Relativistic Hydrodynamics II

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Navier-Stokes hydrodynamics

complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu} = T^{\mu\nu}_{\rm eq} + 2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}$$

again four equations for four unknowns

$$\partial_\mu T^{\mu\nu} = 0$$

THIS SCHEME DOES NOT WORK IN PRACTICE!
ACAUSAL BEHAVIOR + INSTABILITIES!

NEVERTHELESS, THE GRADIENT FORM (14) IS A GOOD APPROXIMATION FOR SYSTEMS APPROACHING LOCAL EQUILIBRIUM

Gradient expansion

complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \underbrace{2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}}_{\text{first order terms in gradients}}$$

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \underbrace{2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}}_{\text{first order terms in gradients}} + \underbrace{\dots\dots\dots\dots\dots\dots}_{\text{second order terms in gradients}} + \dots$$

HYDRODYNAMIC EXPANSION OF THE ENERGY-MOMENTUM TENSOR, ASYMPTOTIC SERIES

M.P. Heller, R. Janik, R. Witaszczyk, PRL 110 (2013) 211602

Non-equilibrium entropy current (Israel-Stewart formalism, still close to local equilibrium)

$$S^\mu = P\beta^\mu - \xi N^\mu + \beta_\lambda T^{\lambda\mu} + Q^\mu .$$

$$d(P\beta^\mu) = N^\mu_{\mathrm{eq}} d\xi - T^{\lambda\mu}_{\mathrm{eq}} d\beta_\lambda ,$$

$$T\partial_\mu S^\mu = \frac{nT}{\varepsilon + P} q^\lambda \partial_\lambda \left(\frac{\mu}{T}\right) + \pi^{\lambda\mu}\partial_{<\mu}u_{\lambda>} - \Pi\theta + T\partial_\mu Q^\mu .$$

Israel-Stewart: new ansatz for the entropy current

$$S^\mu = S^\mu_{\rm NS} - \left(\beta_0 \Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\nu\lambda}\pi^{\nu\lambda}\right)\frac{u^\mu}{2T} - \alpha_0 \frac{\Pi q^\mu}{T} + \alpha_1 \frac{\pi^{\mu\nu} q_\nu}{T}.$$

with the following divergence:

$$\begin{aligned} T\partial_\mu S^\mu = & -\Pi\left[\theta + \beta_0 D\Pi + T\Pi\partial_\lambda\left(\frac{\beta_0 u^\lambda}{2T}\right) + \alpha_0 \partial_\nu q^\nu\right] \\ & -q^\mu\left[\nabla_\mu \ln T - Du_\mu - \beta_1 Dq_\mu - Tq_\mu\partial_\lambda\left(\frac{\beta_1 u^\lambda}{2T}\right) - \alpha_1\partial_\nu \pi^\nu{}_\mu + T\nabla_\mu\left(\frac{\alpha_0 \Pi}{T}\right)\right] \\ & +\pi^{\mu\nu}\left[\sigma_{\mu\nu} - \beta_2 D\pi_{\mu\nu} - T\pi_{\mu\nu}\partial_\lambda\left(\frac{\beta_2 u^\lambda}{2T}\right) + T\nabla_\mu\left(\frac{\alpha_1 q_\nu}{T}\right)\right]. \end{aligned} \tag{2.65}$$

Dynamic equations for dissipative quantities – they are upgraded to new hydrodynamic variables!

With alphas=0

$$\zeta\beta_0 D\Pi + \Pi = -\zeta\theta - \zeta T \Pi \partial_\lambda \left(\frac{\beta_0 u^\lambda}{2T}\right),$$

$$\lambda\beta_1 \Delta^{\mu\nu} D q_\nu + q^\mu = \lambda\left(\nabla^\mu \ln T - D u^\mu\right) - \lambda T q^\mu \partial_\lambda \left(\frac{\beta_1 u^\lambda}{2T}\right),$$

$$2\eta\beta_2 D\pi^{<\mu\nu>} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - 2\eta T \pi^{\mu\nu} \partial_\lambda \left(\frac{\beta_2 u^\lambda}{2T}\right).$$

