Relativistic Hydrodynamics I

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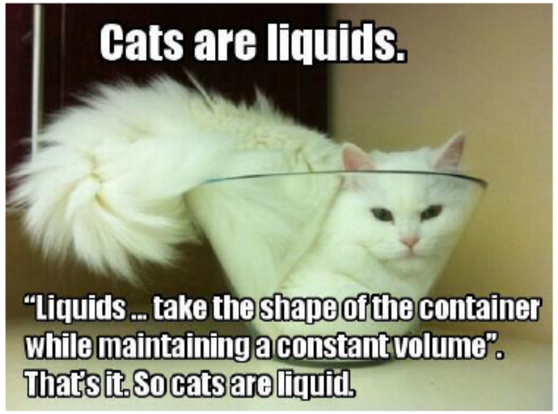
- Traditional and modern concepts of hydrodynamics
- Global vs. local equilibrium, perfect fluid
- Relativistic Navier-Stokes hydrodynamics
  Eckart 1940 Landau and Lifshitz 1959
- Israel and Stewart, 1970-1979,
  BRSSS 2008,
  other kinetic-theory formulations ~2010
- Insights from kinetic theory
- Anisotropic hydrodynamics 2010
- Gradient expansion
- Hydrodynamics with spin
- first-order causal and stable hydrodynamics, 2020
- Summary & conclusions

Basic hydrodynamic concepts

- **hydrodynamics deals with liquids in motion**, it is a subdiscipline of fluid mechanics (fluid dynamics) which deals with both liquids and gases
- liquids, gases, solids and plasmas are states of matter, characterised locally by macroscopic quantities, such as energy density, temperature or pressure
- **states of matter differ typically by compressibility and rigidity**
  liquids are less compressible than gases, solids are more rigid than liquids
  a typical liquid conforms to the shape of its container but retains a (nearly) constant volume independent of pressure

- a natural explanation of different properties of liquids, gases, solids and plasmas is achieved within atomic theory of matter
- **hydrodynamics, similarly to thermodynamics, may be formulated without explicit reference to microscopic degrees of freedom**
- this is important if we deal with strongly interacting matter — in this case neither hadronic nor partonic degrees of freedom seem to be adequate degrees of freedom



Physicist Wins Ig Noble Prize For Study On Whether Cats Should Be Classified As Liquids Or Solids

November 17, 2018

A French physicist has won an Ig Nobel Prize for using mathematical formulas to determine whether cats are liquid or solid.

Photo Credit: @cakes1todough1

The Ig Nobel prizes are awarded every year by Improbable Research, an organization devoted to science and humor. The goal is to highlight scientific studies that first make people laugh, then think.

