PRINCETON CENTER FOR THEORETICAL SCIENCE

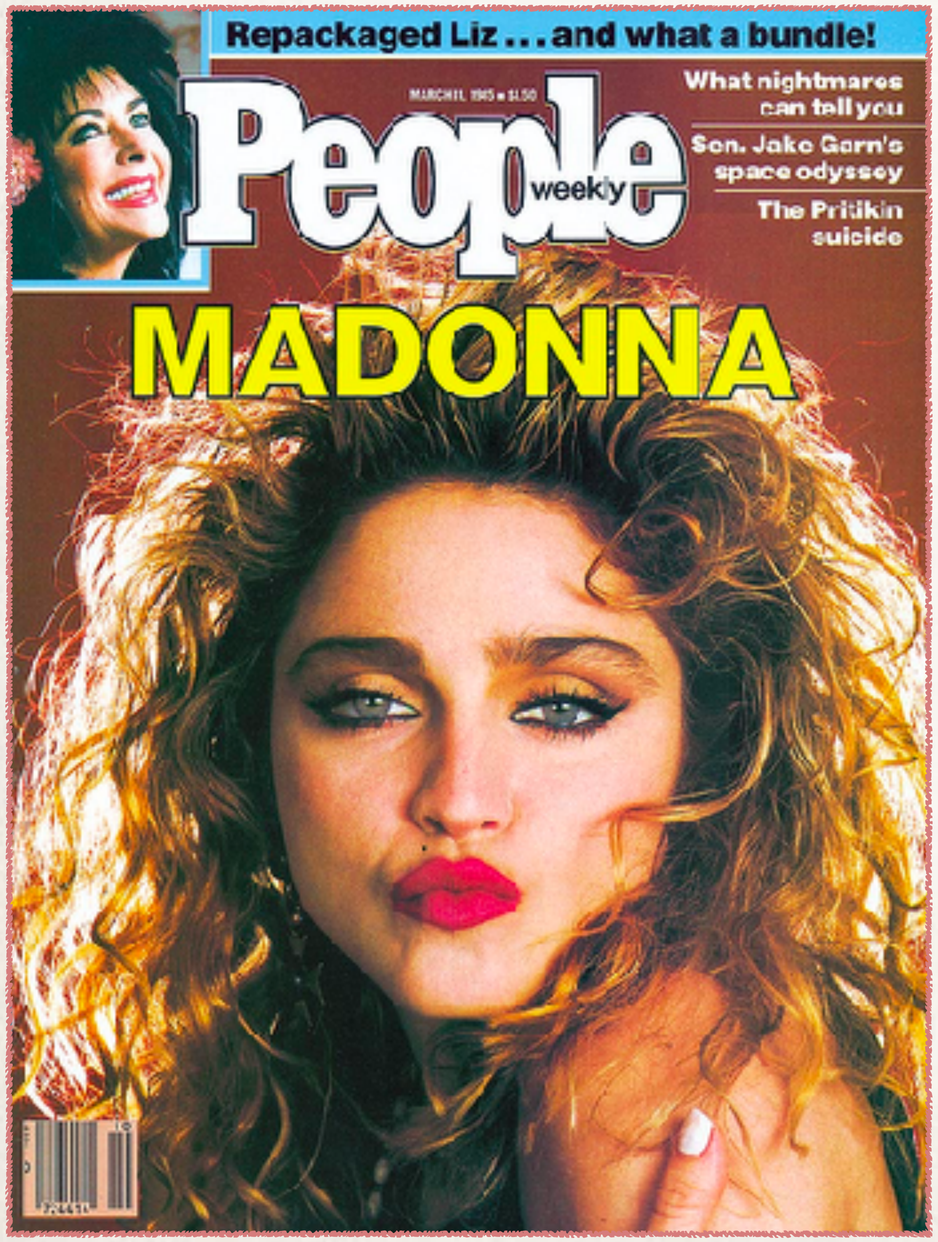
Constraints on SuperFluid DM using Local MW Observables

Oren Slone, Princeton University

arXiv 1812:08169 - M. Lisanti, M. Moschella, N. Outmezguine and O. Slone

Constraining Superfluid DM with MW Dynamics, Same authors

Great things from the 80's



Madonna, 1980

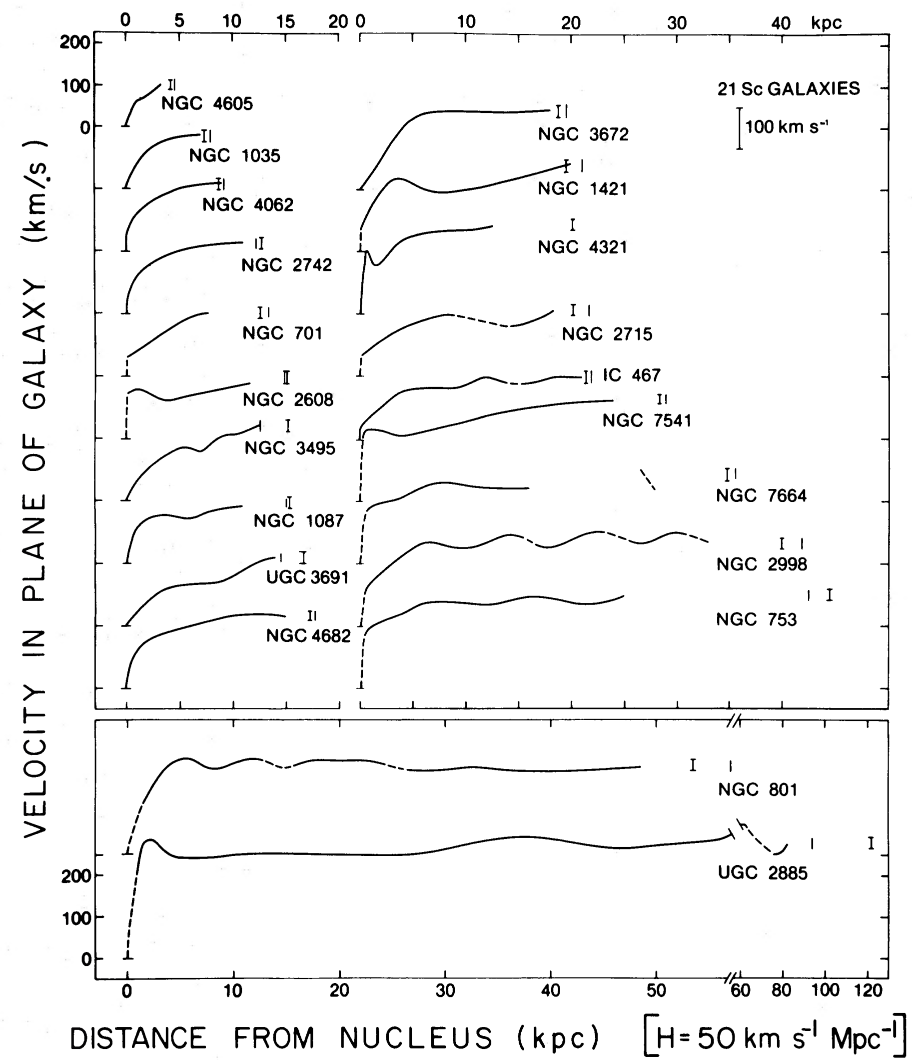
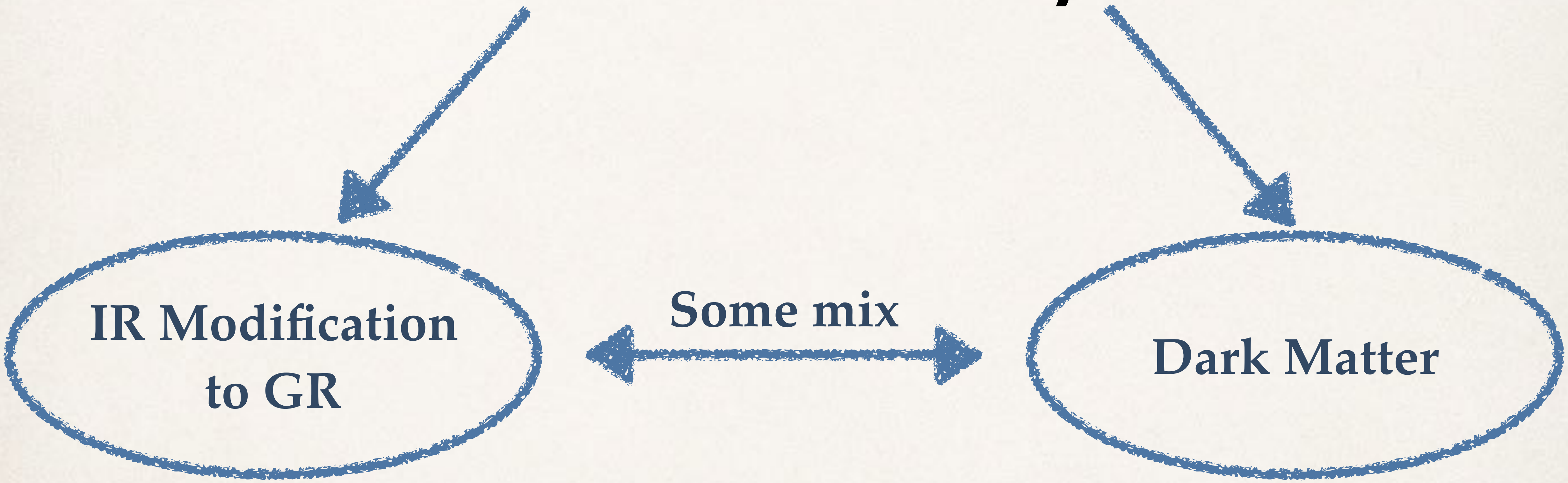


FIG. 5.—Mean velocities in the plane of the galaxy, as a function of linear distance from the nucleus for 21 Sc galaxies, arranged according to increasing linear radius. Curve drawn is rotation curve formed from mean of velocities on both sides of the major axis. Vertical bar marks the location of R_{25}, the isophote of 25 mag arcsec^{-2}; those with upper and lower extensions mark $R^{i,b}$, i.e., R_{25} corrected for inclination and galactic extinction. Dashed line from the nucleus indicates regions in which velocities are not available, due to small scale. Dashed lines at larger R indicates a velocity fall faster than Keplerian.

Vera Rubin, Ford and Thonnard, June 1980

A Naive Solution

$$\nabla^2 \Phi = 4\pi G \rho$$



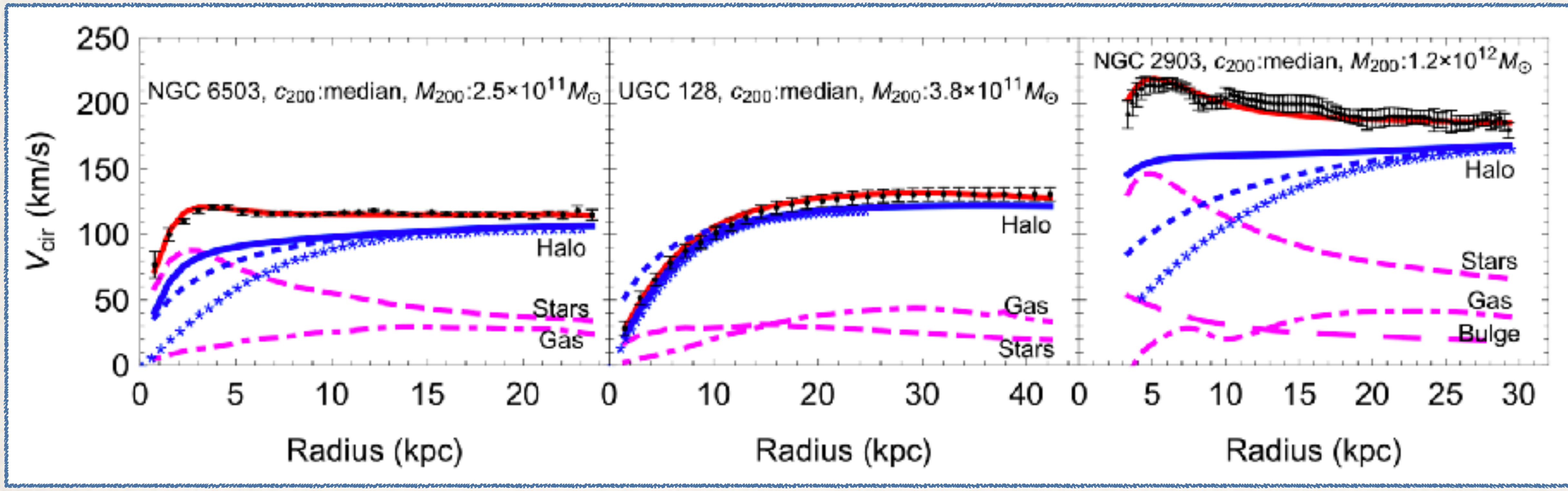
Amazingly: Still not clear-cut on galactic scales

The Missing Mass Problem on Galactic Scales, 2019

- **Flat Rotation Curves**
- **Issues with Small Scales:**
    - Missing Satellites
    - Too Big To Fail
    - Core vs Cusp
- **DM Correlations with Baryons:**
    - Baryonic Tully Fisher
    - and also…

Galaxy Scale Observables

The Diversity Problem



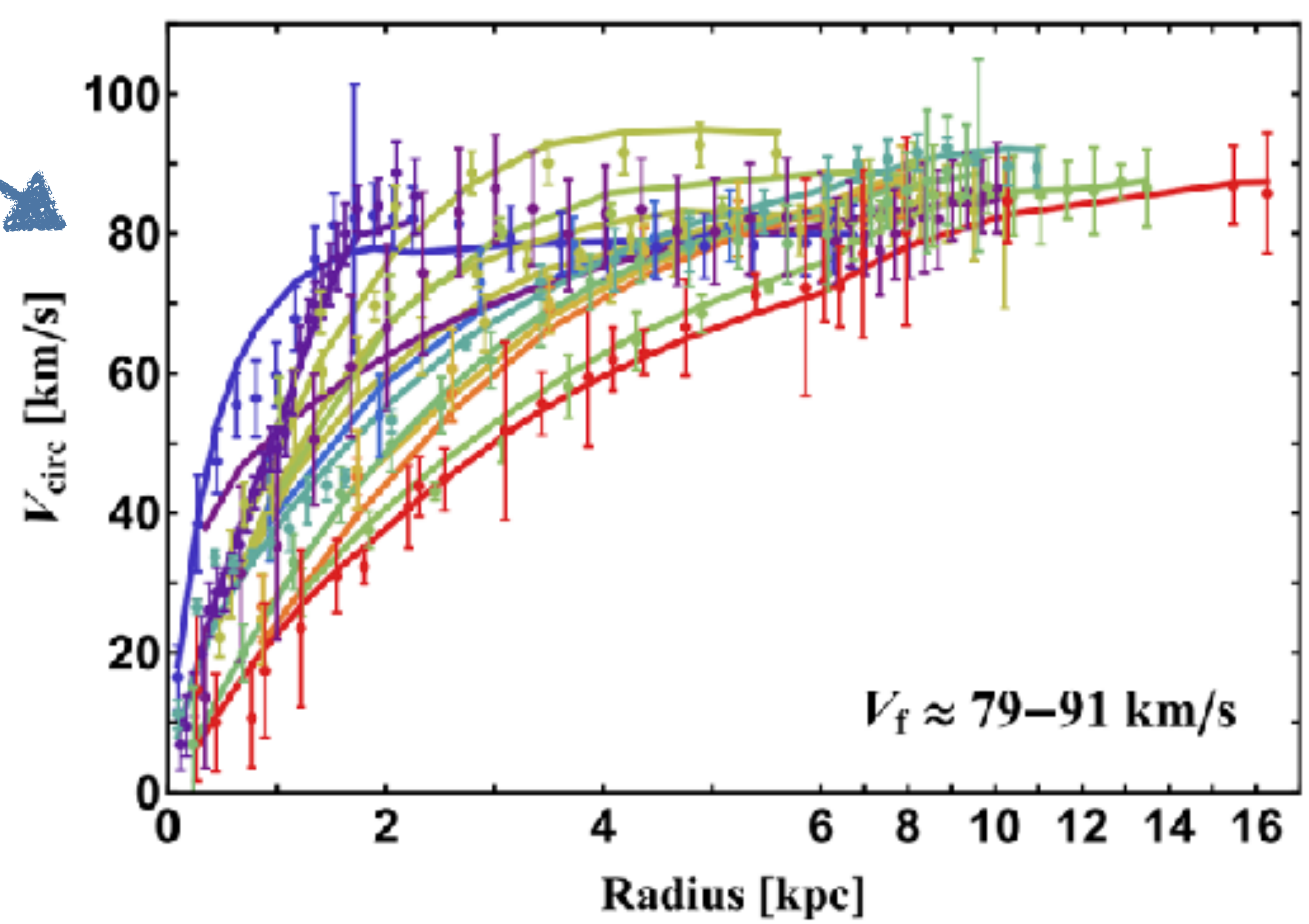
Kamada et. al., 2016

- Diversity of inner rotation curves even for galaxies with similar halo and stellar mass.
- Rotation curves correlate with baryons

Galaxy Scale Observables

The Diversity Problem

DM dominated galaxies!