$$\tau_\Pi = \zeta\beta_0, \quad \beta_\Pi = \frac{1}{\beta_0}, \quad \zeta = \tau_\Pi \beta_\Pi,$$

$$\tau_q = \lambda\beta_1, \quad \beta_q = \frac{1}{\beta_1}, \quad \lambda = \tau_q \beta_q,$$

$$\tau_\pi = 2\eta\beta_2, \quad \beta_\pi = \frac{1}{\beta_2}, \quad 2\eta = \tau_\pi \beta_\pi, ,$$

$$D\Pi + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi\theta - \beta_\Pi T\,\Pi\,\partial_\lambda \left(\frac{u^\lambda}{2\beta_\Pi T}\right),$$

$$\Delta^{\mu\nu} D q_\nu + \frac{q^\mu}{\tau_q} = \beta_q \left(\nabla^\mu \ln T - D u^\mu\right) - \beta_q T q^\mu \partial_\lambda \left(\frac{u^\lambda}{2\beta_q T}\right),$$

$$D\pi^{<\mu\nu>} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \beta_\pi \sigma^{\mu\nu} - \beta_\pi T \pi^{\mu\nu} \partial_\lambda \left(\frac{u^\lambda}{2\beta_\pi T}\right).$$

For conformal systems:

$$D\pi^{<\mu\nu>} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \beta_\pi \sigma^{\mu\nu} - \frac{\pi^{\mu\nu}}{2}\left[\theta - u^\lambda \partial_\lambda \ln(\beta_\pi T)\right]$$
$$= \beta_\pi \sigma^{\mu\nu} - \frac{\pi^{\mu\nu}}{2}\left[\theta - 5u^\lambda \partial_\lambda \ln T\right].$$

$$u^\lambda \partial_\lambda \varepsilon = -(\varepsilon + P)\theta,$$

$$u^\lambda \partial_\lambda \ln T = -\frac{\theta}{3}.$$

$\varepsilon \sim T^4$ and $P = \varepsilon/3$

$$D\pi^{<\mu\nu>} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta.$$

Israel-Stewart equations

Israel-Stewart equations — $\Pi, \pi^{\mu\nu}$ promoted to dynamic variables —
non-hydrodynamic modes are introduced with the appropriate relaxation times τ_Π, τ_π

W. Israel and J.M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Annals of Physics 118 (1979) 341

$$
\begin{aligned}
\dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta \\
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu}
\end{aligned}
$$

1) HYDRODYNAMIC EQUATIONS DESCRIBE BOTH HYDRODYNAMIC AND NON-HYDRODYNAMIC MODES

2) HYDRODYNAMIC MODES CORRESPOND TO GENUINE HYDRODYNAMIC BEHAVIOR

3) NON-HYDRODYNAMIC MODES (TERMS) SHOULD BE TREATED AS REGULATORS OF THE THEORY

4) NON-HYDRODYNAMIC MODES GENERATE ENTROPY

dispersion relations factorize into branches depending on the polarization of the perturbations, with $\vec{k}$ along the x axis

- Sound channel: non-vanishing $\delta u^x, \delta T^{xx}$
- Shear channel: non-vanishing $\delta u^y, \delta T^{xy}$
- Tensor channel: non-vanishing δT^{yz}

For example, in the sound channel one has

$$\omega^3 + \frac{i}{\tau_\pi}\omega^2 - \frac{k^2}{3}\left(1 + 4\frac{\eta/\mathcal{S}}{T\tau_\pi}\right)\omega - \frac{ik^2}{3\tau_\pi} = 0 \tag{26}$$

For small k one finds a pair of hydrodynamic modes (whose frequency tends to zero with k)

$$\omega_H^{(\pm)} = \pm\frac{k}{\sqrt{3}} - \frac{2i}{3T}\frac{\eta}{\mathcal{S}}k^2 + \ldots \tag{27}$$

and a nonhydrodynamic mode

$$\omega_{NH} = -i\left(\frac{1}{\tau_\pi} - \frac{4}{3T}\frac{\eta}{\mathcal{S}}k^2\right) + \ldots \tag{28}$$

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\,\frac{\eta/\mathcal{S}}{T\tau_\pi}}\,, \quad T\tau_\Pi > 2\eta/\mathcal{S}. \tag{29}$$

BRSSS equations

Baier, Romatschke, Son, Starinets, Stephanov (BRSSS)
symmetry arguments due to Lorentz and conformal symmetry, ...

