mathrm{SYM}}(\alpha) = \int d\mu_{2m+1}^{6\mathrm{D}}\,\mathrm{PT}(\alpha)\,\hat{\mathcal{I}}_{(1,1)},$$

where

$$\hat{\mathcal{I}}_{(1,1)} = \mathrm{Pf}'\widehat{A}_n \int d\widehat{\Omega}_F^{(1,1)}.$$

The formulas for $\widehat{A}_n$ and $d\widehat{\Omega}_F^{(1,1)}$ are deduced by considering the soft-gluon limit of $\mathcal{A}_{2m+2}^{(1,1)\,\mathrm{SYM}}(\alpha)$. They inherit an auxiliary puncture $\sigma^\star$ and null vector $p^\star$, associated to the soft particle. Since $p^\star$ appears in the numerator and denominator, it survives in the soft limit, though the amplitude does not depend on the choice.

Verification of the odd-n result

- All symmetries, dimensions, and delta functions are correct.
- The $n = 3$ formula agrees with the result obtained by Dennen, Huang, and Siegel (0910.2688).
- Numerical studies give agreement with Feynman diagrams for $n = 5, 7$.
- The soft-gluon limit of the odd-n amplitudes reproduces the even-n amplitudes.

(2,2) Supergravity amplitudes

Using the KLT double-copy procedure, the $(2,2)$ supergravity amplitudes for $n = 2m + 2$ are given by

$$\mathcal{A}_{2m+2}^{(2,2)\text{ sugra}} = \int d\mu_{2m+2}^{\text{6D}}\, \mathcal{I}_{(1,1)}^{L}\, \mathcal{I}_{(1,1)}^{R},$$

where the Grassmann coordinates in $\mathcal{I}_{(1,1)}^{L}$ and $\mathcal{I}_{(1,1)}^{R}$ are distinct. Similarly, for $n = 2m + 1$

$$\mathcal{A}_{2m+1}^{(2,2)\text{ sugra}} = \int d\mu_{2m+1}^{\text{6D}}\, \widehat{\mathcal{I}}_{(1,1)}^{L}\, \widehat{\mathcal{I}}_{(1,1)}^{R}.$$

As before, the even-n and odd-n formulas are related to one another by the soft-graviton theorem.

(2,0) Supergravity coupled to tensor multiplets

Type IIB superstring theory, compactified on K3, gives a 6D theory with $(2,0)$ supersymmetry, which in the IR consists of $(2,0)$ supergravity coupled to 21 tensor multiplets. The moduli space of the string theory,

$$SO(5,21;\mathbb{Z})\backslash SO(5,21)/(SO(5)\times SO(21)),$$

is parametrized by the 21×5 scalar fields in the tensor multiplets.

At fixed points of the duality group tensor multiplets are replaced by strongly interacting $(2,0)$ CFT's.

This theory can also be obtained as the strong-coupling limit of the heterotic string theory toroidally compactified to 5D. The 5D heterotic coupling constant corresponds to the radius of a circle that decompactifies in the limit.

Our analysis is applicable to the field-theory limit, which is perturbative away from the fixed points.

We have constructed all the tree amplitudes of the field-theory limit and used soft theorems to demonstrate that it has an $SO(5,21)/(SO(5) \times SO(21))$ moduli space.

Other results

Maximal SYM and supergravity amplitudes in 5D have been obtained by straightforward restriction of the 6D formulas to 5D.

Amplitudes for 4D $\mathcal{N} = 4$ SYM on the Coulomb branch have been derived from the massless 6D SYM amplitudes by a less trivial dimensional reduction that gives massive spinor-helicity variables in 4D.

Conclusion

We have obtained all of the tree amplitudes for various supersymmetric theories in 6D: M5-brane, D5-brane, $(1,1)$ SYM, $(2,2)$ supergravity, and $(2,0)$ supergravity coupled to 21 tensor multiplets.

The last three theories have UV completions as string theories without adding any more massless degrees of freedom. It would be interesting to extend our results to the string amplitudes.

Amplitudes of theories with $(1,0)$ supersymmetry also deserve study.