The superstring n-point 1-loop amplitude

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Motivation

- Compute the n-point open superstring correlator at one loop using worldsheet methods

Essential requirements

- Correlator $\mathcal{K}_n(\ell)$ defined by:

$$\mathcal{A}_n = \sum_{\text{top}} C_{\text{top}} \int_{D_{\text{top}}} d\tau \, dz_2 \, dz_3 \, \ldots \, dz_n \int d^D\ell \, |\mathcal{I}_n(\ell)| \, \langle \mathcal{K}_n(\ell) \rangle$$

such that:

1. BRST invariant (ie susy and gauge invariant)

$$Q\mathcal{K}_n(\ell) = 0$$

2. monodromy invariant

$$D\mathcal{K}_n(\ell) = 0$$

Summary of results

- Correlators built from:
  1. kinematic factors in pure spinor superspace
  2. worldsheet functions at genus one surface
- Outcome: a beautiful Lie-polynomial structure

$$\mathcal{K}_n(\ell) = \sum_{r=0}^{n-4} \frac{1}{r!}\Big(V_{A_1}\, T^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} \mathcal{Z}^{m_1\ldots m_r}_{A_1,\ldots,A_{r+4}} + \big[12\ldots n|A_1,\ldots,A_{r+4}\big]\Big)$$

$$+ \text{ corrections}$$

- Duality between BRST and monodromy operators (BRST invariants vs generalized elliptic integrands)

$$Q \leftrightarrow D$$

Examples

- 4 points (Berkovits 2004)

$$\mathcal{K}_4(\ell) = V_1 T_{2,3,4} \mathcal{Z}_{1,2,3,4}$$

- kinematic factor is BRST invariant

$$V_1 T_{2,3,4} \equiv \frac{1}{3}(\lambda A_1)\big((\lambda\gamma^m W_2)(\lambda\gamma^m W_3)F^4_{mn} + \mathrm{cyc}(2,3,4)\big)$$

$$QV_1 T_{2,3,4} = 0$$

- worldsheet functions are monodromy invariant

$$\mathcal{Z}_{1,2,3,4} \equiv 1$$

5 point correlator

$$\mathcal{K}_5(\ell) = V_1\, T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} \\ + V_{12}\, T_{3,4,5} \mathcal{Z}_{12,3,4,5} + (2{\leftrightarrow}3,4,5) \\ + V_1\, T_{23,4,5} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)$$

- kinematic factors $V_A T_{B,C,D}$ and $V_A T^m_{B,C,D,E}$ in pure spinor superspace with covariant BRST variations
- one-loop worldsheet functions $\mathcal{Z}_{A,B,C,D}$ and $\mathcal{Z}^m_{A,B,C,D,E}$ from Kronecker–Einsestein series and loop momentum with covariant monodromy variations

5pt BRST & monodromy invariance

There is a strong interplay between kinematics and worldsheet functions:

- The 5-pt correlator is BRST invariant due to a total derivative:

$$Q\mathcal{K}_5(\ell) = -V_1 V_2 T_{3,4,5}\Big[k_2^m \mathcal{Z}^m_{1,2,3,4,5} + \big[s_{21}\mathcal{Z}_{21,3,4,5} + (1 \leftrightarrow 3,4,5)\big]\Big] + (2 \leftrightarrow 3,4,5) \cong 0$$

- The 5-pt correlator is single valued due to BRST cohomology ids (BRST exact terms)

$$D\mathcal{K}_5(\ell) = \Omega_1\Big(k_1^m V_1 T^m_{2,3,4,5} + \big[V_{12} T_{3,4,5} + 2 \leftrightarrow 3,4,5\big]\Big) + \Omega_2\Big(k_2^m V_1 T^m_{2,3,4,5} + V_{21} T_{3,4,5} + \big[V_1 T_{23,4,5} + 3 \leftrightarrow 4,5\big]\Big) + (2 \leftrightarrow 3,4,5) \cong 0$$