R. Baier, P. Romatschke, D.T. Son, A. O. Starinets, M. A. Stephanov,

Relativistic viscous hydrodynamics, conformal invariance, and holography, JHEP 0804 (2008) 100

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

$$\begin{aligned} \Pi &= 0 \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{\lambda_1}{\tau_\pi \eta^2}\pi^{\langle\mu}_{\ \ \lambda}\pi^{\nu\rangle\lambda} \end{aligned}$$

(+ terms including vorticity and curvature)

Thus, our final expression for the dissipative part of the stress-energy tensor, up to second order in derivatives, is

$$\begin{aligned}\Pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} \\ &\quad + \eta\tau_\Pi \left[{}^{\langle}D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}(\nabla\cdot u) \right] + \kappa \left[R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta \right] \\ &\quad + \lambda_1 \sigma^{\langle\mu}{}_\lambda\sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} .\end{aligned} \tag{3.11}$$

The five new constants are τ_Π, κ, $\lambda_{1,2,3}$. Note that using lowest order relations $\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$, eqs. (3.5) and $D\eta = -\eta\,\nabla\cdot u$, eq. (3.11) may be rewritten in the form

$$\begin{aligned}\Pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \tau_\Pi \left[{}^{\langle}D\Pi^{\mu\nu\rangle} + \frac{d}{d-1}\Pi^{\mu\nu}(\nabla\cdot u) \right] \\ &\quad + \kappa \left[R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta \right] \\ &\quad + \frac{\lambda_1}{\eta^2} \Pi^{\langle\mu}{}_\lambda\Pi^{\nu\rangle\lambda} - \frac{\lambda_2}{\eta} \Pi^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} .\end{aligned} \tag{3.12}$$

This equation is, in form, similar to an equation of the Israel-Stewart theory (see section 6). In the linear regime it actually coincides with the Israel-Stewart theory (6.1). We emphasize, however, that one cannot claim that eq. (3.12) captures all orders in the momentum expansion (see section 6).

DNMR equations

Denicol, Niemi, Molnar, Rischke (DNMR)
simultaneous expansion in the Knudsen number and inverse Reynolds number

approach based on the kinetic theory

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta \underline{+ \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta \underline{- \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}$$

the version of equations shown is for RTA version of the Boltzmann kinetic equation, with neglected vorticity, for standard form of the collision term additional terms (with new kinetic coefficients) appear

shear-bulk coupling $\eta - \zeta$

A series of papers by Amaresh Jaiswal

Relativistic dissipative hydrodynamics from kinetic theory with relaxation time approximation
Amaresh Jaiswal (Tata Inst.). Feb 25, 2013. 5 pp.
Published in **Phys.Rev. C87 (2013) no.5, 051901**

Relaxation-time approximation

$$p^\mu \partial_\mu f = -\frac{u \cdot p}{\tau_R}(f - f_0),$$

$$f_0 = \frac{1}{\exp(\beta\, u \cdot p - \alpha) + r},$$

$$f_1 = f_0 - \frac{\tau_R}{u \cdot p}\, p^\mu \partial_\mu f_0, \quad f_2 = f_0 - \frac{\tau_R}{u \cdot p}\, p^\mu \partial_\mu f_1, \; \ldots,$$

$$f = f_0 + \delta f, \quad \delta f = \delta f^{(1)} + \delta f^{(2)} + \cdots,$$