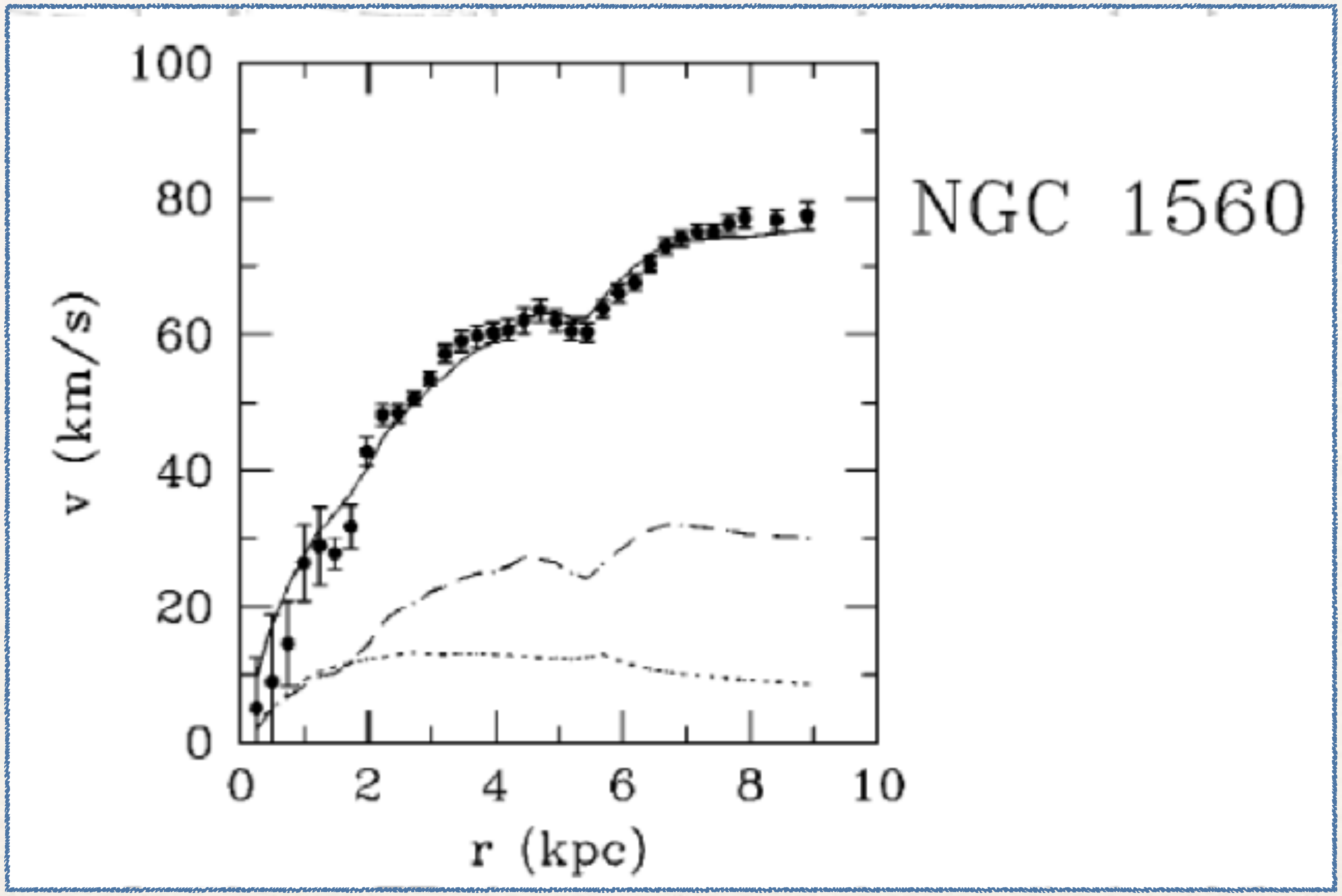


Kamada et. al., 2016

- Low surface brightness - halo is cored
- High surface brightness - halo is cusped
- Self similar if scaled to baryonic scale radius

Galaxy Scale Observables

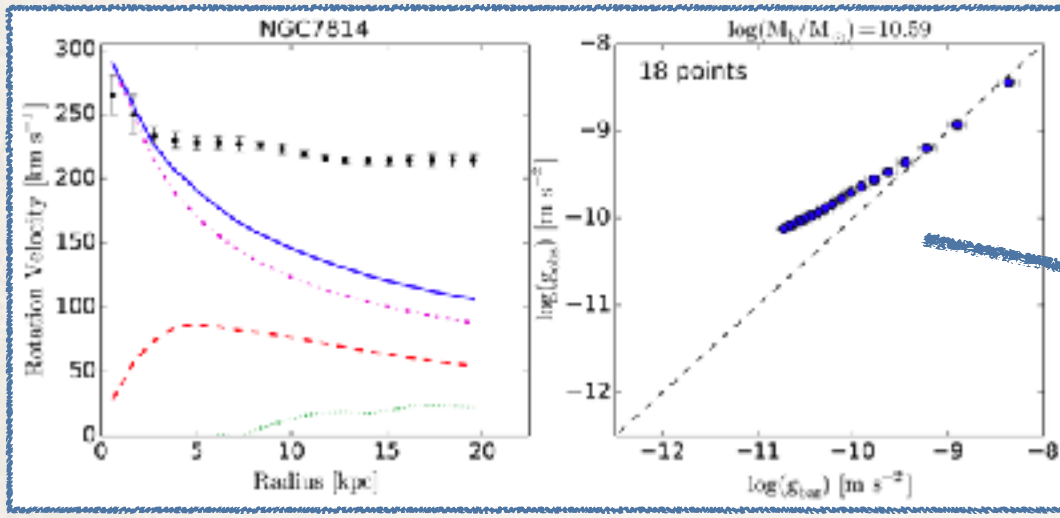
Renzo's Rule



Sancisi, 2003

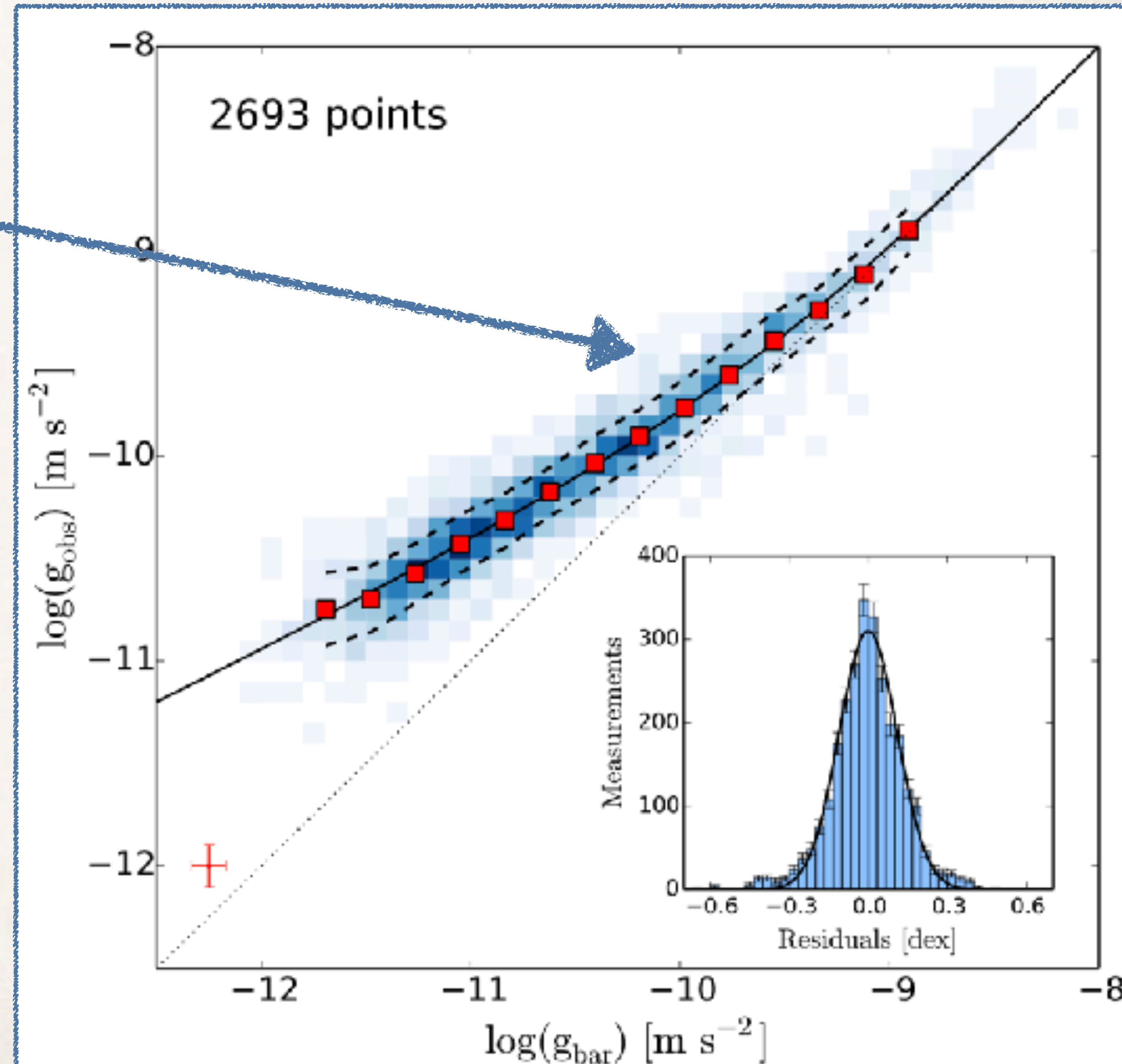
Galaxy Scale Observables

The Radial Acceleration Relation (RAR)



Lelli et. al, 2017

A tight correlation and an acceleration scale appear in rotation curve data from the SPARC catalog



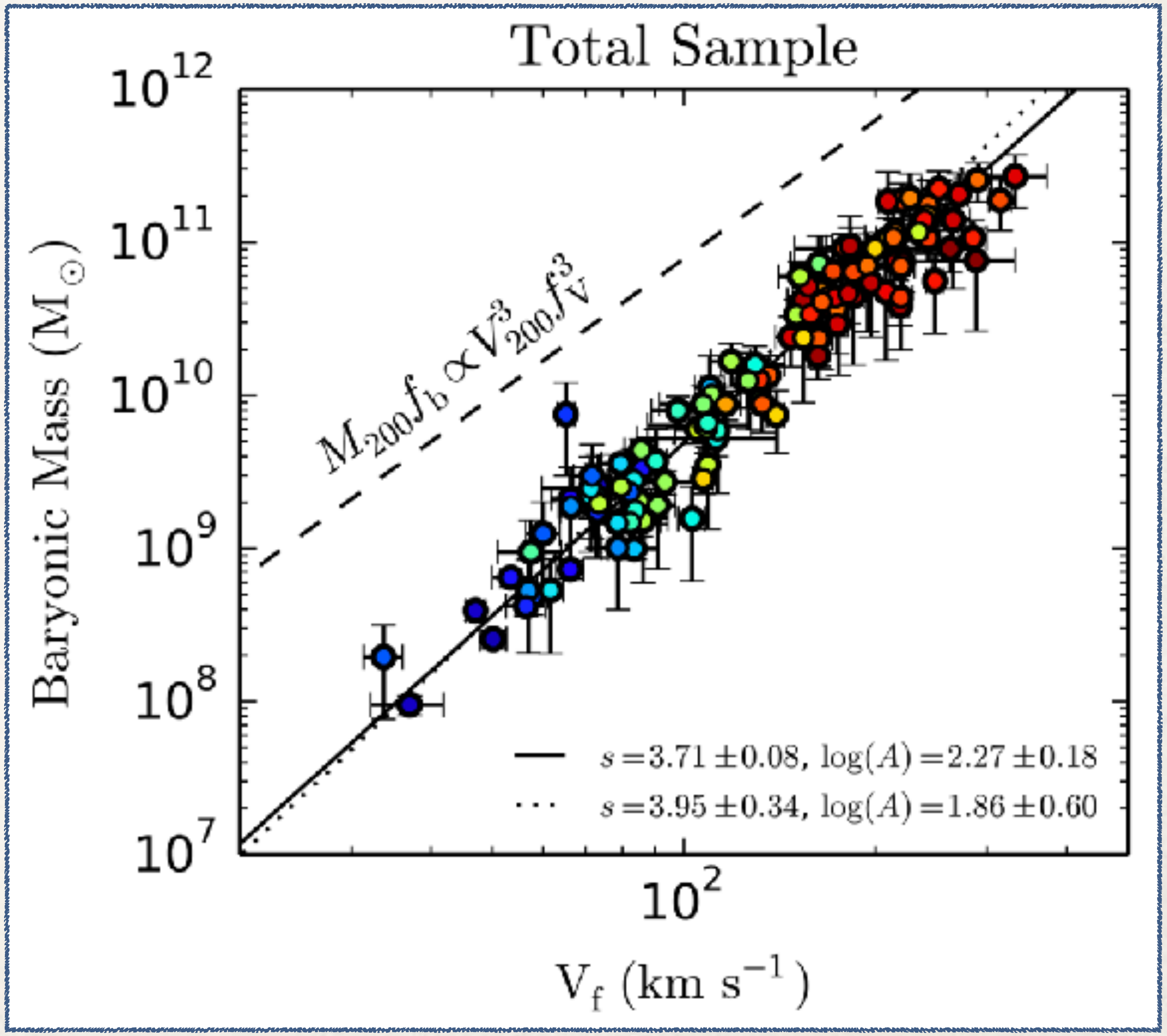
McGaugh, Lelli, 2017

Galaxy Scale Observables

The Baryonic Tully-Fisher Relation

A result of the information in the low end of the RAR

$$g_{\rm obs} \propto \sqrt{g_{\rm bar}} \quad \Rightarrow \quad \frac{V_{\rm f}^2}{R} \propto \frac{\sqrt{GM_{\rm bar}}}{R}$$

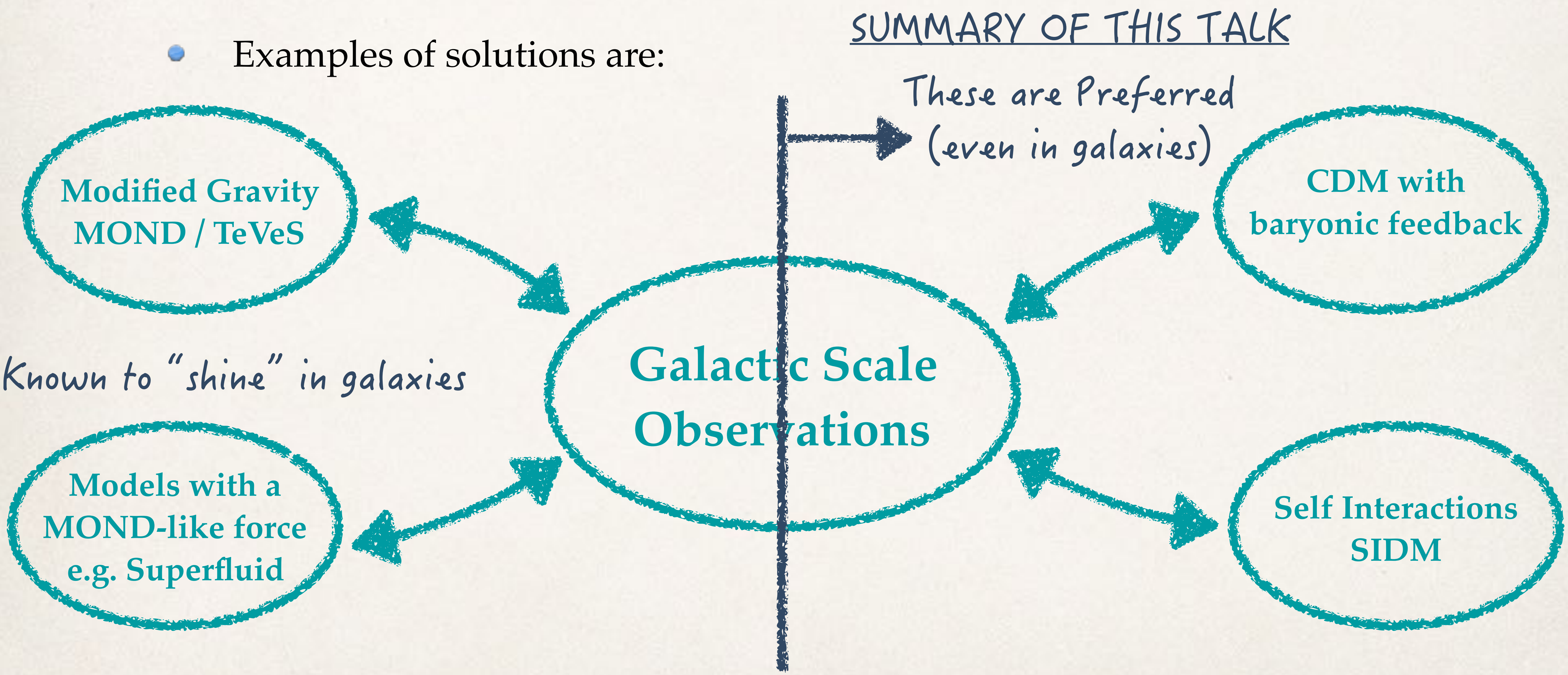


Lelli et. al., 2015

Galaxy Scale Observables

What models resolve these issues?

- Galaxies provide clues that DM correlates with baryons.
- Examples of solutions are:



Or maybe DM mimics MOND on galactic scales?