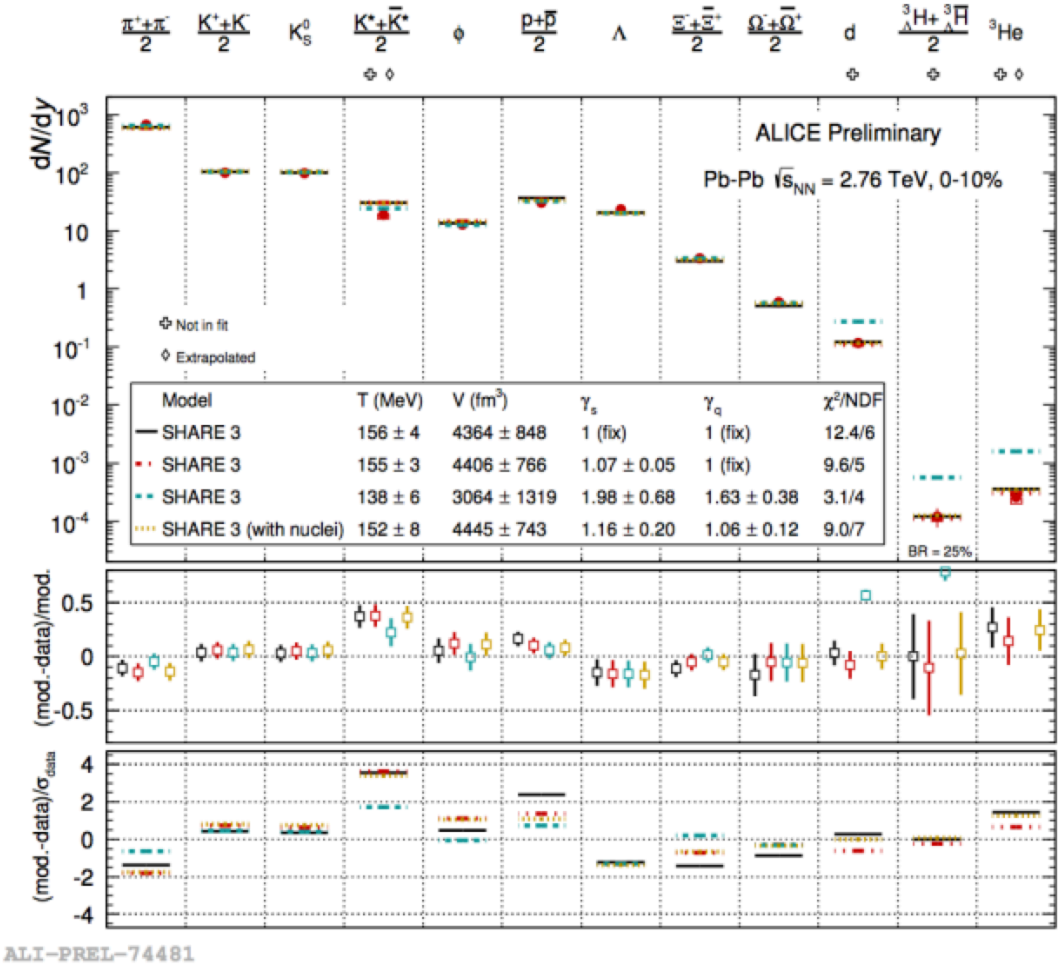
modern POV:

- **genuine hydrodynamic behaviour is a property of physical systems evolving toward equilibrium**
  1) one separates between transient (nonhydrodynamic) and slowly decaying (hydrodynamic) modes, 2) the latter are connected with real hydrodynamic behaviour, 3) typical modern hydrodynamic equations include both of them
- **hydrodynamics (set of hydrodynamic equations) may be formulated without explicit reference to microscopic degrees of freedom**
  1) this is important if we deal with strongly interacting matter —
  in this case neither hadronic nor partonic degrees of freedom seem to be adequate degrees of freedom, 2) such a general formulation of hydrodynamics may be limited – based on the gradient expansion, which does not converge
- **hydrodynamics (set of hydrodynamic equations) may be also constructed in a direct relation to some underlying, microscopic theory**
  1) the most common approaches refer to kinetic theory, 2) new developments based in the AdS/CFT correspondance

- hydrodynamic equations describe the space-time evolution of the energy-momentum tensor components, $T^{\mu\nu}$, seems to be a limited knowledge but ...
- the information about the state of matter is, to large extent, encoded in the structure of its energy-momentum tensor