Examples

- 6 point correlator

$$
\begin{aligned}
\mathcal{K}_6(\ell) = &\frac{1}{2} V_1\, T^{mn}_{2,3,4,5,6} \mathcal{Z}^{mn}_{1,2,3,4,5,6} \\
&+ \big[V_{12}\, T^{m}_{3,4,5,6} \mathcal{Z}^{m}_{12,3,4,5,6} + (2 \leftrightarrow 3,4,5,6) \big] \\
&+ \big[V_1\, T^{m}_{23,4,5,6} \mathcal{Z}^{m}_{1,23,4,5,6} + (2,3|2,3,4,5,6) \big] \\
&+ \big[V_{123}\, T_{4,5,6} \mathcal{Z}_{123,4,5,6} + V_{132}\, T_{4,5,6} \mathcal{Z}_{132,4,5,6} + (2,3|2,3,4,5,6) \big] \\
&+ \big[(V_{12}\, T_{34,5,6} \mathcal{Z}_{12,34,5,6} + \mathrm{cyc}(2,3,4)) + (2,3,4|2,3,4,5,6) \big] \\
&+ \big[(V_1\, T_{2,34,56} \mathcal{Z}_{1,2,34,56} + \mathrm{cyc}(3,4,5)) + (2 \leftrightarrow 3,4,5,6) \big] \\
&+ \big[V_1\, T_{234,5,6} \mathcal{Z}_{1,234,5,6} + V_1\, T_{243,5,6} \mathcal{Z}_{1,243,5,6} + (2,3,4|2,3,4,5,6) \big]
\end{aligned}
$$

- Nice combinatorics of Stirling set and cycle numbers:

$$
\mathcal{K}_6(\ell) = \sum_{r=0}^{2} \frac{1}{r!} \Big(V_{A_1}\, T^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} \mathcal{Z}^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} + \big[12\ldots 6 | A_1, \ldots, A_{r+4} \big] \Big)
$$

6pt anomaly cancellation (Green, Schwarz 84)

- 6pt correlator is not BRST invariant by itself
- However BRST variation is a total derivative on moduli space

$$Q\mathcal{K}_6(\ell) = -\frac{1}{2}V_1 Y_{2,3,4,5,6}\mathcal{Z}^{mm}_{1,2,3,4,5,6} = -2\pi i\, V_1 Y_{2,3,4,5,6}\frac{\partial}{\partial\tau}\log\mathcal{I}_6(\ell) \cong 0$$

where $Y_{2,3,4,5,6}$ is the anomaly kinematic factor (CM, Berkovits 2006)

$$Y_{2,3,4,5,6} \equiv \frac{1}{2}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_5\gamma_{mnp}W_6)$$

- To show this need identities for τ derivatives of the Kronecker-Eisenstein series, several BRST variations etc
- So anomaly cancels after summing over one-loop topologies for $SO(32)$ (Green, Schwarz 84)

Derivations

Pure spinor amplitude prescription at one loop

$$\mathcal{A}_1 = \int_{\text{moduli}} \Big\langle (\mu, b)(PCs) V^1(0) \int dz\ U^2 \cdots \int dz\ U^n \Big\rangle$$

- vertex operators using SYM superfields $A_\alpha(x,\theta)$, $A_m(x,\theta)$, $W^\alpha(x,\theta)$ and $F_{mn}(x,\theta)$

$$V = \lambda^\alpha A_\alpha(x,\theta),$$
$$U = \partial\theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} F_{mn}$$

- CFT calculation: zero modes and OPEs
- OPEs among vertices organized using multiparticle superfields with covariant BRST variations (CM, Schlotterer '14)
- b ghost and PCOs complications bypassed by completing the known parts of the correlators from OPEs to BRST-invariant and single-valued answers

SYM description in 10D

- Single-particle (i is particle label) (Witten'86)

$$K_i \in \{A^i_\alpha,\ A^m_i,\ W^\alpha_i,\ F^{mn}_i\}$$

- Multiparticle (B is a "word" with particle labels)

$$K_B \in \{A^B_\alpha,\ A^m_B,\ W^\alpha_B,\ F^{mn}_B\}$$

- Inspired by OPE computations and defined recursively, eg

$$\begin{aligned} W^\alpha_1 &= W^\alpha_1 \\ W^\alpha_{12} &= \frac{1}{4}(\gamma^{mn}W^2)^\alpha F^1_{mn} + W^\alpha_2(k^2\cdot A^1) - (1\leftrightarrow 2) \\ W^\alpha_{123} &= -(k^{12}\cdot A^3)W^\alpha_{12} + \frac{1}{4}(\gamma^{rs}W^3)^\alpha F^{12}_{rs} - (12\leftrightarrow 3) \\ &\quad + \frac{1}{2}(k^1\cdot k^2)\big[W^\alpha_2(A^1\cdot A^3) - (1\leftrightarrow 2)\big] \end{aligned}$$