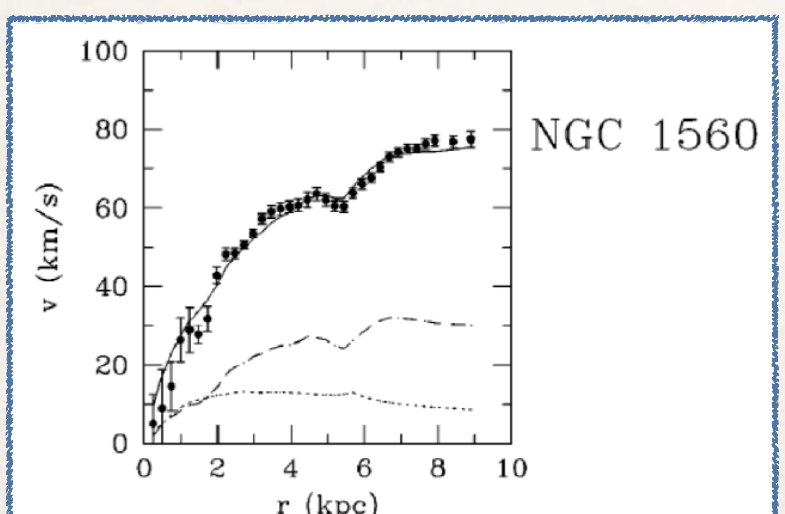
Phenomenology of the Solutions

$$\nabla^2 \Phi = 4\pi G \rho$$

IR Modification to GR

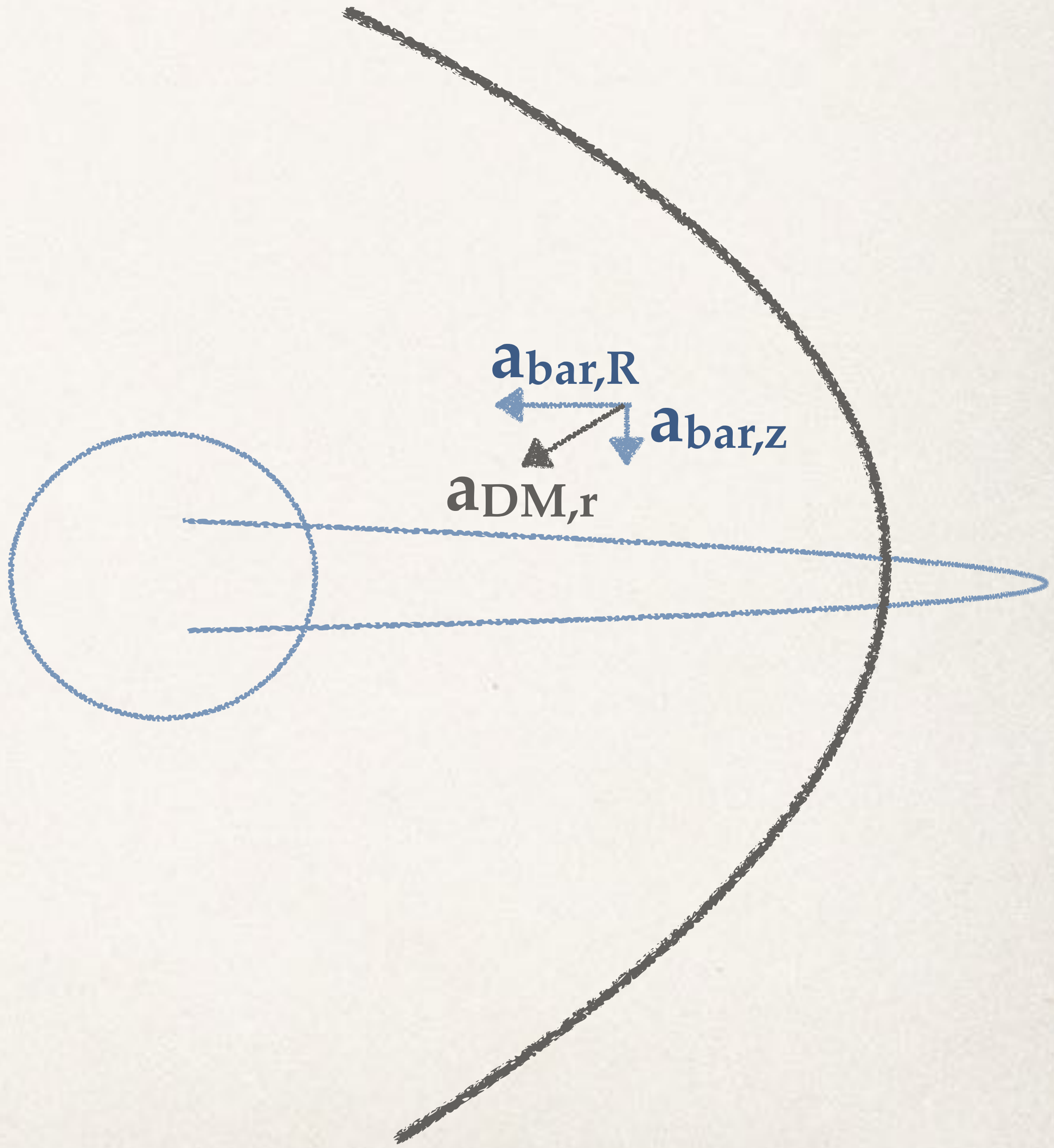
Some mix

Dark Matter



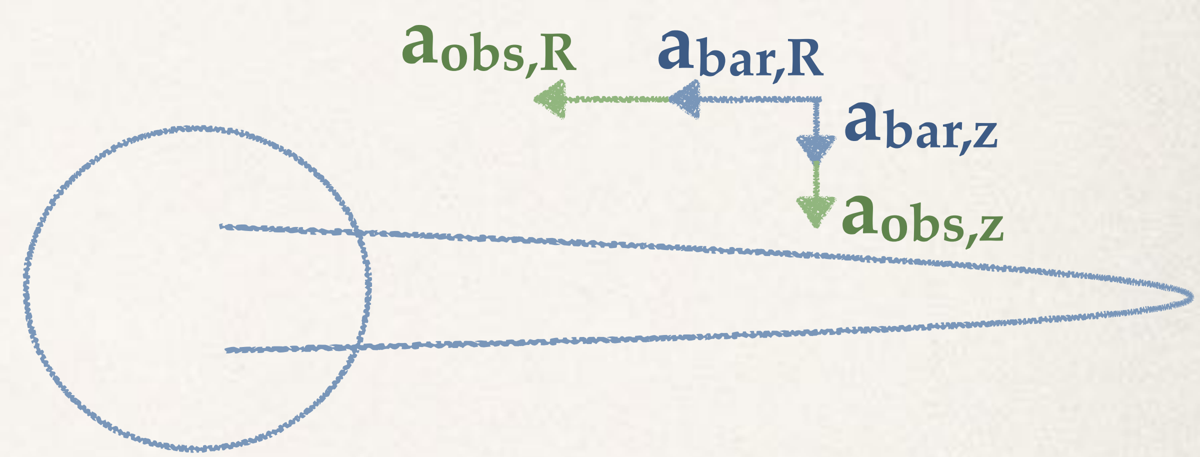
Dark Matter Pheno

- Galactic dynamics driven by an extended DM halo
- Halo shape is weakly constrained by measurements
- NFW-like profile probable from N-body simulations
- Amplifies acceleration via additional density profile



MOND-Like Pheno

- **Galactic dynamics driven purely by baryons**
- Most simple example is a scalar enhancement to Newtonian gravity
- Designed to reproduce flat rotation curves:
- MOND-like forces amplify acceleration:



$$\Phi \propto \log r \to a \propto \frac{1}{r} \to v_c \propto \mathrm{const}$$

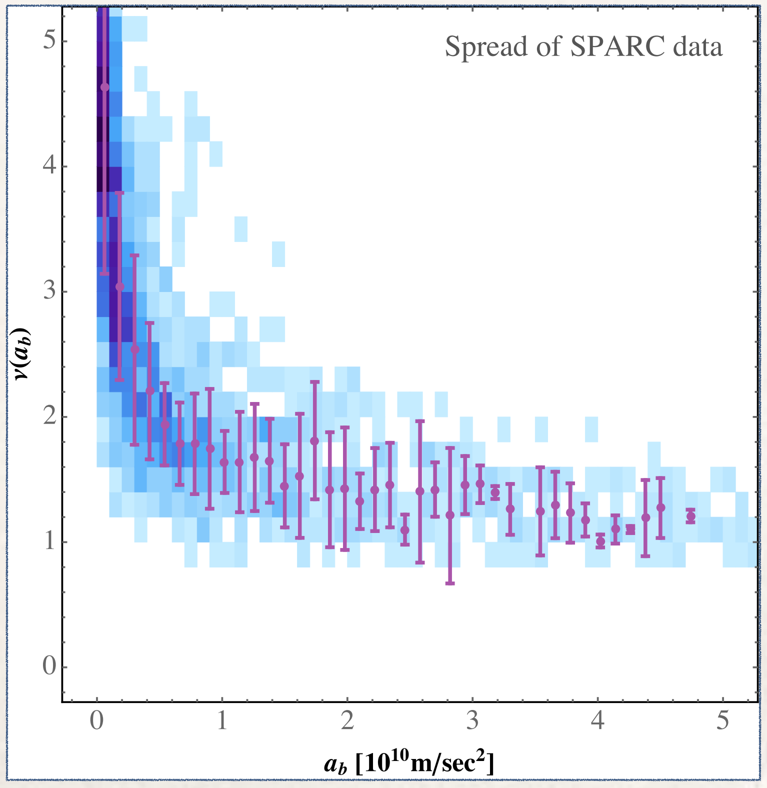
$$a = \begin{cases} a_{\mathrm{N}} & a \gg a_0 \\ \sqrt{a_0 a_{\mathrm{N}}} & a \ll a_0 \end{cases}$$

Newtonian acceleration

$$a_{\mathrm{N}} \propto \frac{1}{r^2}$$

MOND-like forces

- MOND-like theories: MOND, QuMOND, TeVeS, AQUAL, Superfluid DM
- All try to reproduce rotation curves: $\Phi \propto \log r \rightarrow a \propto \frac{1}{r} \rightarrow v_c \propto \text{const}$
- All reduce to: $\boldsymbol{a} = \nu\left(\frac{a_{\mathrm{N}}}{a_0}\right)\boldsymbol{a}_{\mathrm{N}}$
- With an interpolation function with asymptotes: $\nu\left(x_{\mathrm{N}}\right) = \begin{cases} x_{\mathrm{N}}^{-1/2} & x_{\mathrm{N}} \ll 1 \\ 1 & x_{\mathrm{N}} \gg 1 \end{cases}$



Lisanti, Moschella, Outmezguine, OS (PRELIMINARY)

For example:

$$\hat{\nu}_\alpha(x_{\mathrm{N}}) = \left(1 - e^{-x_{\mathrm{N}}^{\alpha/2}}\right)^{-\frac{1}{\alpha}}$$

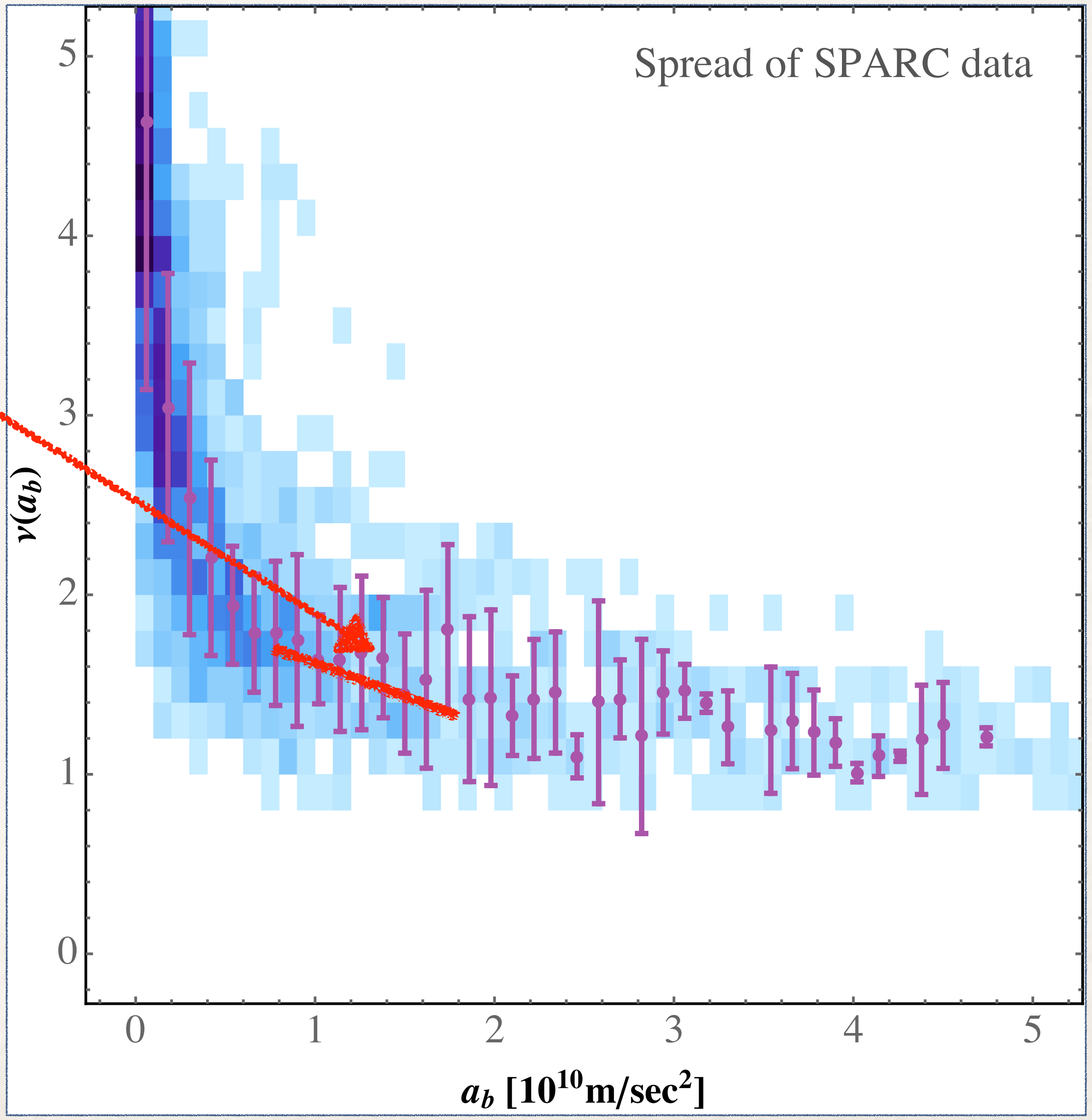
McGaugh, et. al. 2016

MOND-like forces

Solar acceleration happens to live here

Local measurements are sensitive only to small deviation in acceleration

$$\boldsymbol{a} = \nu\left(\frac{a_{\rm N}}{a_0}\right)\boldsymbol{a}_{\rm N} \;\Rightarrow\; \mathbf{a} = (\nu_0 + \nu_1 a_{\rm N})\,\mathbf{a}_{\rm N}$$



Lisanti, Moschella, Outmezguine, OS (PRELIMINARY)

What can we do?

1. Ask a **model independent** question:
    - Can local MW measurements fit a generic model that results in a MOND-like force?

Anything that mimics MOND

2. Test a **specific realization**:
    - e.g. A specific interpolation function
    - e.g. Superfluid dark matter

(Test MOND-like models where they're supposed to shine!)

Intro to Superfluid DM

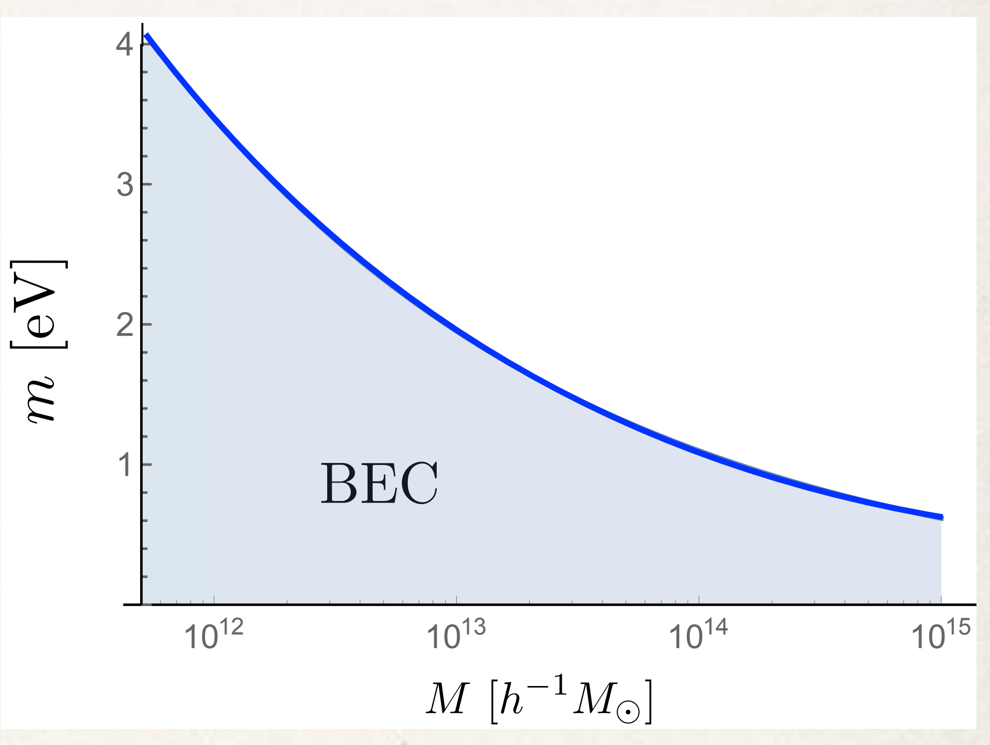
Justin Khoury, Lasha Berezhiani

- Consider a light scalar DM particle with mass m.
- Require condensation to a state where the relevant DOF are phonons:
- An overlapping de Broglie wavelength:

$$\frac{1}{mv} \geq \left(\frac{m}{\rho_{\rm vir}}\right)^{1/3} \quad \Rightarrow m \lesssim 2\text{eV}$$

- With a critical temperature:

$$T_c \approx \frac{1}{3} m v^2 \approx \text{few} \left(\frac{\text{eV}}{m}\right)^{5/3} \text{mK}$$

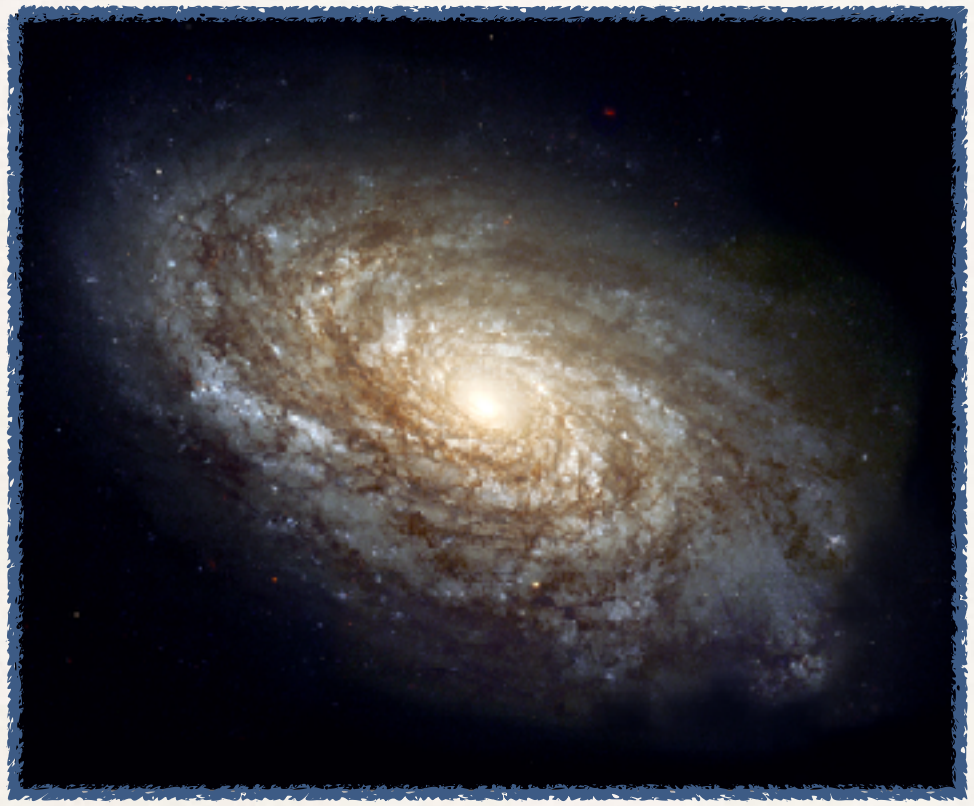


Berezhiani, Khoury, 2015

Superfluid DM

$$T \approx m v_{\mathrm{vir}}$$

Galaxies



$$T_{\mathrm{gal}} \approx 0.1\mathrm{mK}$$

Super Fluid Phase

MOND-Like Emergent Force

Galaxy Clusters

$$T_{\mathrm{cluster}} \approx 10\mathrm{mK}$$

Cold DM

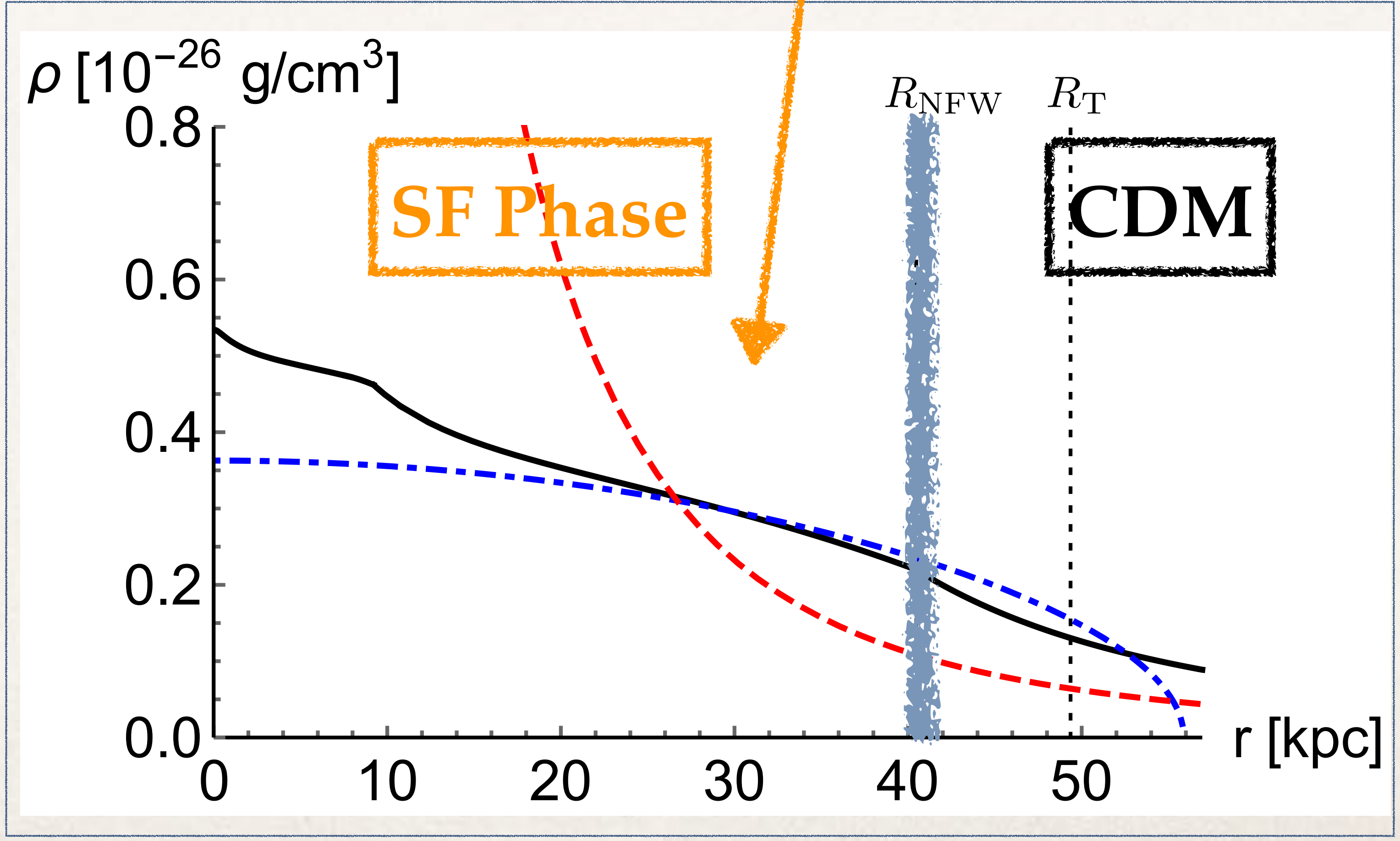
Standard DM Dynamics

Superfluid DM

$$\mathcal{L}_{\mathrm{DM},\,T=0} = \frac{2\Lambda(2m)^{3/2}}{3} X\sqrt{|X|} \qquad \mathcal{L}_{\mathrm{int}} = \alpha\Lambda \frac{\phi}{M_{\mathrm{Pl}}} \rho_{\mathrm{b}}$$