equation of state, kinetic coefficients (viscosities), …

Thermal fit to hadron multiplicity ratios

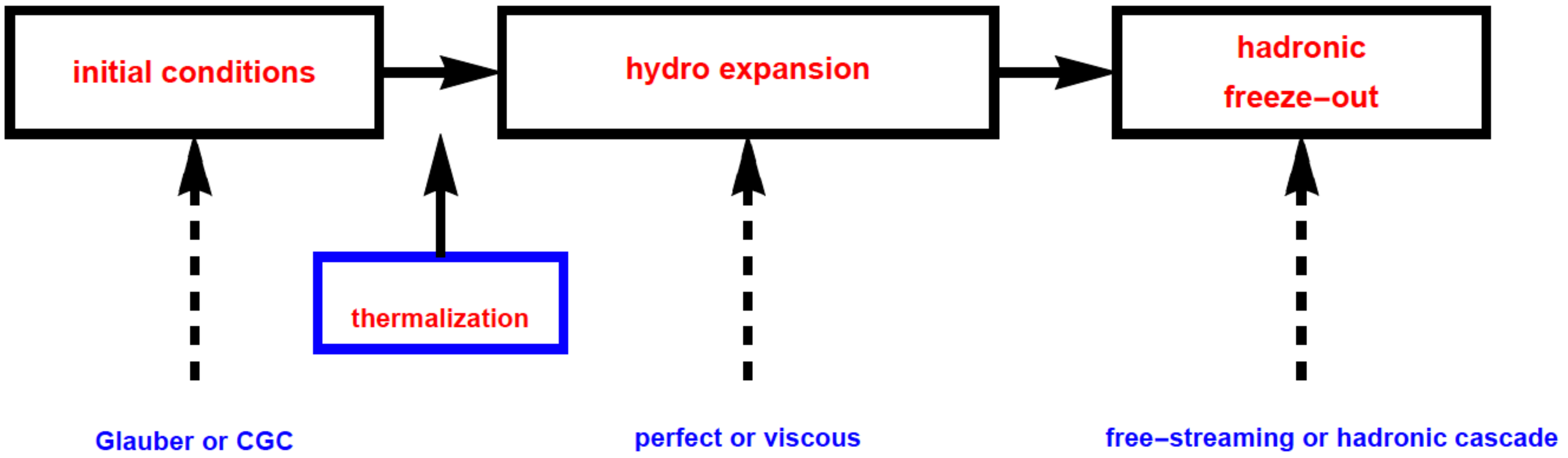


M. Floris, Nucl. Phys. A931 (2014) c103

F. Becattini, P. Braun-Munzinger, W. Broniowski, J. Cleymans, WF, M. Gaździcki, J. Rafelski, H. Satz, J. Stachel, ….

status quo ante, Karpacz School in 2012

STANDARD MODEL (MODULES) of HEAVY–ION COLLISIONS



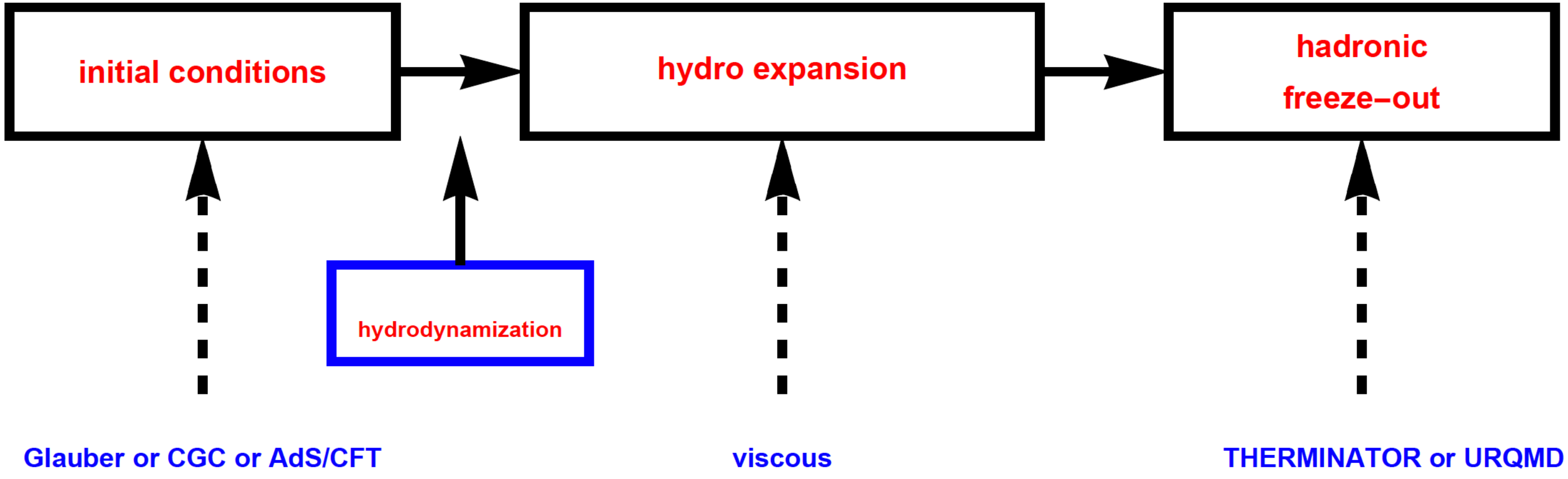
NEW: FLUCTUATIONS IN THE INITIAL STATE / EVENT–BY–EVENT HYDRO / FINAL–STATE FLUCTUATIONS

EQUATION OF STATE?