Generalized SYM equations of motion

- Superfields in K_B satisfy generalized SYM EOMs, eg

$$
\begin{aligned}
D_\alpha W_1^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^1_{mn} \\
D_\alpha W_{12}^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^{12}_{mn} \\
&\quad + (k^1 \cdot k^2)(A^1_\alpha W_2^\beta - A^2_\alpha W_1^\beta) \\
D_\alpha W_{123}^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^{123}_{mn} \\
&\quad + (k^1 \cdot k^2)\big[A^1_\alpha W_{23}^\beta + A^{13}_\alpha W_2^\beta - (1 \leftrightarrow 2)\big] \\
&\quad + (k^{12} \cdot k^3)\big[A^{12}_\alpha W_3^\beta - (12 \leftrightarrow 3)\big] \,,
\end{aligned}
$$

- In general:

$$
D_\alpha W_P^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^P_{mn} + \sum_{\substack{P=XjY \\ Y=R\sqcup S}} (k_X \cdot k_j)\big[A^{XR}_\alpha W^\beta_{jS} - A^{jR}_\alpha W^\beta_{XS}\big] \,,
$$

- Similar EOMs for $A^B_\alpha, A^m_B, F^{mn}_B$

Generalized Jacobi symmetries

- The superfields K_B satisfy generalized Jacobi symmetries

$$
\begin{aligned}
0 &= K_{12} + K_{21}, \\
0 &= K_{123} + K_{231} + K_{312}, \quad \text{(Jacobi identity)} \\
0 &= K_{1234} - K_{1243} + K_{3412} - K_{3421} \\
0 &= K_{A\ell(B)} + K_{B\ell(A)}
\end{aligned}
$$

- $\ell(A)$ is the Dynkin operator (left-to-right nested brackets)
- These are the same symmetries obeyed by nested commutators

$$
K_{1234\ldots p} \equiv K_{\ell(P)} = K_{[\ldots[[[1,2],3],4],\ldots,p]}
$$

- BCJ identities/numerators are natural in this framework
- BRST operator is $\lambda^\alpha D_\alpha$ so multiparticle superfields lead to (a rich) BRST algebra, cohomology identities etc

Zero-mode prescription and building blocks

- An analysis of the PS prescription leads to a zero-mode contribution of $d_\alpha d_\beta N^{mn}$ from the vertices (Berkovits '04)
- Four points

$$K_4 = \langle V_1 U_2 U_3 U_4 \rangle_{ddN} = \langle V_1 T_{2,3,4} \rangle$$

where

$$T_{2,3,4} = \frac{1}{3}(\lambda\gamma^m W_2)(\lambda\gamma^m W_3)F_4^{mn} + \mathrm{cyc}(2,3,4)$$

- Higher points: multiparticle version (CM, Schlotterer '12)

$$T_{A,B,C} = \frac{1}{3}(\lambda\gamma^m W_A)(\lambda\gamma^m W_B)F_C^{mn} + \mathrm{cyc}(A,B,C)$$

- at 5pts

$$V_{12} T_{3,4,5}, \qquad V_1 T_{23,4,5} + \mathrm{perm}$$

- Also tensorial generalization ($V_A T^{mn\dots}_{B,C,D,E,\dots}$)

One-loop superstring correlators

Recalling: Lie polynomials

A Lie polynomial is an expression written in terms of nested commutators

Ree theorem

If $\mathcal{Z}_P$ satisfies shuffle symmetries $\mathcal{Z}_{A \sqcup B} = 0$ and t^{p_i} are non-commutative indeterminates then

$$\sum_P \mathcal{Z}_{p_1 p_2 p_3 \ldots} t^{p_1} t^{p_2} t^{p_3} \ldots$$

is a Lie polynomial

- Example: $\mathcal{Z}_{12}$ satisfies shuffle if it is antisymmetric, so

$$\mathcal{Z}_{12} t^1 t^2 + \mathcal{Z}_{21} t^2 t^1 = \mathcal{Z}_{12}[t^1, t^2]$$

is a Lie polynomial

Lessons from tree-level (CM, Schlotterer, Stieberger 2011)