$$X = \mu - m\Phi + \dot{\phi} - (\vec{\nabla}\phi)^2/2m$$

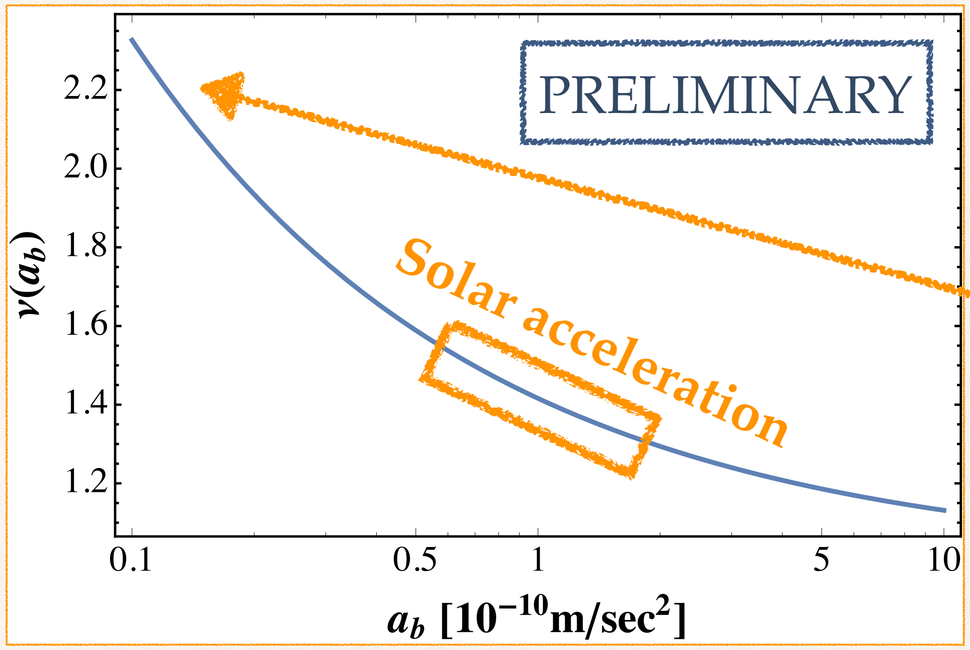
$$\rho_{\mathrm{SF}} = \frac{\partial \mathcal{L}}{\partial \Phi}$$



Berezhiani, Famaey, Khoury, 2017

Superfluid DM

E.O.M. for ϕ



Lisanti, Moschella, Outmezguine, OS (PRELIMINARY)

$$|\vec{\nabla}\phi|\vec{\nabla}\phi \simeq \alpha M_{\rm Pl}\vec{a}_{\rm b}$$

$$\vec{a}_\phi = \alpha \frac{\Lambda}{M_{\rm Pl}} \vec{\nabla}\phi \,.$$

$$a_\phi = \sqrt{\frac{\alpha^3\Lambda^2}{M_{\rm Pl}} a_{\rm b}} \,.$$

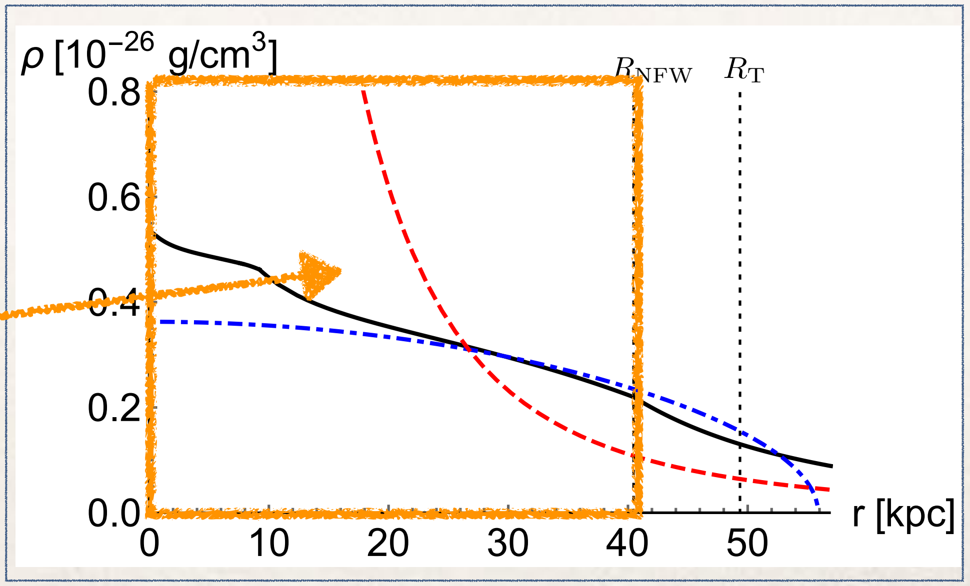
$$a_0 = \frac{\alpha^3\Lambda^2}{M_{\rm Pl}}$$

DM

+

Emergent MOND

$$\vec{a} = \vec{a}_{\rm b} + \vec{a}_{\rm DM} + \vec{a}_{\rm phonon}$$

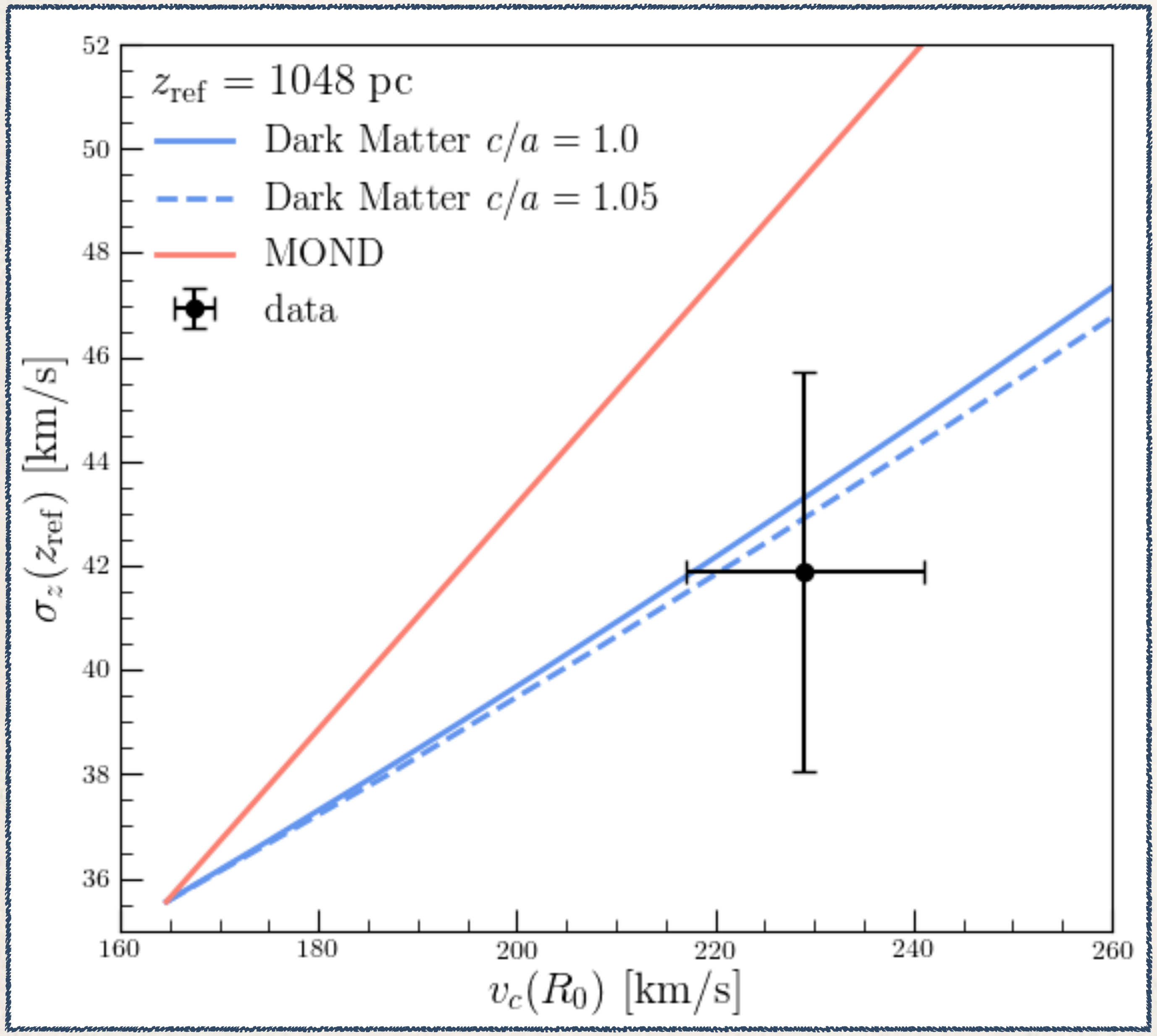


Berezhiani, Famaey, Khoury, 2017

Constraining These Models

Local MW Observations Provide Differentiating Power

Compare accelerations in the R and z directions:



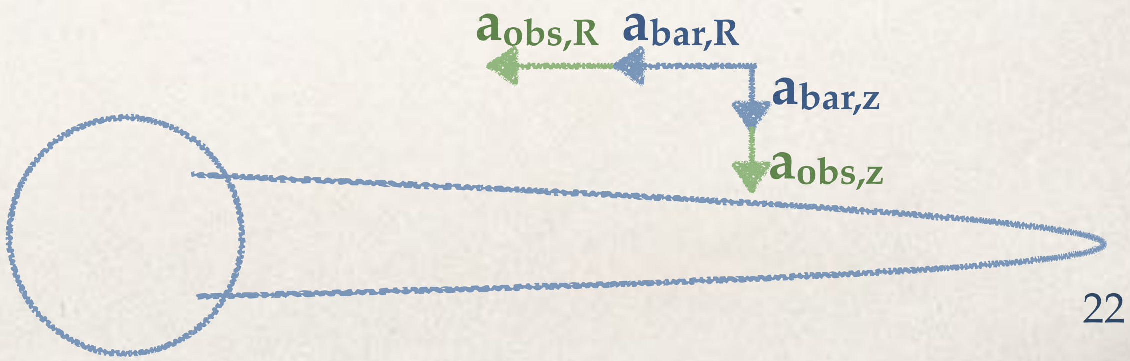
1812.08169 - Lisanti, Moschella, Outmezguine, O.S.

- Data requires amplification in a_R but essentially none in a_z.
- A spherical DM halo does precisely this:

$$\boldsymbol{a}_{\rm DM} \approx -G\frac{M(R_0)}{R_0^2}\left(1, \frac{z}{R_0}\right)$$

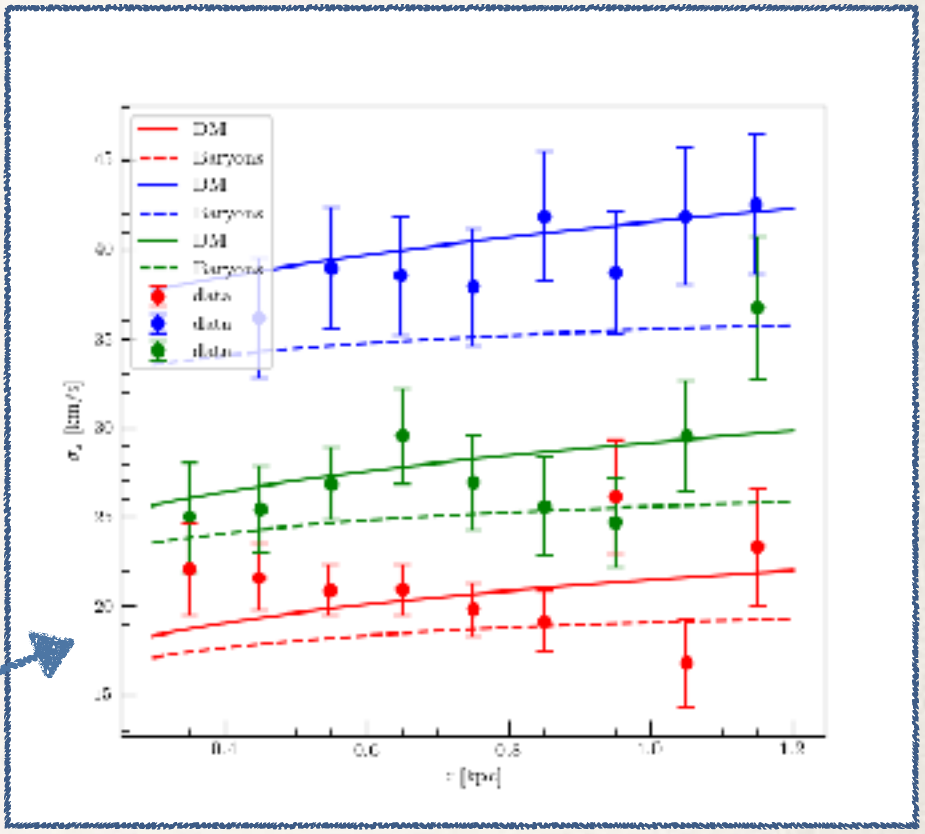
- A slightly prolate halo is slightly better.
- A MOND-like force amplifies a_R too little or a_z too much:

$$\frac{a_z}{a_R} = \frac{a_{z,\rm N}}{a_{R,\rm N}}\Big|_{\rm disk}$$



Local MW Observations Provide Differentiating Power

- In principle: measure **a** and $\mathbf{a_N}$ and you're done!
- However measurements are imperfect:
  - Baryonic profile is not perfectly measured.
  - Accelerations are not directly measured. Velocities and velocity dispersions are.
  - Superfluid DM is slightly less simple than MOND.
- Therefore: Adopt a **Bayesian Approach**



Lisanti, Moschella, Outmezguine, O.S., 2018
Data from Zhang et. al., 2013

Local MW Observations Provide Differentiating Power

Bayesian Approach

- Given a model: $\mathcal{M} = \mathrm{DM, MG}$
- With parameters: $\boldsymbol{\theta}_{\mathcal{M}}$
- Construct a likelihood function: $\mathcal{L}(\boldsymbol{\theta}_{\mathcal{M}}) \propto \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\left(\frac{X_{j,\mathrm{obs}} - X_j(\boldsymbol{\theta}_{\mathcal{M}})}{\delta X_{j,\mathrm{obs}}}\right)^2\right]$
- $\mathbf{X}_{\mathrm{obs}}$: a set of measured values imposed as constraints
- $\mathbf{X}(\boldsymbol{\theta}_{\mathcal{M}})$: the corresponding model predictions
- Impose reasonable priors on $\boldsymbol{\theta}_{\mathcal{M}}$ and recover posterior distributions

Analysis Procedure:
TESTING a MOND-like force vs DM

Analysis Procedure

Milky Way Model

MOND-like Force
Taylor expansion of ν
or SF model

Dark Matter
A generalized NFW
profile

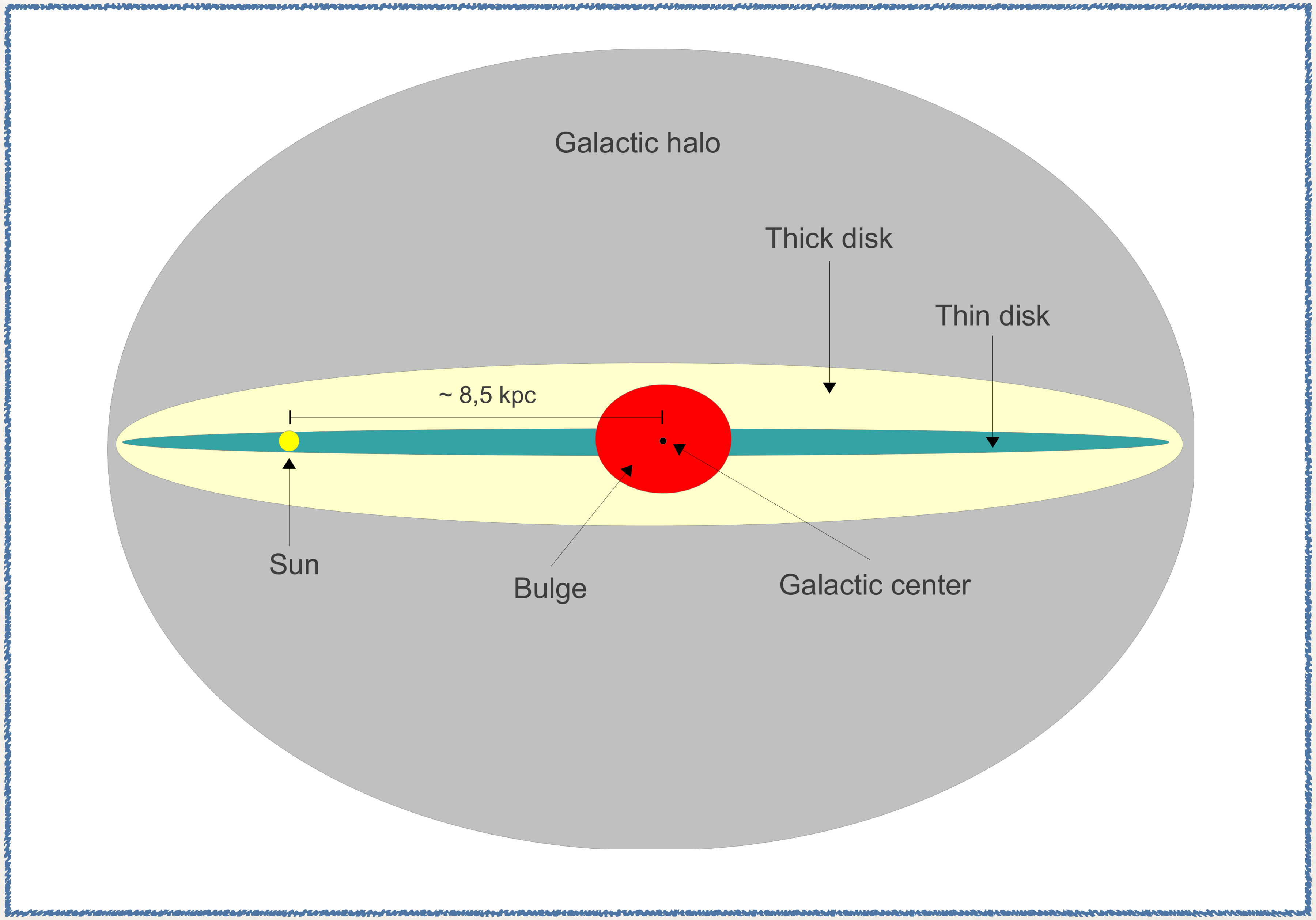
Model baryonic profile:

- Double exponential stellar disk
- Hernquist stellar bulge
- Double exponential gas disk

Perform a Markov Chain Monte Carlo analysis
and fit parameters to MW measurements

Analysis Procedure

Baryonic Density Profiles



$$\rho_B = \rho_{*,\mathrm{bulge}} + \rho_{*,\mathrm{disk}} + \rho_{g,\mathrm{disk}}$$

Analysis Procedure

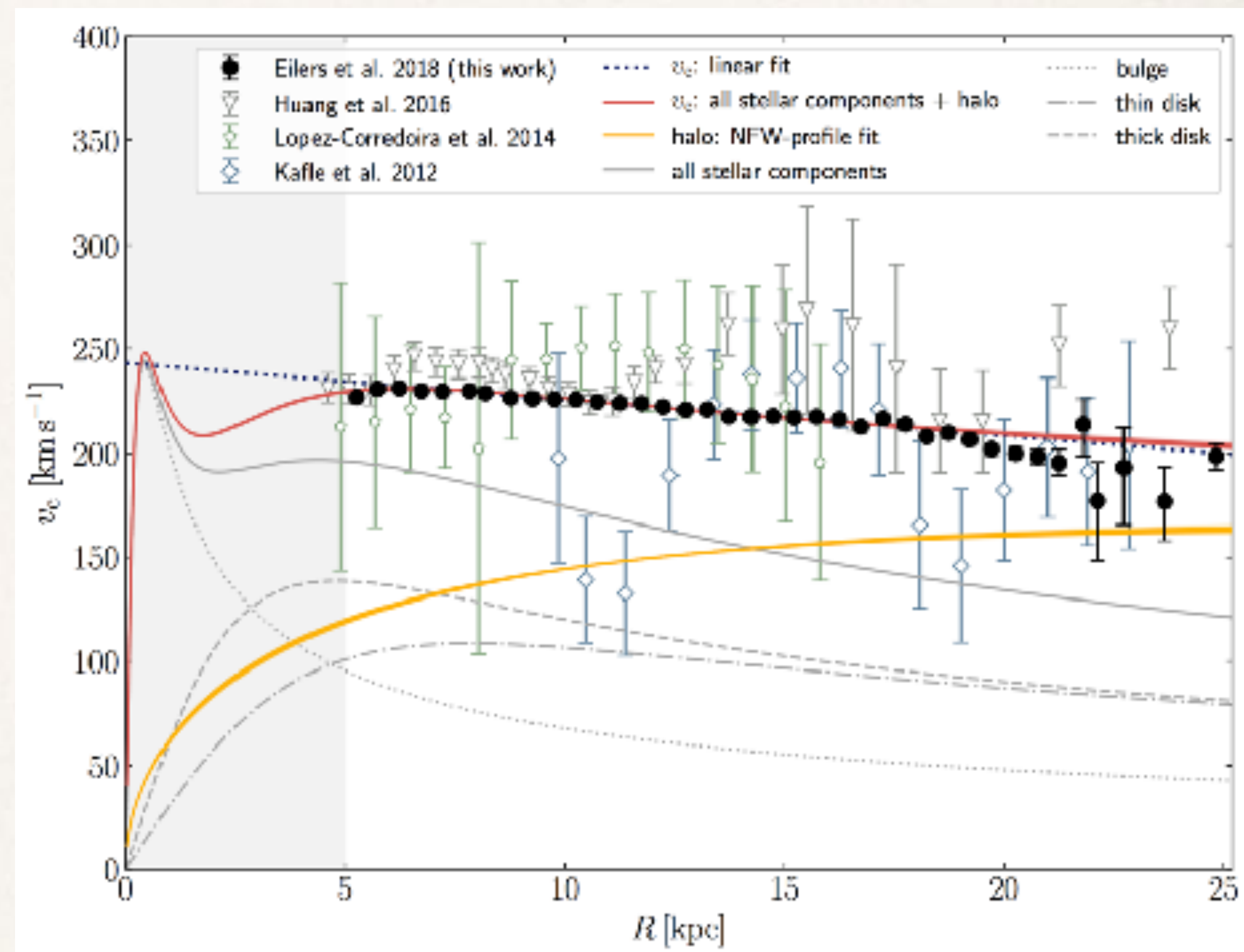
Milky Way Observables

- **Local stellar surface density**
- **Local gas surface density**
- **MW scale radius**
- **MW bulge mass**
- **MW rotation curve**
- **Slope of the rotation curve**
- **The vertical acceleration**

Analysis Procedure

Milky Way Observables

- Local stellar surface density
- Local gas surface density
- MW scale radius
- MW bulge mass
- **MW rotation curve**
- **Slope of the rotation curve**
- The vertical acceleration



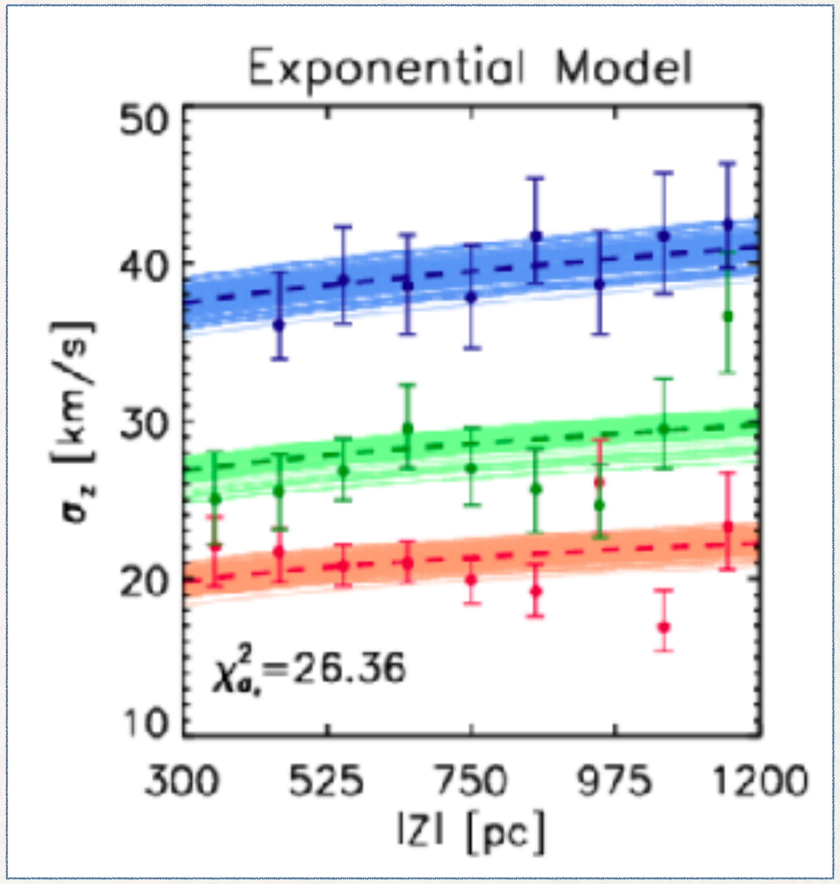
Eilers et. al., 2018

$$v_c(R) = \sqrt{R \cdot a(R)}\Big|_{z=0}$$

Analysis Procedure

Milky Way Observables

- Local stellar surface density
- Local gas surface density
- MW scale radius
- MW bulge mass
- MW rotation curve
- Slope of the rotation curve
- **The vertical acceleration**
  Inferred from 9000 K-dwarfs in the SEGUE sub-survey of the SDSS



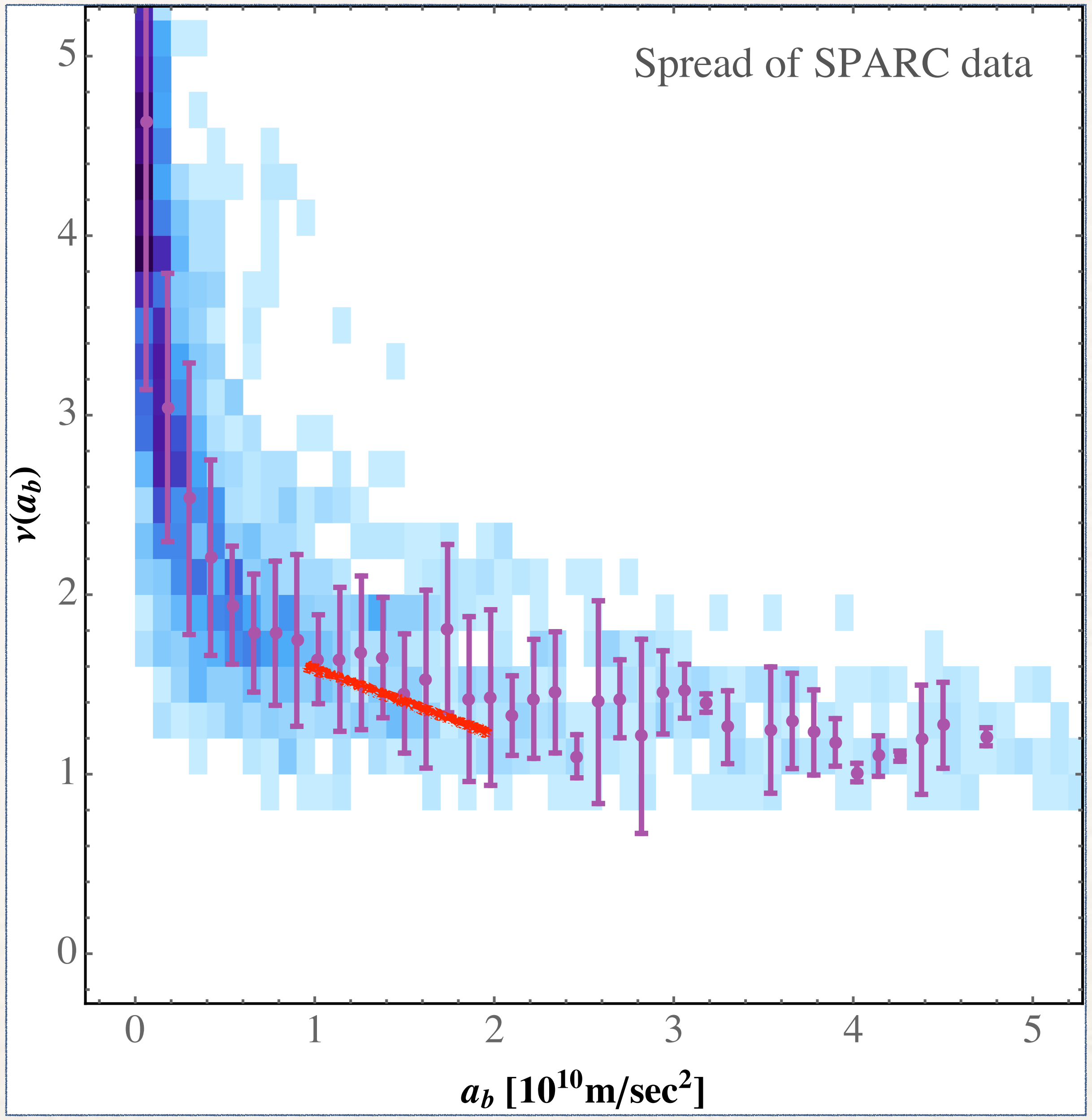
Zhang et. al., 2013

$$\sigma_{i,z}(z)^2 = \frac{n_i(0)\,\sigma_{i,z}(0)^2}{n_i(z)} + \frac{1}{n_i(z)} \int_0^z n_i(z')\,a_z(z')\,dz'$$

RESULTS

Results for any MOND-like Model

FIT ONLY LOCAL ROTATION CURVE

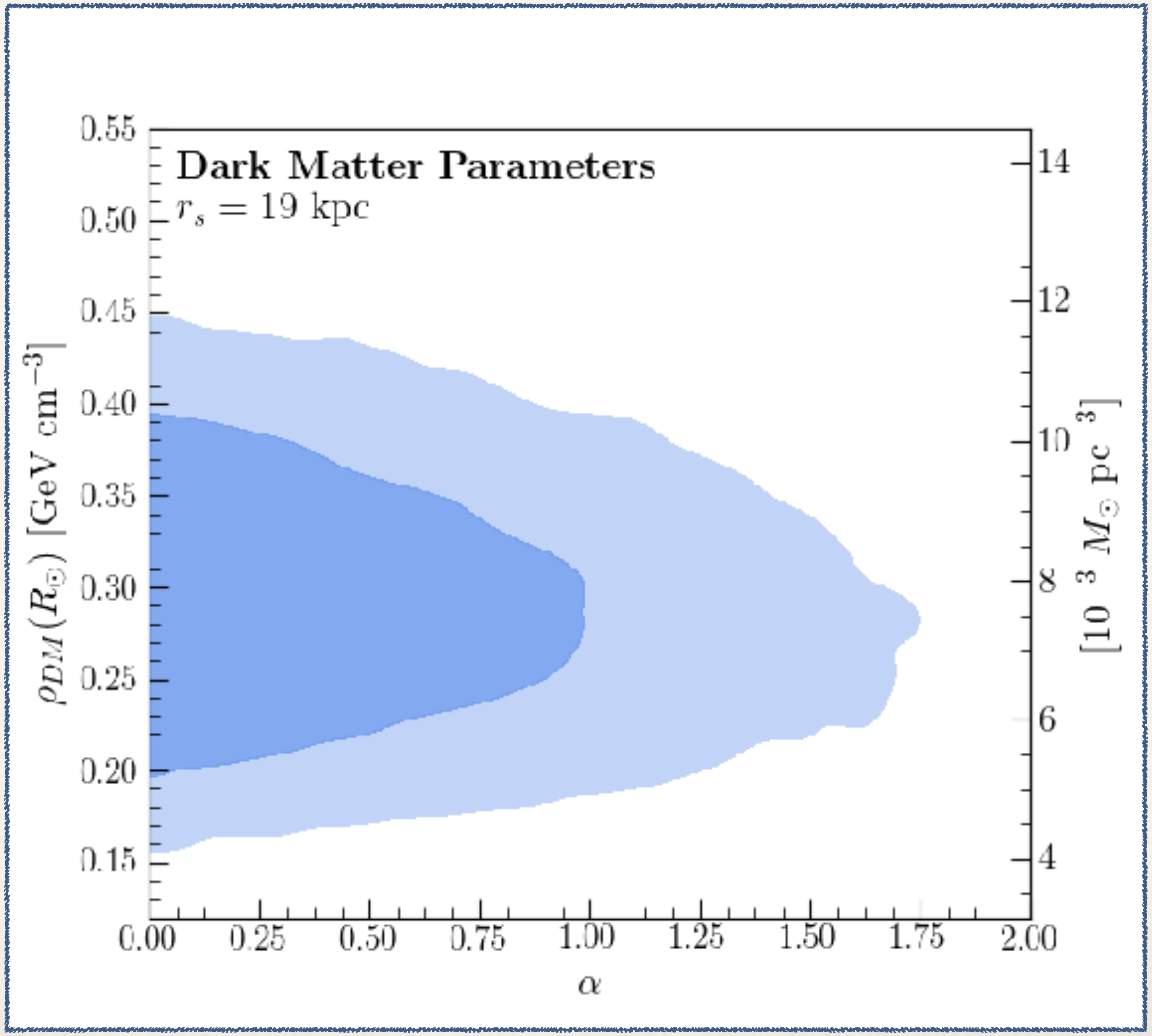


Lisanti, Moschella, Outmezguine, OS (PRELIMINARY)

$$\boldsymbol{a} = \nu\left(\frac{a_N}{a_0}\right)\boldsymbol{a}_N \rightarrow \mathbf{a} = (\nu_0 + \nu_1 a_N)\,\mathbf{a}_N$$