VISCOSITY?

status quo

STANDARD MODEL (MODULES) of HEAVY–ION COLLISIONS



FLUCTUATIONS IN THE INITIAL STATE / EVENT–BY–EVENT HYDRO / FINAL–STATE FLUCTUATIONS

EQUATION OF STATE = lattice QCD

1 < VISCOSITY < 3 times the lower bound

Global equilibrium

The equilibrium energy-momentum tensor in the **fluid rest-frame** is given by

$$T^{\mu\nu}_{\mathrm{EQ}} = \begin{vmatrix} \mathcal{E}_{\mathrm{EQ}} & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathcal{E}_{\mathrm{EQ}}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathcal{E}_{\mathrm{EQ}}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathcal{E}_{\mathrm{EQ}}) \end{vmatrix} \tag{1}$$

assumption: the equation of state is known, so that the pressure $\mathcal{P}$ is a given function of the energy density $\mathcal{E}_{\mathrm{EQ}}$

in an arbitrary frame of reference

$$T^{\mu\nu}_{\mathrm{EQ}} = \mathcal{E}_{\mathrm{EQ}} u^\mu u^\nu - \mathcal{P}(\mathcal{E}_{\mathrm{EQ}})\Delta^{\mu\nu}, \tag{2}$$

where u^μ is a constant velocity, and $\Delta^{\mu\nu}$ is the operator that projects on the space orthogonal to u^μ, namely

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad \Delta^{\mu\nu} u_\nu = 0. \tag{3}$$

Local equilibrium – perfect fluid

The energy-momentum tensor of a perfect fluid is obtained by allowing the variables $\mathcal{E}$ and u^μ to depend on the spacetime point x

$$T^{\mu\nu}_{\rm eq}(x) = \mathcal{E}(x)u^\mu(x)u^\nu(x) - \mathcal{P}(\mathcal{E}(x))\Delta^{\mu\nu}(x) \tag{4}$$

the subscript “eq” refers to local thermal equilibrium.

local effective temperature $T(x)$ is determined by the condition that the equilibrium energy density at this temperature agrees with the non-equilibrium value of the energy density, namely

$$\mathcal{E}_{\rm EQ}(T(x)) = \mathcal{E}_{\rm eq}(x) = \mathcal{E}(x) \tag{5}$$

Perfect fluid

$T(x)$ and $u^\mu(x)$ are fundamental fluid/hydrodynamic variables

the relativistic perfect-fluid energy-momentum tensor is the most general symmetric tensor which can be expressed in terms of these variables without using derivatives.

dynamics of the perfect fluid theory is provided by the conservation equations of the energy-momentum tensor

$$\partial_\mu T^{\mu\nu}_{\rm eq} = 0 \tag{6}$$

four equations for the four independent hydrodynamic fields – a self-consistent (hydrodynamic) theory

DISSIPATION DOES NOT APPEAR!

$$\partial_\mu (\mathcal{S} u^\mu) = 0 \tag{7}$$

entropy conservation follows from the energy-momentum conservation and the form of the energy-momentum tensor

Perfect-fluid hydrodynamics:

$$D\varepsilon = (\varepsilon + P)\theta$$

$$(\varepsilon + P)Du^\lambda = \nabla^\lambda P,$$

First line represents entropy conservation
While the second one is a relativistic
Euler fluid equation:

$$T(D\sigma + \sigma\theta) + \mu(Dn + n\theta) = 0$$

$$T\partial_\mu(\sigma u^\mu) + \mu\partial_\mu(n u^\mu) = 0.$$

Thermodynamic identities:

$$dE = TdS - PdV + \mu dN.$$

$$E + PV = TS + \mu N.$$

Switching to densities (convenient in hydro):

$$d\varepsilon = Td\sigma + \mu dn$$

$$\varepsilon + P = T\sigma + \mu n,$$

$$\varepsilon = \frac{E}{V}, \quad \sigma = \frac{S}{V}, \quad n = \frac{N}{V}.$$

$$dP = \sigma dT + nd\mu.$$

Entropy current (a direct consequence of previous definitions)

$$S^\mu_{\text{eq}} = \sigma u^\mu = P\beta^\mu - \xi N^\mu_{\text{eq}} + \beta_\lambda T^{\lambda\mu}_{\text{eq}}$$

$$dS^\mu_{\text{eq}} = -\xi dN^\mu_{\text{eq}} + \beta_\lambda dT^{\lambda\mu}_{\text{eq}}.$$

$$d(P\beta^\mu) = N^\mu_{\text{eq}} d\xi - T^{\lambda\mu}_{\text{eq}} d\beta_\lambda,$$