- n-point disk correlator can be rewritten in a suggestive way:

$$\mathcal{K}_n^{\rm tree} = \sum_{AB=23\ldots n-2} \big(\mathcal{Z}_{1A}^{\rm tree} V_{1A}\big)\big(\mathcal{Z}_{n-1,B}^{\rm tree} V_{n-1,B}\big) V_n + {\rm perm}(23\ldots n-2)\,.$$

1. Worldsheet functions satisfy shuffle symmetries

$$\mathcal{Z}_{123\ldots p}^{\rm tree} \equiv \frac{1}{z_{12}z_{23}\ldots z_{p-1,p}} \qquad \longrightarrow \qquad \mathcal{Z}_{A\shuffle B}^{\rm tree} = 0$$

2. associated kinematics satisfy generalized Jacobi symmetries

$$V_P \equiv \lambda^\alpha A_\alpha^P \qquad \longrightarrow \qquad V_{A\ell(B)} + V_{B\ell(A)} = 0$$

- This has the same structure of a Lie polynomial!

$$\sum_P \mathcal{Z}_P^{\rm tree} V_P$$

Ansatz for one-loop correlators

Tree-level reinterpretation key to unlock the one-loop correlators

1. Assume Lie-polynomial structure for one-loop correlators:

$$\mathcal{K}_n \to \sum \mathcal{Z}_{A,B,C,D} V_A T_{B,C,D} + \cdots$$

2. kinematic factors $V_A T_{B,C,D}$ satisfying generalized Jacobi symmetries
3. one-loop worldsheet functions $\mathcal{Z}_{A,B,C,\ldots}$ satisfying shuffle symmetries

- Singular behaviour of $\mathcal{Z}_{A,B,\ldots}$ as vertices collide is known from OPEs
- Unlike at tree-level, OPEs don't determine the complete functions as regular pieces are not fixed by singularities

The shuffle-symmetry requirement was very helpful in fixing the functions

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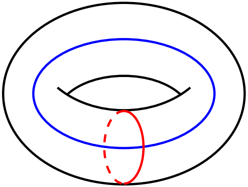
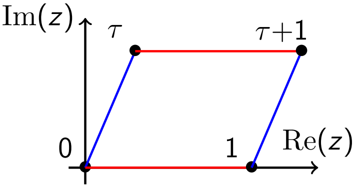
Elliptic Functions





Kronecker–Einsenstein series

- The Kronecker–Eisenstein series is defined as

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(\alpha,\tau)\theta_1(z,\tau)} \equiv \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z,\tau) \qquad (1)$$

- $\theta_1(z,\tau)$ is Jacobi odd theta function
- Expansion in α defines meromorphic functions (Brown, Levin)

$$\begin{aligned} g^{(0)}(z,\tau) &= 1 \\ g^{(1)}(z,\tau) &= \partial_z \ln \theta_1(z,\tau) \\ 2g^{(2)}(z,\tau) &= (\partial_z \ln \theta_1(z,\tau))^2 + \partial_z^2 \ln \theta_1(z,\tau) - \frac{\theta_1'''(0,\tau)}{3\theta_1'(0,\tau)} \end{aligned}$$

- Notation: $g_{ij}^{(n)} \equiv g^{(n)}(z_i - z_j, \tau)$

Kronecker-Eisenstein coefficient functions

- $g^{(1)}(z,\tau) = \partial \log \theta_1(z,\tau)$ is the genus-one generalization of tree-level $1/z$ function
- $g^{(n)}(z,\tau)$ for $n \geq 2$ have no singularities on the surface as $z \to 0$
- $g^{(n)}(z,\tau)$ are single-valued around a-cycles
- monodromies around b-cycles given by

$$Dg_{ij}^{(n)} = \Omega_{ij} g_{ij}^{(n-1)}$$

  where D is a monodromy operator
- $g_{ij}^{(n)}$ satisfy Fay identities, eg

$$g_{12}^{(1)} g_{23}^{(1)} + g_{12}^{(2)} + \mathrm{cyc}(1,2,3) = 0$$

- can argue that $D\ell^m = \sum_i \Omega_i k_i^m$

Shuffle symmetric functions

- PS zero-mode rules and OPEs imply at low multiplicities

$$\mathcal{Z}_{1,2,3,4} = 1$$
$$\mathcal{Z}_{12,3,4,5} = g^{(1)}_{12}, \qquad \mathcal{Z}^m_{1,2,3,4,5} = \ell^m$$

- $\mathcal{Z}_{12,3,4,5}$ is antisymmetric in [12], so it obeys shuffle symmetry
- Casting the 4 and 5-pt correlators in Lie-polynomial form we get

$$\begin{aligned}\mathcal{K}_4(\ell) &= V_1 T_{2,3,4} \mathcal{Z}_{1,2,3,4}\\ \mathcal{K}_5(\ell) &= V_1 T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[V_{12} T_{3,4,5} \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big]\\ &\quad + \big[V_1 T_{23,4,5} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\big]\end{aligned}$$

- what about 6 points?