Results of MCMC Scans

Dark Matter Parameters



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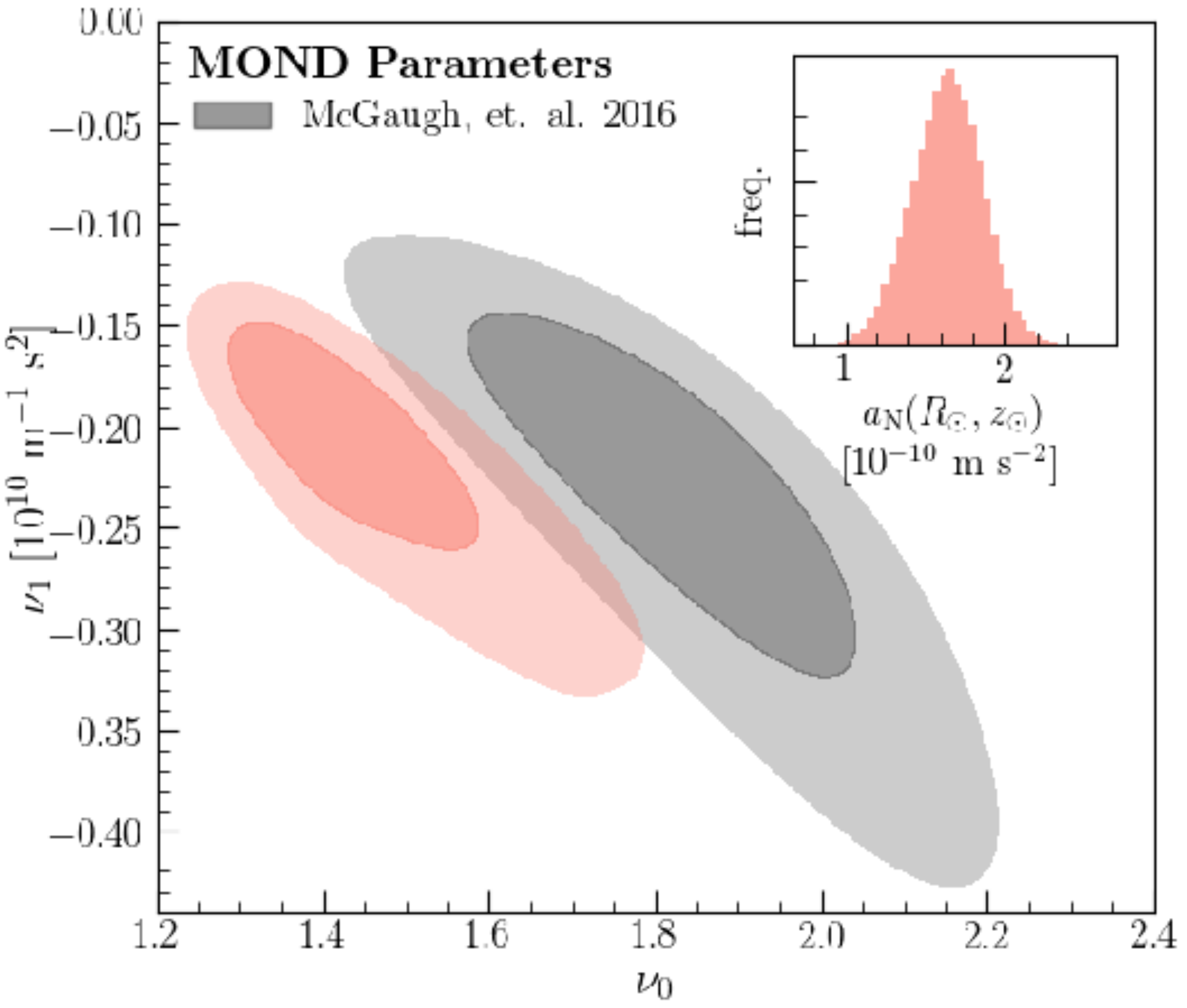
Results of MCMC Scans

MOND Parameters



Interpolation function fitted to RAR:

$$\nu(a_{\rm N}/a_0) = \frac{1}{1 - e^{-\sqrt{a_{\rm N}/a_0}}}$$

with

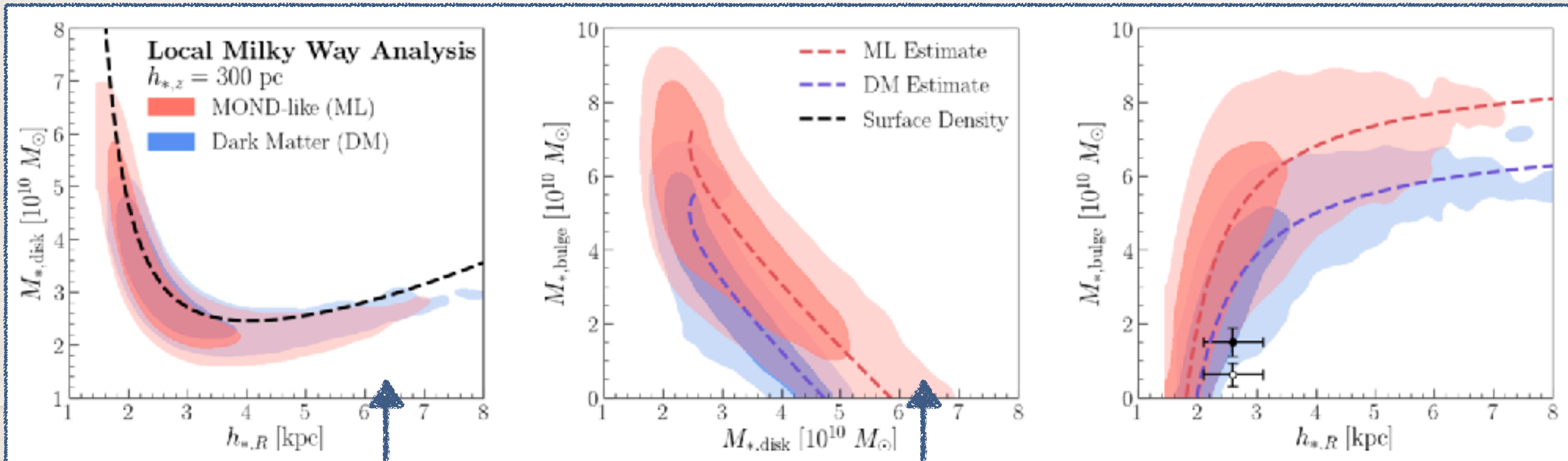
$$a_0 = 1.20 \pm 0.24 \times 10^{-10}\ \mathrm{m\ s^{-2}}$$

Excluded at 95% confidence

1812.08169 - Lisanti, Moschella, Outmezguine, O.S.

Results of MCMC Scans

Tension with MW Observations



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Driven by stellar surface density constraint

$$M_{*,\mathrm{disk}} = \frac{2\pi h_{*,R}^2 \Sigma_{*,\mathrm{obs}}^{z_{\max}} \exp(R_\odot/h_{*,R})}{1 - \exp(-z_{\max}/h_{*,z})}$$

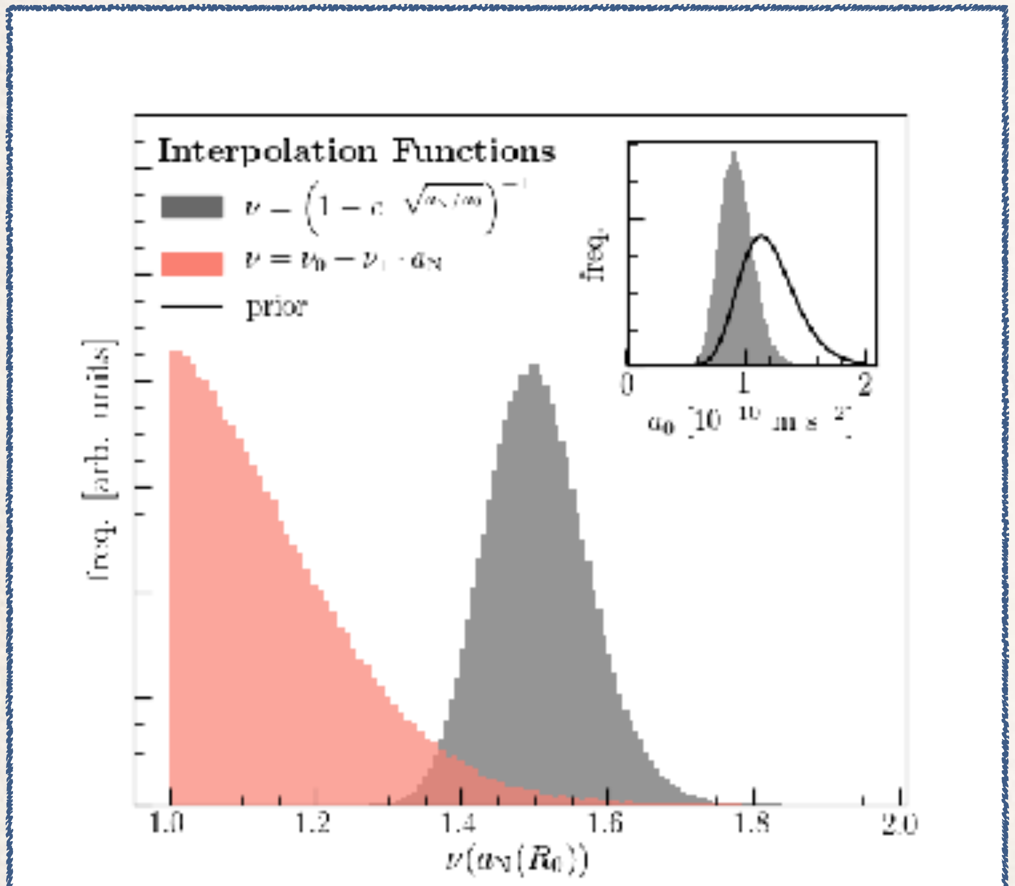
Driven by local value of rotation curve constraint

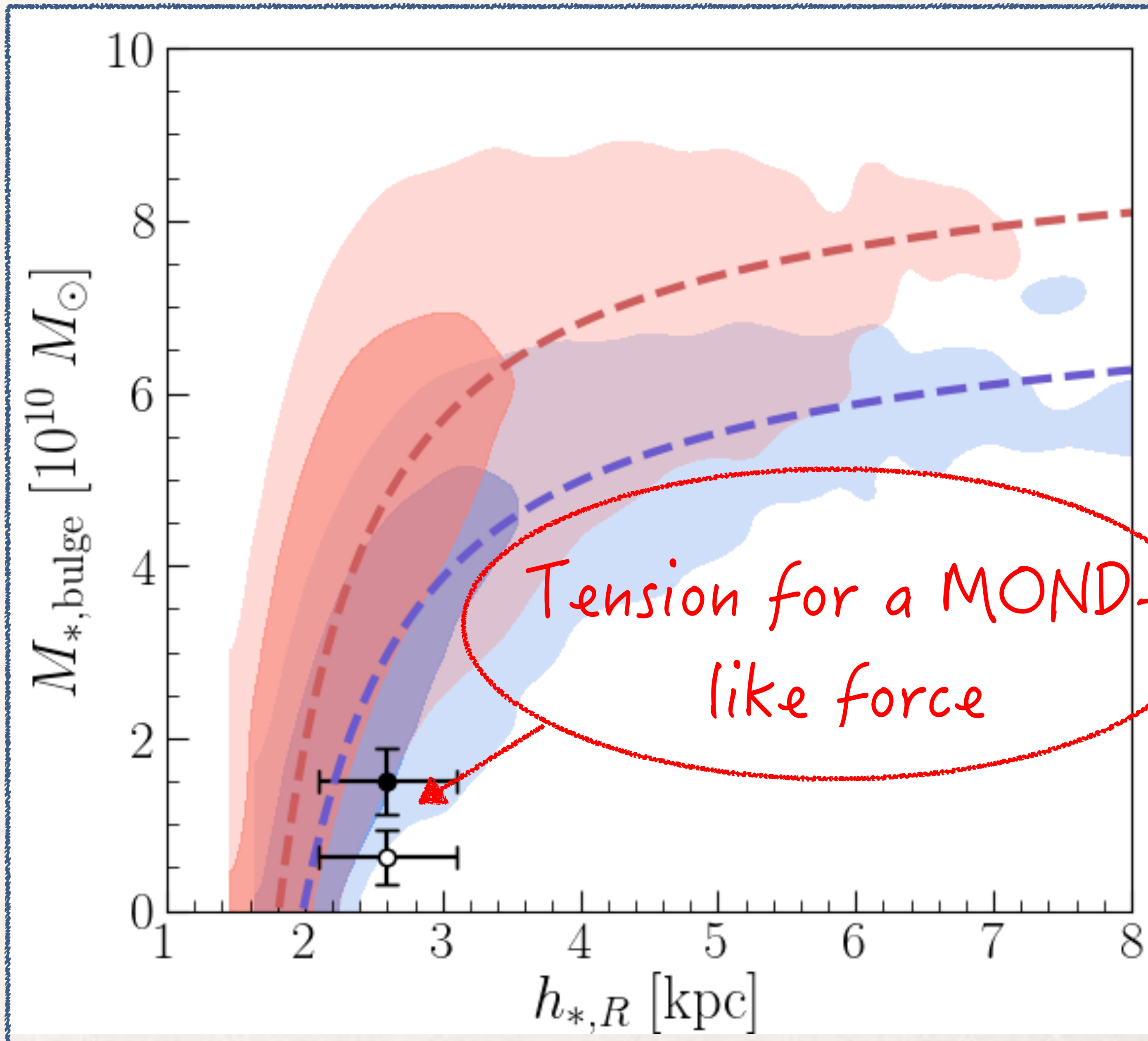
$$v_c(R) = \sqrt{R \cdot a(R)}\Big|_{z=0}$$

Results of MCMC Scans

Stellar Scale Radius *vs* Stellar Bulge Mass

Driven by local surface density and rotation curve





1812.08169 - Lisanti, Moschella, Outmezguine, O.S.

Results of MCMC Scans

Bulge Mass is Poorly Constrained

Reference	$M_\star^B \pm 1\sigma$ (10^{10} $M_\odot$)	R_0 assumed (kpc)	Constraint type	β[a]	$M_\star^B \pm 1\sigma (R_0 = 8.33\text{kpc})$ (10^{10} $M_\odot$)
Kent (1992)	1.69 ± 0.85	8.0	Dynamical	1	1.76 ± 0.88
Dwek et al. (1995)	2.11 ± 0.81	8.5	Photometric	2	2.02 ± 0.78
Han & Gould (1995)	1.69 ± 0.85	8.0	Dynamical	1	1.76 ± 0.88
Blum (1995)	2.63 ± 1.32	8.0	Dynamical	1	2.74 ± 1.37
Zhao (1996)	2.07 ± 1.03	8.0	Dynamical	1	2.15 ± 1.08
Bissantz et al. (1997)	0.81 ± 0.22	8.0	Microlensing	0	0.81 ± 0.22
Freudenreich (1998)[b]	0.48 ± 0.65	...	Photometric	...	0.48 ± 0.65
Dehnen & Binney (1998)	0.61 ± 0.38	8.0	Dynamical	1/2	0.62 ± 0.38
Sevenster et al. (1999)	1.60 ± 0.80	8.0	Dynamical	1	1.66 ± 0.83
Klypin et al. (2002)	0.94 ± 0.29	8.0	Dynamical	1	0.98 ± 0.31
Bissantz & Gerhard (2002)[c]	0.84 ± 0.09	8.0	Dynamical	1	0.87 ± 0.09
Han & Gould (2003)	1.20 ± 0.60	8.0	Microlensing	0	1.20 ± 0.60
Picaud & Robin (2004)	0.54 ± 1.11	8.5	Photometric	0	0.54 ± 1.11
Hamadache et al. (2006)	0.62 ± 0.31	None	Microlensing	0	0.62 ± 0.31
Wyse (2006)	1.00 ± 0.50	None	Historical review	0	1.00 ± 0.50
López-Corredoira et al. (2007)	0.60 ± 0.30	8.0	Photometric	2	0.65 ± 0.33
Calchi Novati et al. (2008)	1.50 ± 0.38	8.0	Microlensing	0	1.50 ± 0.38
Widrow et al. (2008)	0.90 ± 0.11	7.94	Dynamical	1	0.95 ± 0.12

Bland-Hawthorn, Gerhard (2016), Licquia, Newman (2015)

Conservative Range: $0 < M_{*,\text{bulge}} < 2 \times 10^{10} M_\odot$

Reference Value: $M_{*,\text{bulge}} = 1.50 \pm 0.38 \times 10^{10} M_\odot$

Results of MCMC Scans

Comparison between the Theories

Naming Convention	Functional Form	Prior for Scan	ΔBIC
Taylor Expansion	$\nu(a_{\rm N}) = \nu_0 + \nu_1 a_{\rm N}$	$\nu(a_{\rm N}) > 1$ or 1.3	4.1 or 7.5
RAR [7]	$\nu(a_{\rm N}) = \left(1 - e^{-\sqrt{a_{\rm N}/a_0}}\right)^{-1}$	$a_0 = \text{LOGNORMAL}\,(1.20, 0.24^2)$	10.4
Simple [27, 52]	$\nu(a_{\rm N}) = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{a_{\rm N}/a_0}}\right)$	$a_0 = \text{LOGNORMAL}\,(1.2, 0.4^2)$	9.6
Standard [27, 52]	$\nu(a_{\rm N}) = \sqrt{\frac{1}{2}\left(1 + \sqrt{1 + \left(\frac{2}{a_{\rm N}/a_0}\right)^2}\right)}$	$a_0 = \text{LOGNORMAL}\,(1.2, 0.4^2)$	4.8

Bayesian Information Criterion:
(a proxy for the Bayes Evidence)

$$\text{B.I.C.} = k \log n - 2 \log \hat{\mathcal{L}}$$

k: number of model parameters
n: number of data points
$\hat{\mathcal{L}}$: maximum likelihood

Results for SuperFluid DM

$$\mathcal{L}_{\rm DM,\,T=0} = \frac{2\Lambda(2m)^{3/2}}{3} X\sqrt{|X|} \qquad \mathcal{L}_{\rm int} = \alpha\Lambda\frac{\phi}{M_{\rm Pl}}\rho_{\rm b}$$

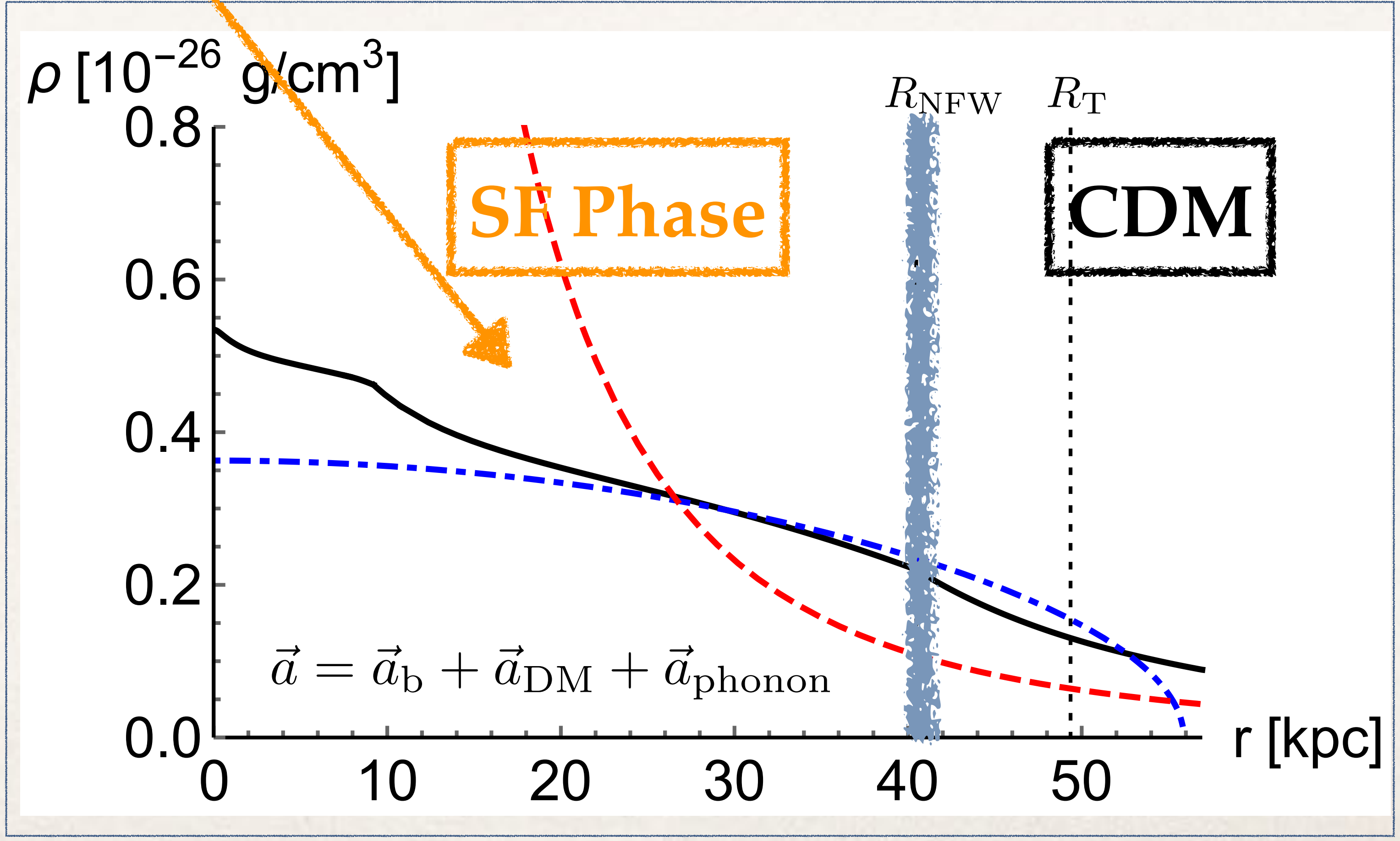
$$X = \mu - m\Phi + \dot{\phi} - (\vec{\nabla}\phi)^2/2m$$