Four-temperature vector

$$\beta^\mu = \frac{u^\mu}{T}, \quad \beta = \sqrt{\beta^\lambda \beta_\lambda} = \frac{1}{T},$$

Chemical potential scaled by temperature

$$\xi = \frac{\mu}{T}.$$

Non-equilibrium entropy current (Israel-Stewart formalism, still close to local equilibrium)

$$S^\mu = P\beta^\mu - \xi N^\mu + \beta_\lambda T^{\lambda\mu} + Q^\mu.$$

Tensor decompositions (with respect to the flow vector

Particle/baryon current

$$\begin{aligned} N^\mu &= N_\alpha g^{\alpha\mu} = N_\alpha \left(g^{\alpha\mu} - u^\alpha u^\mu + u^\alpha u^\mu\right) \\ &= N_\alpha \left(\Delta^{\alpha\mu} + u^\alpha u^\mu\right) = n u^\mu + V^\mu, \end{aligned}$$

Energy-momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= T_{\alpha\beta} g^{\alpha\mu} g^{\beta\mu} = T_{\alpha\beta} \left(\Delta^{\alpha\mu} + u^\alpha u^\mu\right)\left(\Delta^{\beta\nu} + u^\beta u^\nu\right) \\ &= \varepsilon u^\mu u^\nu + W^\mu u^\nu + W^\nu u^\mu + \Pi'^{\mu\nu}, \end{aligned}$$

$$\varepsilon = T_{\alpha\beta} u^\alpha u^\beta,$$

$$W^\mu = T_{\alpha\beta} u^\beta \Delta^{\alpha\mu}$$

$$\begin{aligned} \Pi'^{\mu\nu} &= T_{\alpha\beta} \Delta^{\alpha\mu} \Delta^{\beta\nu} \\ &= \frac{1}{2} T_{\alpha\beta} \left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right) + \frac{1}{3} T_{\alpha\beta} \Delta^{\mu\nu}\Delta^{\alpha\beta}. \end{aligned}$$

$$\Pi'^{\mu\nu} = T^{\alpha\beta} \Delta^{\mu\nu}_{\alpha\beta} + \frac{1}{3} T_{\alpha\beta} \Delta^{\alpha\beta} \Delta^{\mu\nu} \equiv \pi^{\mu\nu} - (P + \Pi)\Delta^{\mu\nu}.$$

$$\Pi_{\alpha\beta} = \pi_{\alpha\beta} - \Pi\Delta_{\alpha\beta}$$

$$\begin{aligned} T^{\mu\nu} &= \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \Pi^{\mu\nu} \\ &= \varepsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}. \end{aligned}$$

Eckart vs. Landau-Lifschitz (hydro) frames, V=0 or W=0

$$N^\mu = N^\mu_{\text{eq}} + V^\mu, \qquad T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} - \Pi\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}.$$

Non-equilibrium entropy current (Israel-Stewart formalism, still close to local equilibrium)

$$S^\mu = \sigma u^\mu - \xi V^\mu + \frac{W^\mu}{T} + Q^\mu \qquad S^\mu = sN^\mu + \frac{q^\mu}{T} + Q^\mu,$$

Heat-flux four-vector

$$q^\mu = W^\mu - \frac{\varepsilon + P}{n} V^\mu = W^\mu - wV^\mu.$$

Entropy production

$$\partial_\mu S^\mu = -\left(N^\mu - N^\mu_{\mathrm{eq}}\right)\partial_\mu \xi + \left(T^{\lambda\mu} - T^{\lambda\mu}_{\mathrm{eq}}\right)\left(\partial_\mu \beta_\lambda\right) + \partial_\mu Q^\mu .$$

$$T\partial_\mu S^\mu = -TV^\mu \partial_\mu \xi + W^\lambda \left(Du_\lambda - \frac{1}{T}\nabla_\lambda T\right) + \pi^{\lambda\mu}\partial_\mu u_\lambda - \Pi\theta + T\partial_\mu Q^\mu .$$