- We need a shuffle-symmetric one-loop counterpart of the tree-level

$$\mathcal{Z}^{\text{tree}}_{123} = \frac{1}{z_{12}z_{23}}$$

- However, both

$$g^{(1)}_{12}g^{(1)}_{23} + \frac{1}{2}(g^{(2)}_{12} + g^{(2)}_{23})$$

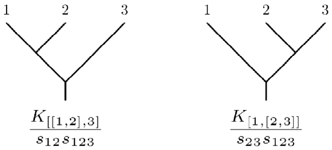
  and

$$g^{(1)}_{12}g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}$$

  satisfy shuffle symmetries in $P = 123$ (using Fay ids)
- Which one to use at six points?
- A new (double-copy) duality comes to the rescue! BRST invariants vs elliptic functions

BRST invariants

Berends–Giele supercurrents



$$\mathcal{K}_{123} \equiv \frac{K_{123}}{s_{12}s_{123}} + \frac{K_{321}}{s_{23}s_{123}}$$

- Defined from all planar binary trees dressed with propagators and K_B

$$\mathcal{K}_B \in \{\mathcal{A}^B_\alpha, \mathcal{A}^m_B, \mathcal{W}^\alpha_B, \mathcal{F}^{mn}_B\}$$

- Satisfy simple EOMs

$$D_\alpha \mathcal{W}^\beta_B = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathcal{F}^B_{mn} + \sum_{XY=B}\big(\mathcal{A}^X_\alpha \mathcal{W}^\beta_Y - \mathcal{A}^Y_\alpha \mathcal{W}^\beta_X\big)$$

- Berends-Giele supercurrents satisfy shuffle symmetries

$$\mathcal{K}_{A\sqcup B} = 0, \quad \forall A, B \neq \emptyset$$

Scalar BRST invariants

- Define (λ^α is a pure spinor)

$$M_B \equiv \lambda^\alpha \mathcal{A}_\alpha$$
$$M_{A,B,C} \equiv \frac{1}{3}(\lambda\gamma_m \mathcal{W}_A)(\lambda\gamma_n \mathcal{W}_B)\mathcal{F}_C^{mn} + (C \leftrightarrow A, B)\,.$$

- BRST variations ($Q = \lambda^\alpha D_\alpha$)

$$QM_B = \sum_{XY=B} M_X M_Y$$
$$QM_{A,B,C} = \sum_{XY=A} \big(M_X M_{Y,B,C} - M_Y M_{X,B,C}\big) + (A \leftrightarrow B, C)\,,$$

Scalar BRST invariants

- BRST invariants: $QC_{1|A,B,C} = 0$
- Recursive construction (CM, Schlotterer'14)

$$C_{1|2,3,4} = M_1 M_{2,3,4}$$
$$C_{1|23,4,5} = M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}$$
$$C_{1|A,B,C} = \text{general formula known}$$

- Generalization for arbitrary tensor ranks (CM, Schlotterer 2014)
- Simplest vector BRST invariant

$$C^m_{1|2,3,4,5} = M_1 M^m_{2,3,4,5} + \left[k^m_2 M_{12} M_{3,4,5} + (2 \leftrightarrow 3,4,5)\right]$$

BRST cohomology identities

- BRST invariants satisfy BRST cohomology identities
- Momentum contractions:

$$k_2^m C^m_{1|2,3,4,5} + \big[s_{23} C_{1|23,4,5} + (3 \leftrightarrow 4,5) \big] = 0$$

- Change of basis:

$$\begin{aligned} C_{2|34,1,5} &= C_{1|34,2,5} + C_{1|23,4,5} - C_{1|24,3,5} \\ C_{2|13,4,5} &= -C_{1|23,4,5} \\ C^m_{2|1,3,4,5} &= C^m_{1|2,3,4,5} + \big[k_3^m C_{1|23,4,5} + (3 \leftrightarrow 4,5) \big] \end{aligned}$$

- Rich mathematical structure: free Lie algebra

Worldsheet functions/BRST-invariants duality

Worldsheet function/BRST-invariants duality

A happy surprise!