E.O.M. for ϕ

↓

Interpolation function

$$\rho_{\rm SF} = \frac{\partial \mathcal{L}}{\partial \Phi}$$



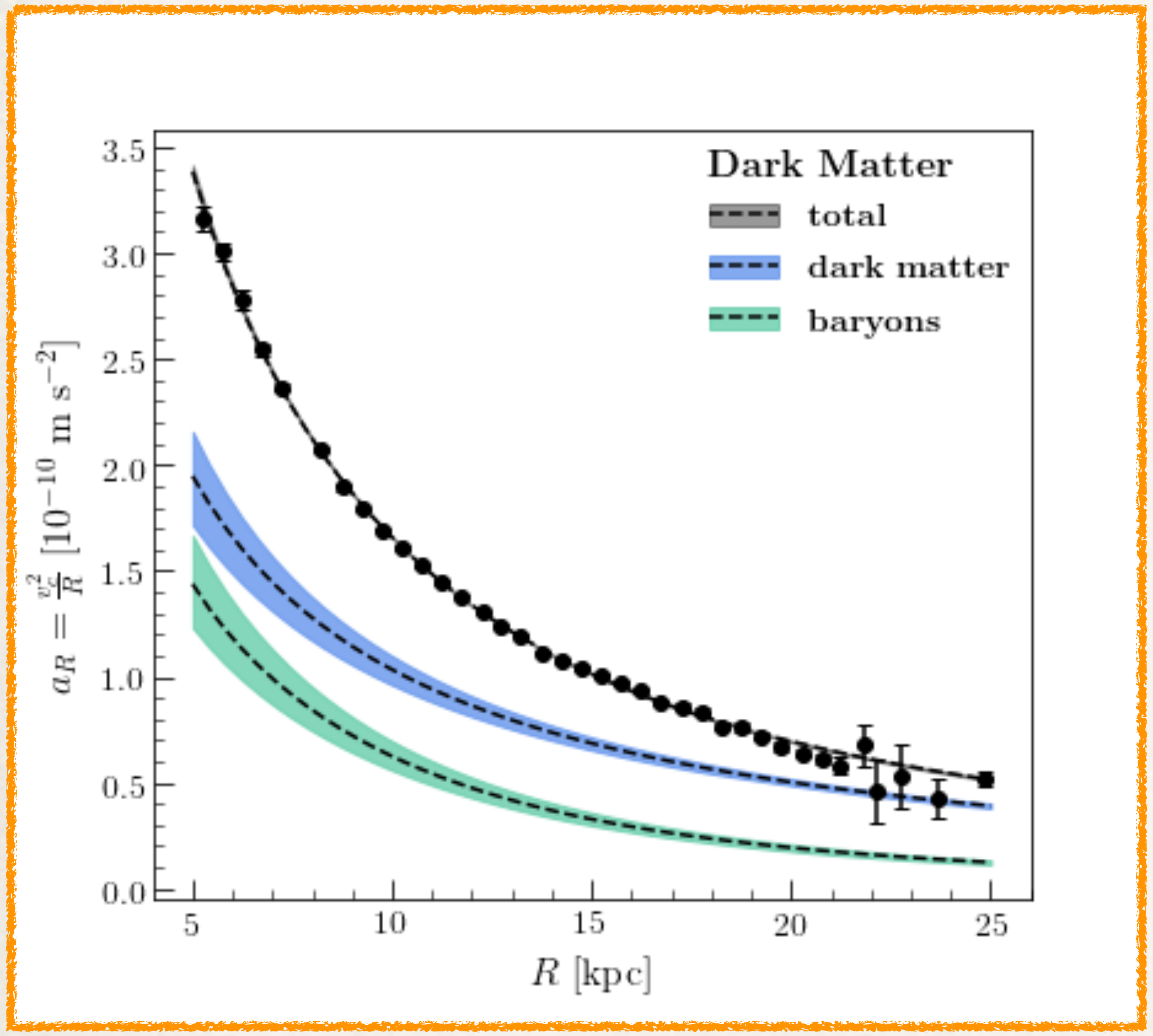
$$a_0 = \frac{\alpha^3\Lambda^2}{M_{\rm Pl}}$$

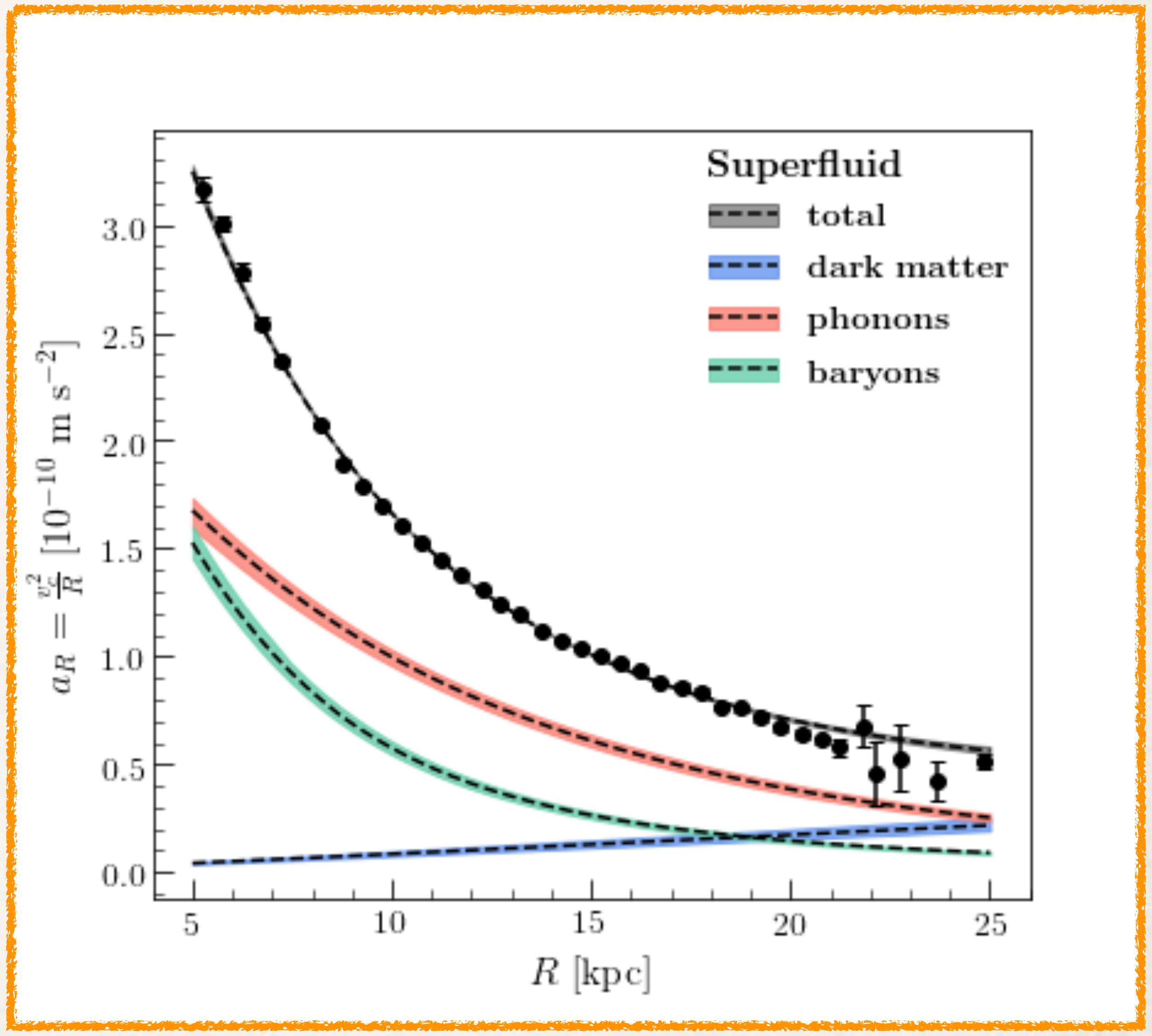
FIT FULL ROTATION CURVE

Berezhiani, Famaey, Khoury, 2017

Results for SuperFluid DM

Rotation Curve

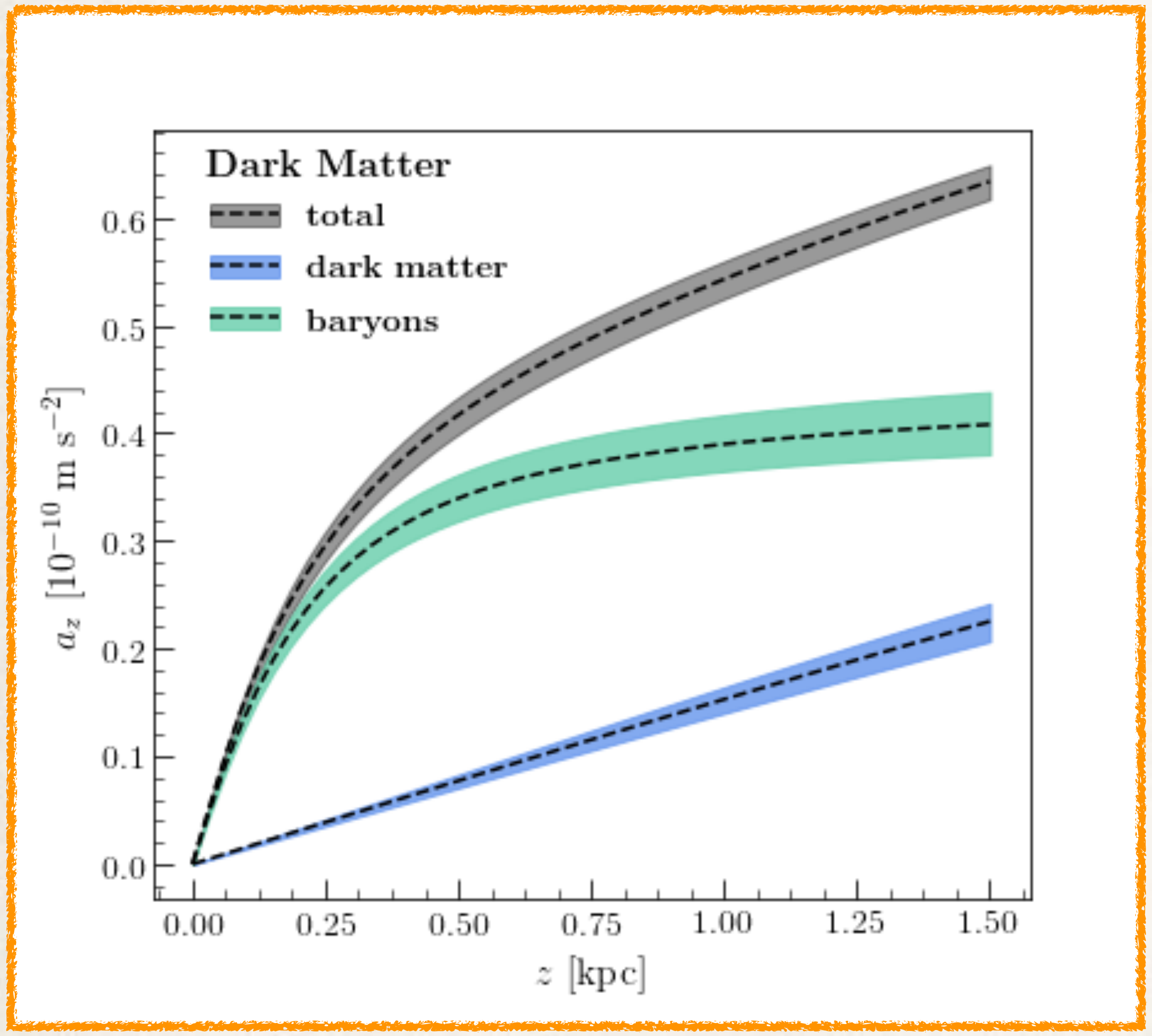


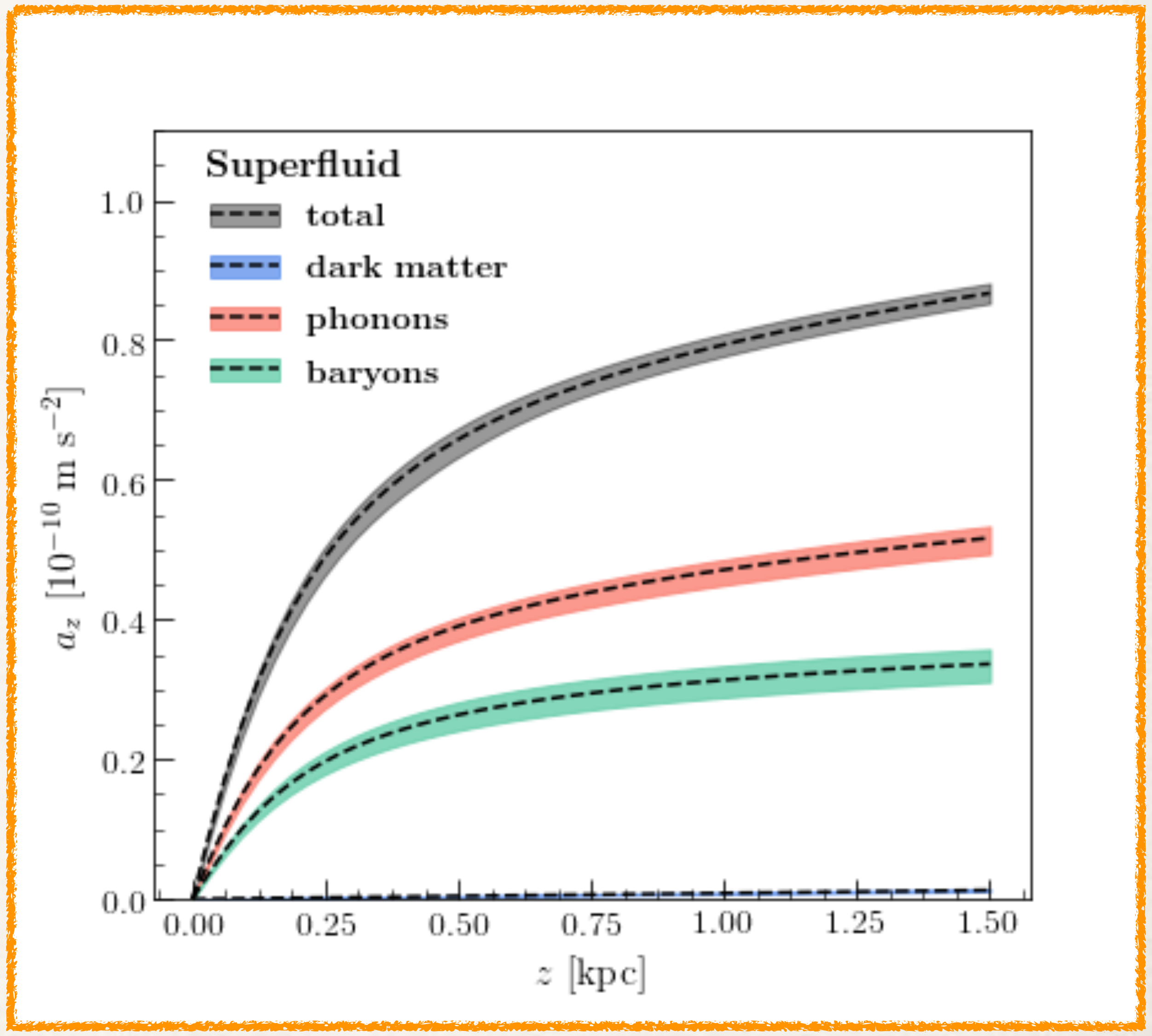


Lisanti, Moschella, Outmezguine, O.S. (PRELIMINARY)

Results for SuperFluid DM

Vertical Acceleration

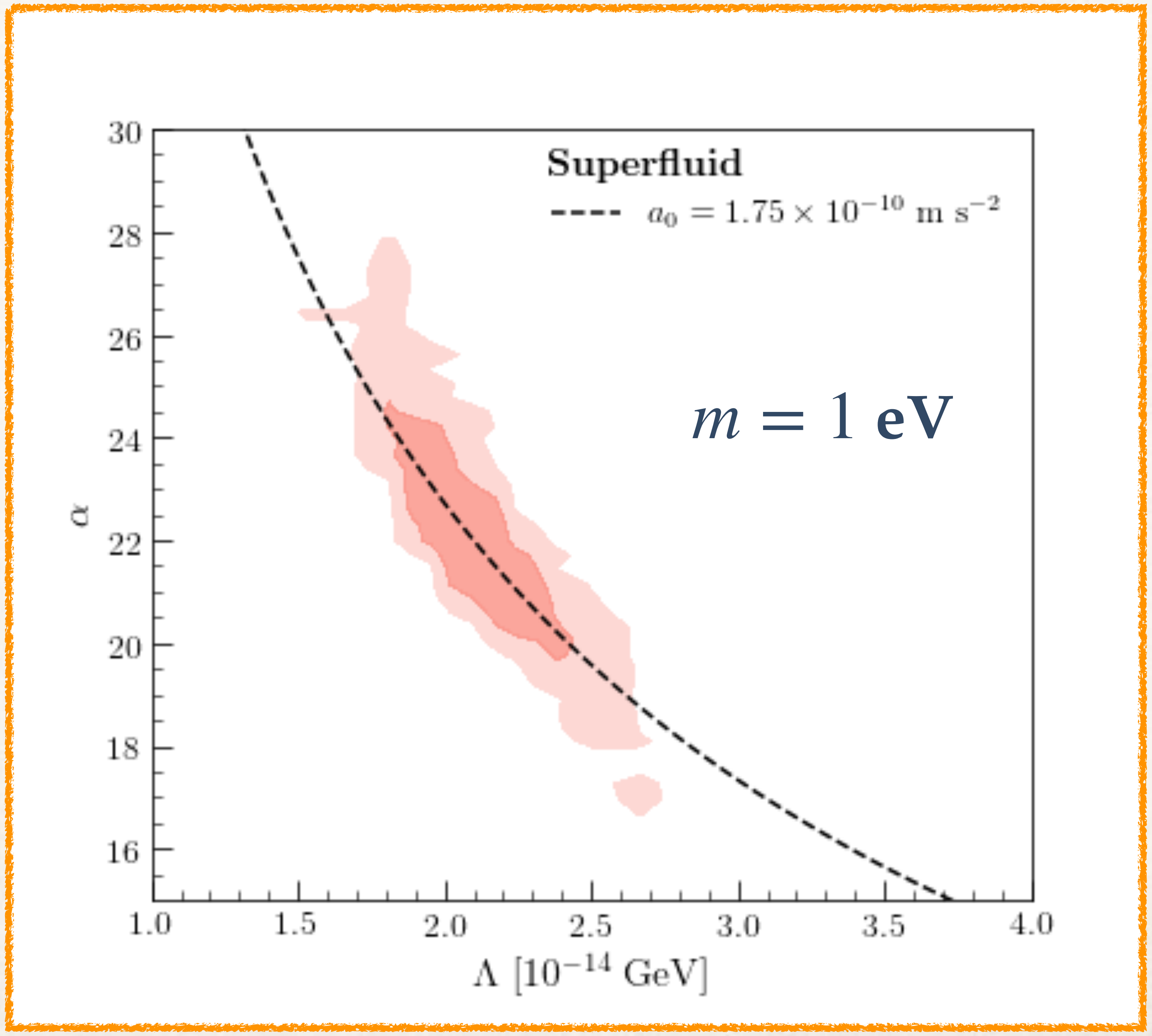




Lisanti, Moschella, Outmezguine, O.S. (PRELIMINARY)