$$\delta f^{(1)} = -\frac{\tau_R}{u \cdot p}\, p^\mu \partial_\mu f_0,$$

$$\delta f^{(2)} = \frac{\tau_R}{u \cdot p} p^\mu p^\nu \partial_\mu \left(\frac{\tau_R}{u \cdot p} \partial_\nu f_0 \right).$$

$$\Pi = -\frac{\Delta_{\alpha\beta}}{3} \int dp\; p^\alpha p^\beta \left[-\frac{\tau_R}{u.p}\, p^\gamma \partial_\gamma (f_0 + \bar{f}_0) \right],$$

$$n^\mu = \Delta^\mu_\alpha \int dp\; p^\alpha \left[-\frac{\tau_R}{u.p}\, p^\gamma \partial_\gamma (f_0 - \bar{f}_0) \right],$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp\; p^\alpha p^\beta \left[-\frac{\tau_R}{u.p}\, p^\gamma \partial_\gamma (f_0 + \bar{f}_0) \right].$$

Anisotropic hydrodynamics

Thermodynamic formulation

R. Ryblewski, WF

PRC 83, 034907 (2011), JPG 38 (2011) 015104

1. energy-momentum conservation
   $\partial_\mu T^{\mu\nu} = 0$
2. ansatz for the entropy source, e.g.,
   $\partial(\sigma U^\mu) \propto (\lambda_\perp - \lambda_\parallel)^2/(\lambda_\perp \lambda_\parallel)$

Kinetic-theory formulation

M. Martinez, M. Strickland

NPA 848, 183 (2010), NPA 856, 68 (2011)

1. first moment of the Boltzmann equation = energy-momentum conservation
2. zeroth moment of the Boltzmann equation = specific form of the entropy source

3. Generalized form of the equation of state based on the Romatschke-Strickland (RS) form

generalization of equilibrium/isotropic distributions, frequently used in the studies of anisotropic quark-gluon plasma (here as a modified Boltzmann distribution in the local rest frame)

$$f_{RS} = \exp\left(-\sqrt{\frac{p_\perp^2}{\lambda_\perp^2} + \frac{p_\parallel^2}{\lambda_\parallel^2}}\right) = \exp\left(-\frac{1}{\lambda_\perp}\sqrt{p_\perp^2 + x\, p_\parallel^2}\right) = \exp\left(-\frac{1}{\Lambda}\sqrt{p_\perp^2 + (1+\xi)\, p_\parallel^2}\right)$$

anisotropy parameter $x = 1 + \xi = \left(\frac{\lambda_\perp}{\lambda_\parallel}\right)^2$ and transverse-momentum scale $\lambda_\perp = \Lambda$

4. Energy-momentum tensor (with single anisotropy parameter)

$$T^{\mu\nu} = (\varepsilon + P_\perp)\, U^\mu U^\nu - P_\perp\, g^{\mu\nu} - (P_\perp - P_\|) Z^\mu Z^\nu$$

$\varepsilon(\sigma, x)$ — energy density, $P_\perp(\sigma, x)$ — transv. pressure, $P_\|(\sigma, x)$ — long. pressure
alternatively one may use: $\varepsilon(\Lambda, \xi)$, $P_\perp(\Lambda, \xi)$, $P_\|(\Lambda, \xi)$

U — flow four-vector, Z — beam four-vector, $U^2 = 1$, $Z^2 = -1$, $U \cdot Z = 0$

this form of $T^{\mu\nu}$ follows from the covariant version of RS

$$f_{RS} = \exp\left(-\frac{1}{\Lambda}\sqrt{(p \cdot U)^2 + \xi\, (p \cdot Z)^2}\right), \quad U = (t/\tau, 0, 0, z/\tau), \quad Z = (z/\tau, 0, 0, t/\tau)$$

5. Several applications have been made to describe the heavy-ion data within this framework

Two expansion methods

Kinetic-theory formulation

Perturbative approach

Bazov, Heinz, Strickland
PRC 90, 044908 (2014)

$$f = f_{RS} + \delta f$$

- the leading order is still described by the Romatschke-Strickland form (accounting for the difference between the longitudinal and transverse pressures)
- advanced methods of traditional viscous hydrodynamics are used to restrict the form of the correction δf and to derive aHydro equations — non-trivial dynamics included in the transverse plane and, more generally, in (3+1)D