Using hydrodynamic equations in the leading order (perfect-fluid equations) one gets:

$$T\partial_\mu S^\mu = \frac{nT}{\varepsilon + P} q^\lambda \partial_\lambda \left(\frac{\mu}{T}\right) + \pi^{\lambda\mu}\partial_{<\mu} u_{\lambda>} - \Pi\theta + T\partial_\mu Q^\mu .$$

Basic kinetic coefficients

Entropy is positive if:

$$q^\lambda = -\frac{\lambda n T^2}{\varepsilon + P} \nabla^\lambda \left(\frac{\mu}{T}\right),$$

$$\Pi = -\zeta \partial_\mu u^\mu = -\zeta \theta,$$

$$\pi_{\lambda\mu} = 2\eta \partial_{<\lambda} u_{\mu>} = 2\eta \sigma_{\lambda\mu},$$

$$\partial_\mu S^\mu = -\frac{q^\lambda q_\lambda}{\lambda T^2} + \frac{\pi^{\lambda\mu}\pi_{\lambda\mu}}{2\eta T} + \frac{\Pi^2}{\zeta T},$$

with positive kinetic/transport coefficients
heat conductivity, bulk viscosity, shear viscosity:

$$\lambda, \zeta, \eta \geq 0.$$

Navier-Stokes hydrodynamics

Claude-Louis Navier, 1785–1836, French engineer and physicist
Sir George Gabriel Stokes, 1819–1903, Irish physicist and mathematician

C. Eckart, Phys. Rev. 58 (1940) 919

L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon, New York, 1959

complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu} \tag{8}$$

where $\Pi^{\mu\nu} u_\nu = 0$, which corresponds to the Landau definition of the hydrodynamic flow u^μ

$$T^\mu{}_\nu u^\nu = \mathcal{E}\, u^\mu = \mathcal{E}_{\rm eq}\, u^\mu . \tag{9}$$

It proves useful to further decompose $\Pi^{\mu\nu}$ into two components,

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Pi \Delta^{\mu\nu}, \tag{10}$$

which introduces the **bulk viscous pressure** Π (the trace part of $\Pi^{\mu\nu}$) and the **shear stress tensor** $\pi^{\mu\nu}$ which is symmetric, $\pi^{\mu\nu} = \pi^{\nu\mu}$, traceless, $\pi^\mu{}_\mu = 0$, and orthogonal to u^μ, $\pi^{\mu\nu} u_\nu = 0$.

in the Navier-Stokes theory, the **bulk pressure** and **shear stress tensor** are given by the gradients of the flow vector

$$\Pi = -\zeta\, \partial_\mu u^\mu, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{11}$$

Here ζ and η are the bulk and shear viscosity coefficients, respectively, and $\sigma^{\mu\nu}$ is the shear flow tensor defined as

$$\sigma^{\mu\nu} = 2\,\Delta^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta, \tag{12}$$

where the projection operator $\Delta^{\mu\nu}_{\alpha\beta}$ has the form

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu{}_\alpha\Delta^\nu{}_\beta + \Delta^\mu{}_\beta\Delta^\nu{}_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}. \tag{13}$$

Shear flow and shear stress tensors

$\sigma^{\mu\nu}$ – shear flow tensor, $\pi^{\mu\nu}$ – shear stress tensor

$$\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta, \quad \Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta} + \Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\alpha}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$$

$\sigma^{\mu\nu}$ is symmetric, ortoghonal to u, and traceless

$$\sigma^{\mu\nu} = \sigma^{\nu\mu}, \quad \sigma^{\mu\nu}u_\mu = \sigma^{\mu\nu}u_\nu = 0, \quad \sigma^{\mu}{}_{\mu} = 0$$

in the local rest frame where $u^\mu = (1, 0, 0, 0)$

$$\sigma^{\mu\nu} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_{XX} & \sigma_{XY} & \sigma_{XZ} \\ 0 & \sigma_{YX} = \sigma_{XY} & \sigma_{YY} & \sigma_{YZ} \\ 0 & \sigma_{ZX} = \sigma_{XZ} & \sigma_{ZY} = \sigma_{YZ} & \sigma_{ZZ} = -(\sigma_{XX} + \sigma_{YY}) \end{vmatrix}$$