Results for SuperFluid DM

Super Fluid Model Parameters

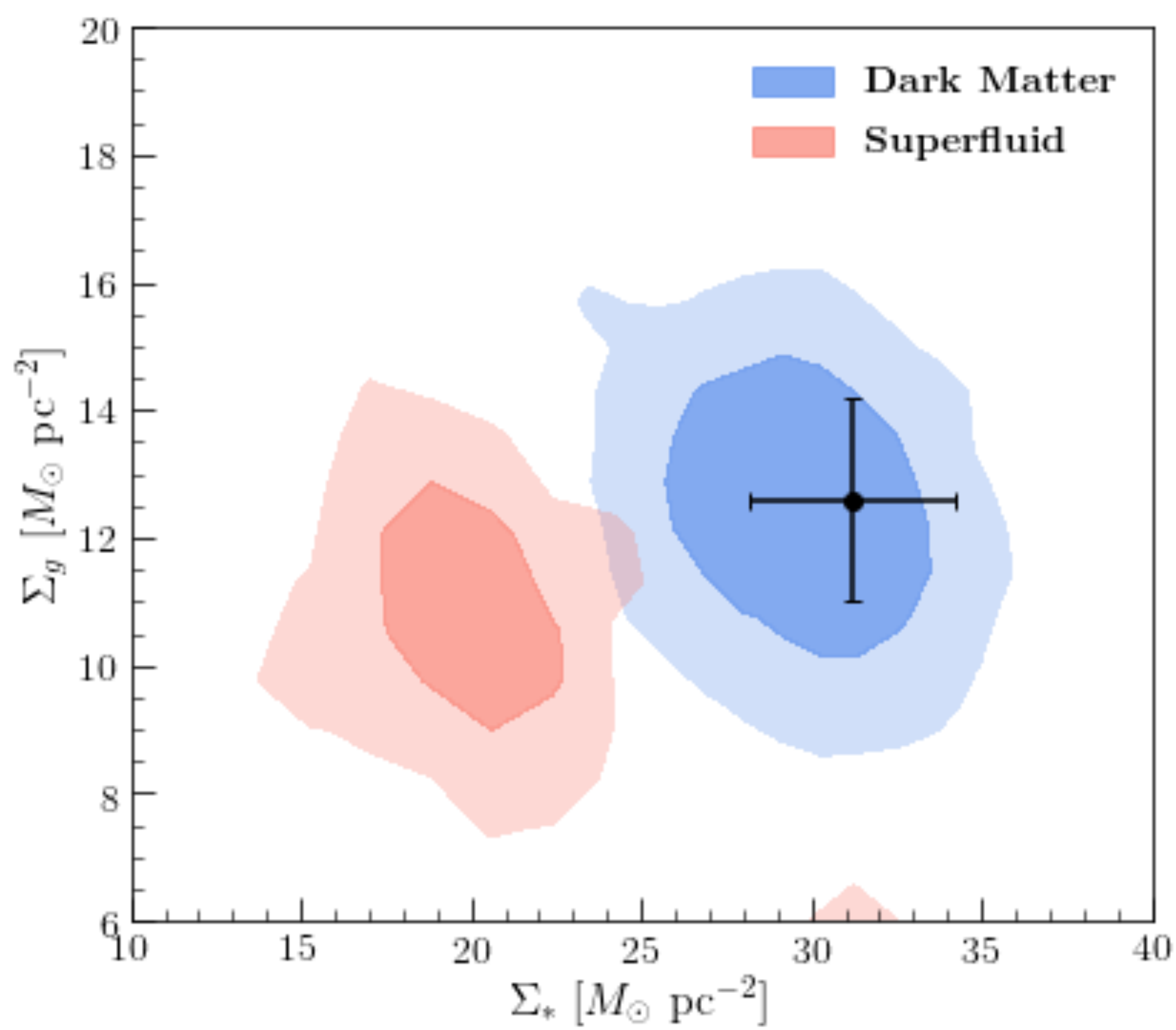


$$a_0 = \frac{\alpha^3 \Lambda^2}{M_{\rm Pl}}$$

Lisanti, Moschella, Outmezguine, O.S. (PRELIMINARY)

Results for SuperFluid DM

Surface Density and ΔBIC



Lisanti, Moschella, Outmezguine, O.S. (PRELIMINARY)

χ^2	DM	SF
vcirc_gaia	54.5	89.8
hr	1.95	0.65
mbulge	0.04	6.56
sden_stars	0.10	13.5
sden_gas	0.14	0.74
sigz_poor	1.78	4.96
sigz_rich	12.0	18.9
sigz_intermediate	8.59	20.1
lognu_poor	11.9	23.7
lognu_rich	2.23	8.82
lognu_intermediate	7.21	20.2
total	100.5	207.9

ΔBIC > 60

Summary of the Results

- Local accelerations only ✔
- Unconstrained I.F. ✔

ΔBIC ≈ 4

POSITIVE EVIDENCE (with $\nu \approx 1$)

- Local accelerations only ✔
- Unconstrained I.F. ✘

ΔBIC ≈ 10

STRONG EVIDENCE

- Local accelerations only ✘
- Unconstrained I.F. ✘

ΔBIC ≈ 60

VERY STRONG EVIDENCE

Outlook

Future Work

- Extend analysis to other theories for which baryons predict accelerations, e.g.:
  - Strongly Interacting DM (Famaey, Khoury & Penco, 2018)
  - Emergent Gravity (Verlinde, 2016)
  - SIDM (Kamada, Kaplinghat, Pace and Yu, 2017)
- Extend the analysis to more precise data-sets (Gaia)

Conclusions

- Standard lore is “MOND-like forces work on Galactic Scales”. This is not precisely true.
- Local MW measurements seem to prefer a Dark Matter theory over a scalar enhancement of gravity (e.g. MOND or Superfluid DM).
- Better measurements will make this statement more precise.

A strictly MOND-like force has trouble simultaneously explaining rotation curves and velocity dispersions… so, probably something else

THANK YOU

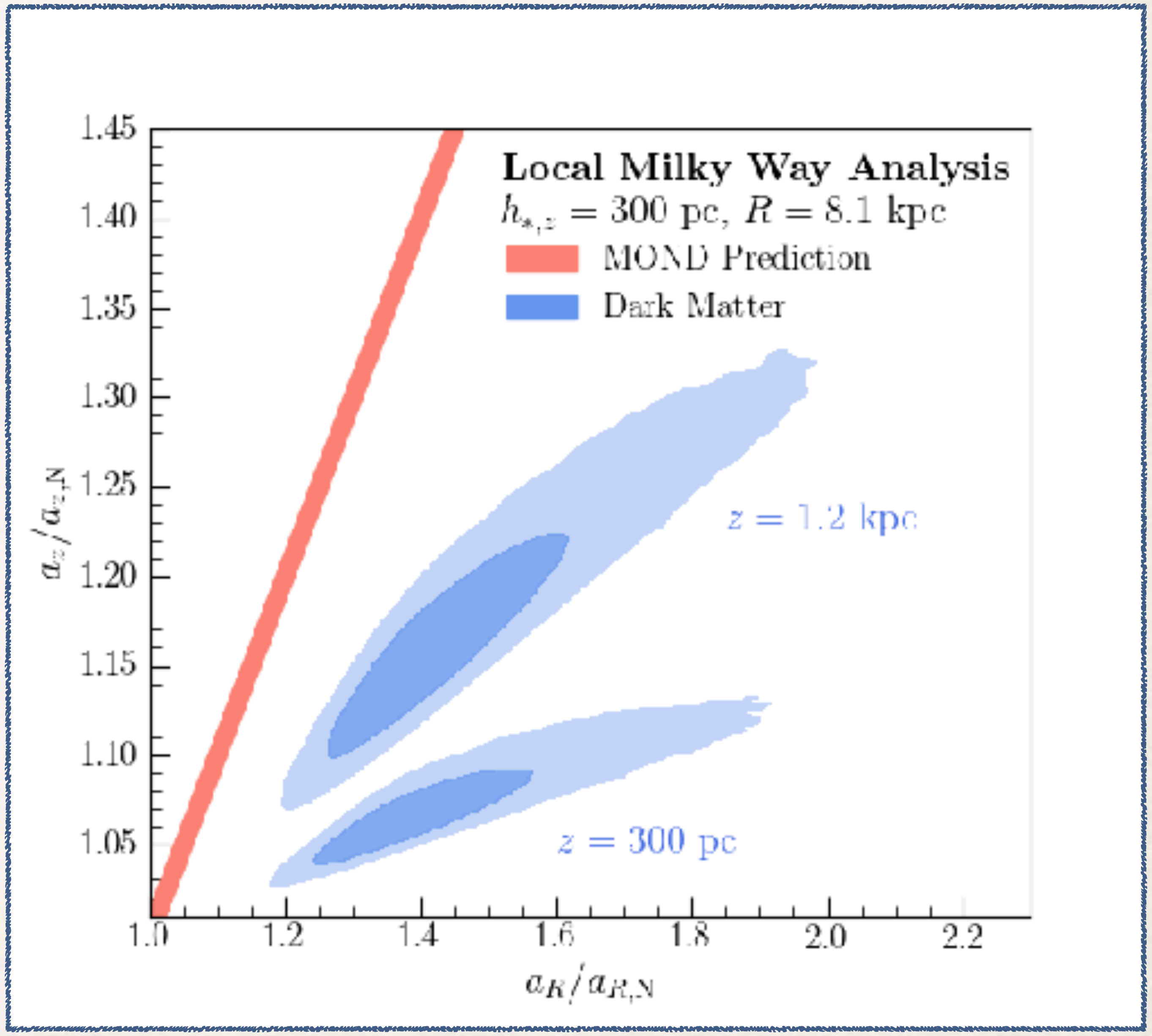
Results of MCMC Scans

Tension between models for **<u>any</u>** Scalar Enhancement

Each axis is the local enhancement of acceleration in the R/z directions

or

an independent measurement of the local value of the interpolation function



1812.08169 - Lisanti, Moschella, Outmezguine, O.S.

Some general comments (and more on MOND-like forces)

Some Comments

- Could be done for any model where dynamics are predicted locally by baryons

- The starting point could have been something of the form:

Example of a MONDian Poisson Equation

$$\nabla \left(\mu \left(\frac{|\nabla \Phi|}{a_0} \right) \nabla \Phi \right) = 4\pi G \rho \quad \rightarrow \quad \Phi \propto \log r$$

Inverse of interp. func.

- This equation is non-linear and difficult to calculate

- Is VERY model dependent

- Starting from an acceleration relation can map onto other theories

MOND / Superfluid DM
Non-Linear Effects

- Non-linear effects must be accounted for!
- Potential problems include:
  - A possible non-trivial correction to the acceleration relation.
  - Small perturbations to a smooth potential can cause large effects.

MOND / Superfluid DM
A Divergenceless Field

Poisson Equation:

$$\nabla\left(\nabla\Phi_{\mathrm{N}}\right) = 4\pi G\rho$$

MONDian Poisson Equation:

$$\Phi \propto \log r$$

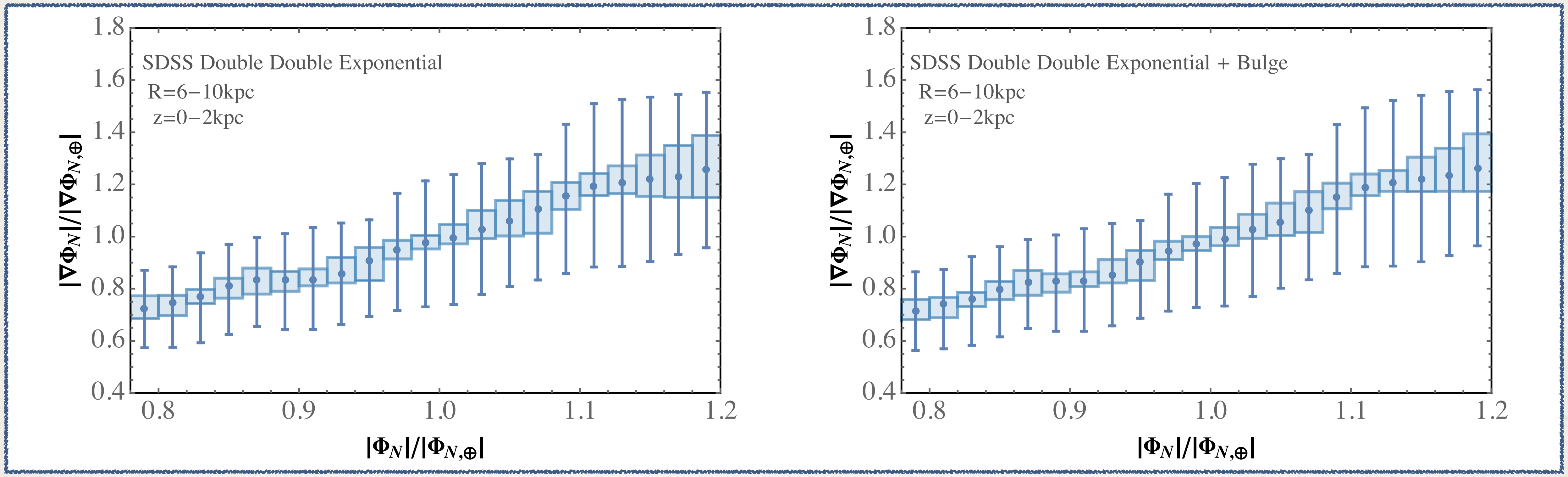
$$\nabla\left(\mu\left(\frac{|\nabla\Phi|}{a_0}\right)\nabla\Phi\right) = 4\pi G\rho$$

Inverse of

Acceleration Relation known up to a divergenceless field:

$$\boldsymbol{a} = \nu\left(\frac{a_{\mathrm{N}}}{a_0}\right)\boldsymbol{a}_{\mathrm{N}} + \boldsymbol{S}$$

MOND
A Divergenceless Field

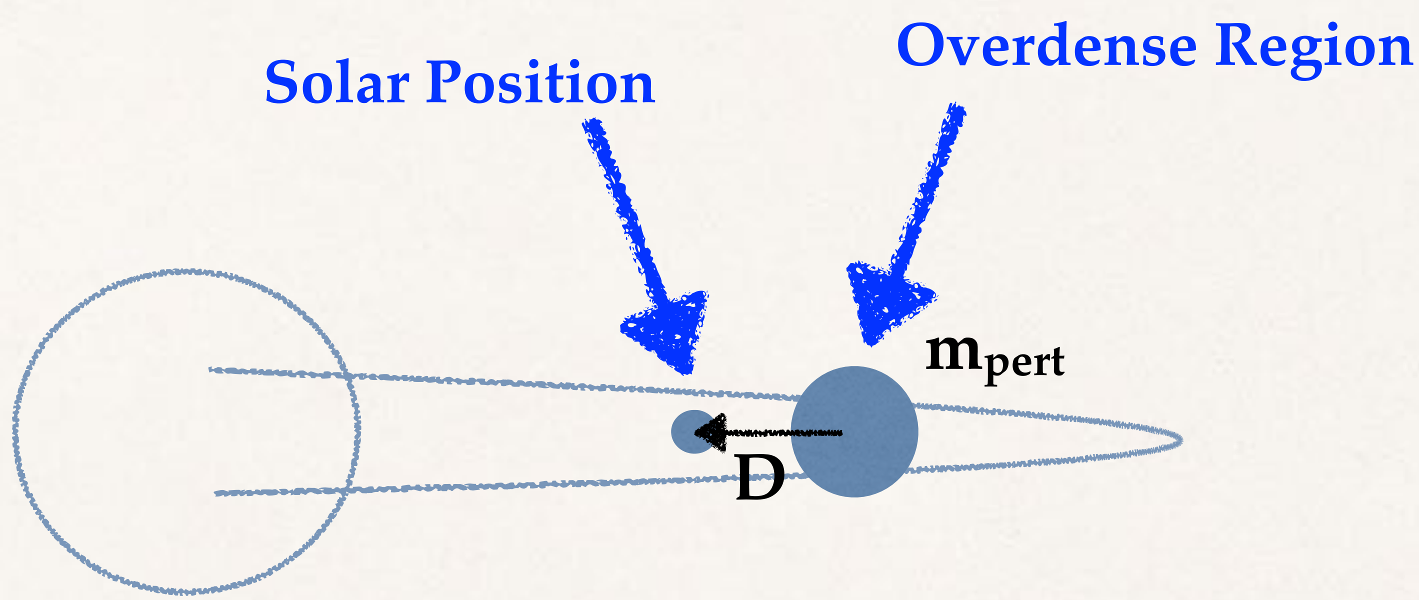


Can be shown that S=0 for 1D symmetrical potentials, or:

$$\nabla|\nabla\Phi_N| \times \nabla\Phi_N = 0$$

$$|\nabla\Phi_N| = f(\Phi_N)$$

MOND / Superfluid DM
Small Perturbations



The External Field Effect (EFE) is small as long as:

$$D \gg 0.1\ \text{kpc} \times \left[\nu\left(\frac{a_{\rm N,BG}}{a_0}\right) \cdot \frac{m_{\rm pert}}{10^7\, M_\odot} \cdot \frac{2\cdot 10^{-10}\ \text{m/s}^2}{a_{\rm loc}} \right]^{1/2}$$

$$a_{\rm loc} = \frac{v_c^2}{R_0} \approx 2\cdot 10^{-10}\ \text{m/s}^2.$$

MOND
So for a local MW study:

Using $$\boldsymbol{a} = \nu\left(\frac{a_{\mathrm{N}}}{a_0}\right)\boldsymbol{a}_{\mathrm{N}}$$

with $$\nu(x_{\mathrm{N}}) \to \nu_0 + \nu_1 \cdot x_{\mathrm{N}}$$

- A good local approximation.
- Holds for many MOND-like theories.
- Independent of specific interpolation function.