Non-perturbative approach

Nopoush, Ryblewski, Strickland,Tinti, WF

$$f = f_{\rm aniso} + \dots$$

- all effects due to anisotropy included in the leading order, in the generalised RS form

1. (1+1)D conformal case, two anisotropy parameters
2. (1+1)D non-conformal case, two anisotropy parameters + one bulk parameter
3. full (3+1)D case, five anisotropy parameters +

Non-perturbative approach

Boost-invariant and cylindrically symmetric expansion, (1+1)D non-perturbative approach

as much as possible, the momentum anisotropy is included in the leading order

$$f(x,p) = f_{\rm aniso} = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

1. Conformal case, two anisotropy parameters
   L. Tinti, WF, Phys.Rev. C89 (2014) 034907

$$\Xi^{\mu\nu} = U^\mu U^\nu + \xi^{\mu\nu}$$
$$u_\mu \xi^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0$$
$$\xi^{\mu\nu} = \mathrm{diag}(0, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} \equiv (\xi_x, \xi_y, \xi_z) \quad \text{(in the local rest frame)}$$

2. Non-conformal case, two anisotropy parameters + one bulk parameter
   M. Nopoush, R. Ryblewski, M. Strickland, Phys. Rev. C 90 (2014) 014908

equations of motion for ξ_x, ξ_y, Φ, λ, T, u^r for (1+1)d case are obtained by taking moments of the Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f = p^\mu \frac{u_\mu}{\tau_{\rm eq}}(f^{\rm eq} - f) \qquad \rightarrow \qquad \partial_{\mu_1} \int dP\, p^{\mu_1} \ldots p^{\mu_{n+1}} f = u_{\mu_1} \int dP\, p^{\mu_1} \ldots p^{\mu_n} \frac{1}{\tau_{\rm eq}}(f^{\rm eq} - f)$$

0th moment (1 eq.)

$$\partial_\mu N^\mu = \frac{u_\mu}{\tau_{\rm eq}}(N^\mu_{\rm eq} - N^\mu)$$

1st moment (2 eq.)

$$u_\nu \partial_\mu T^{\mu\nu} = u_\nu \frac{u_\mu}{\tau_{\rm eq}}(T^{\mu\nu}_{\rm eq} - T^{\mu\nu})$$

Landau matching condition for the energy (1 eq.)

$$u_\mu T^{\mu\nu}_{\rm eq} = u_\mu T^{\mu\nu}$$

2nd moment (2 eq.)

$$X^i_\mu X^i_\nu \partial_\lambda \Theta^{\lambda\mu\nu} = X^i_\mu X^i_\nu \frac{u_\lambda}{\tau_{\rm eq}} \left(\Theta^{\lambda\mu\nu}_{\rm eq} - \Theta^{\lambda\mu\nu}\right)$$

two linear combinations of these equations with $i = 0, 1, 2, 3$

X, Y defined in addition to U and Z

...

3A. (3+1) dimensional framework for leading order anisotropic hydrodynamics
L. Tinti, Phys. Rev. C92 (2015) 014908

Testing different formulations of leading order anisotropic hydrodynamics
L. Tinti, R. Ryblewski, W. Florkowski, M. Strickland, Nucl. Phys. A946 (2016) 29

$$\Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi$$
$$u_\mu \xi^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0 \quad \text{(5 parameters in } \xi^{\mu\nu}\text{)}$$

3B. Anisotropic matching principle for the hydrodynamic expansion, L. Tinti, arXiv:1506.07164

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu f_{\rm aniso}(x,p) = \int dP\, p^\mu p^\nu f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

Instead of looking at the moments we can derive first the equations for the pressure corrections, following DNMR (Denicol, Niemi, Molnar, Rischke) strategy used for viscous hydrodynamics

This is the latest development for the leading order, that may be supplemented by NLO terms following the approach by Heinz et al.