- One can show that

$$E_{1|23,4,5} = \mathcal{Z}_{1,23,4,5} + \mathcal{Z}_{12,3,4,5} - \mathcal{Z}_{13,2,4,5}$$

  is is single valued, $DE_{1|23,4,5} = 0$

- Seen this combinatoric pattern before: 5-pt BRST invariant

$$C_{1|23,4,5} = M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}$$

  satisfying $QC_{1|23,4,5} = 0$

- Duality: elliptic functions vs BRST invariants (CM, Schlotterer '17)

$$E_{1|23,4,5} \longleftrightarrow C_{1|23,4,5}$$

$$DE_{1|23,4,5} = 0 \longleftrightarrow QC_{1|23,4,5} = 0$$

Worldsheet function/BRST invariant duality

- Tensorial generalization (CM, Schlotterer '18)
- Simplest example. From the BRST invariant

$$C^m_{1|2,3,4,5} = M_1 M^m_{2,3,4,5} + \left[k^m_2 M_{12} M_{3,4,5} + (2 \leftrightarrow 3,4,5)\right]$$

satisfying $QC^m_{1|2,3,4,5} = 0$ one is led to define

$$E^m_{1|2,3,4,5} = \mathcal{Z}^m_{1,2,3,4,5} + \left[k^m_2 \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\right]$$

which happens to be single valued

$$DE^m_{1|2,3,4,5} = 0$$

Worldsheet function/BRST invariant duality

- Using the Jacobi theta functions and integration by parts can show

$$k_2^m E^m_{1|2,3,4,5} + \left[s_{23} E_{1|23,4,5} + (3 \leftrightarrow 4,5)\right] = 0$$

- We have seen an identity of identical structure for the BRST invariants:

$$k_2^m C^m_{1|2,3,4,5} + \left[s_{23} C_{1|23,4,5} + (3 \leftrightarrow 4,5)\right] = 0$$

- Similarly, identical symmetry relations hold for the GEIs

$$\begin{aligned} E_{2|34,1,5} &= E_{1|34,2,5} + E_{1|23,4,5} - E_{1|24,3,5} \\ E_{2|13,4,5} &= -E_{1|23,4,5} \\ E^m_{2|1,3,4,5} &= E^m_{1|2,3,4,5} + \left[k_3^m E_{1|23,4,5} + (3 \leftrightarrow 4,5)\right] , \end{aligned}$$

- Duality between elliptic functions and BRST invariants!

Bootstraping worldsheet functions

- This duality can be exploited to derive higher-point worldsheet functions!
- Inspired by the BRST variation written in terms of BRST invariants

$$QM_{123,4,5} = C_{1|23,4,5} - C_{3|12,4,5}$$

assume the following monodromy variation of the 6pt worldsheet function

$$D\mathcal{Z}_{123,4,5,6} = \Omega_1 E_{1|23,4,5,6} - \Omega_3 E_{3|12,4,5,6}$$

where the elliptic functions $E_{1|23,4,5,6}$ are obtained from 5pt functions using the combinatorics of 5pt BRST invariants

Higher-point one-loop correlators

- There is a unique solution:

$$\mathcal{Z}_{123,4,5,6} = g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}$$

- This is the function we should use in 6pt ansatz!
- Can solve all the other functions similarly: require the monodromy variations of $\mathcal{Z}^{mn\ldots}_{A,B,C,\ldots}$ to match the BRST variation of the corresponding Berends-Giele superfield $M_A M^{mn\ldots}_{B,C,\ldots}$

Higher-point one-loop correlators

- This structure generalizes to n-points including refined and anomalous superfields (the “corrections” from the first slide)

$$\mathcal{K}_n(\ell) \equiv \sum_{d=0}^{\lfloor \frac{n-4}{2} \rfloor} (-1)^d \mathcal{K}_n^{(d)}(\ell) + \mathcal{K}_n^Y(\ell)$$

- Leads to BRST-invariant and single-valued 7-pt correlator
- Puzzle at 8-points: modular form of weight four $G_4(\tau)$ remains in the BRST variation
- Probably requires a new class of term that we missed, but the Lie-polynomial structure of the correlator should be the same