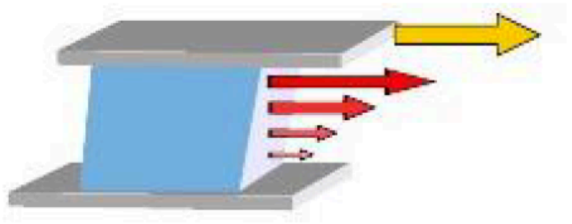
5 independent parameters in $\sigma^{\mu\nu}$
similarly for $\pi^{\mu\nu}$, since $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$

Viscosity

shear viscosity η

$\Downarrow$

reaction to a change of shape

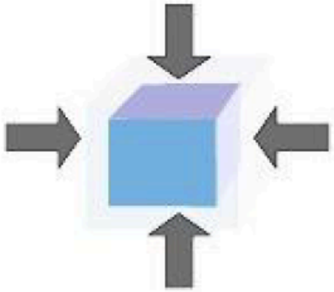


$$\pi^{\mu\nu}{}_{\mathrm{Navier-Stokes}} = 2\eta\,\sigma^{\mu\nu}$$

bulk viscosity ζ

$\Downarrow$

reaction to a change of volume



$$\Pi_{\mathrm{Navier-Stokes}} = -\zeta\theta$$

bulk viscosity and pressure vanish for conformal fluids

$$0 = T^{\mu}{}_{\mu} = \underbrace{\mathcal{E} - 3\mathcal{P}}_{=0} - 3\Pi + \underbrace{\pi^{\mu}{}_{\mu}}_{=0} = -3\Pi, \quad \Pi = 0$$

QGP shear viscosity: large or small?

John Mainstone (Wikipedia)

Pitch

Start 1927

1^{st} drop 1938

8^{th} drop 2000

$\eta \sim 2\,10^8$ Pa s

$\sim 10^{11}\eta_{H_2O}$

Wikipedia: The ninth drop touched the eighth drop on 17 April 2014. However, it was still attached to the funnel. On 24 April 2014, Prof. White decided to replace the beaker holding the previous eight drops before the ninth drop fused to them. While the bell jar was being lifted, the wooden base wobbled and the ninth drop snapped away from the funnel.

$$\eta_{\rm qgp} > \eta_{\rm pitch}$$

$$\eta_{\rm qgp} \sim 10^{11}\ \text{Pa s}, \qquad (\eta/s)_{\rm qgp} < 3/(4\pi) \qquad \text{(from experiment)}$$

Navier-Stokes hydrodynamics

complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + 2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu} \tag{14}$$

again four equations for four unknowns

$$\partial_\mu T^{\mu\nu} = 0 \tag{15}$$

THIS SCHEME DOES NOT WORK IN PRACTICE!
ACAUSAL BEHAVIOR + INSTABILITIES!

NEVERTHELESS, THE GRADIENT FORM (14) IS A GOOD APPROXIMATION FOR SYSTEMS APPROACHING LOCAL EQUILIBRIUM

W. A. Hiscock and L. Lindblom, "Generic instabilities in first-order dissipative relativistic fluid theories," *Phys.Rev.* **D31** (1985) 725–733.

L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon, New York, 1959

10 out of 671 pages about relativistic hydrodynamics, mostly outdated

Eckart-Landau theory leads to acausal behavior

New theories of relativistic hydrodynamics in the LHC era
Wojciech Florkowski (Cracow, INP & Darmstadt, EMMI & Jan Kochanowski U.), Michal P. Heller (Potsdam, Max Planck Inst. & Warsaw, Inst. Nucl. Studies), Michal Spalinski (Warsaw U., Bialystok & Warsaw, Inst. Nucl. Studies). Jul 7, 2017. 53 pp.
Published in **Rept.Prog.Phys. 81 (2018) no.4, 046001**

Relativistic hydrodynamics for spin-polarized fluids
Wojciech Florkowski, Radoslaw Ryblewski (Cracow, INP), Avdhesh Kumar. Nov 11, 2018. 50 pp.
Published in **Prog.Part.Nucl.Phys. 108 (2019) 103709**