Superstring Perturbation Theory

① https: home.icts.res.in/~sen/

Course on Superstring Perturbation Theory

② arxiv: 1703.06410 (sections 2,3)

String theory: Elementary constituents of matter are one dimensional objects (strings)

- Radical idea, but why?

① String theory contains gravity and possibly the standard model of particle phy

② Unlike QFTs, string theory is free fr ultraviolet divergences.

$\rightsquigarrow$ UV finite, unambiguous theory of quantum gravity.

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Goal: To give a glimpse of how string theory works. (skip many derivations)
Exercises can be done based on the results that we provide.

Starting point: Action of a relativistic point particle.

$$S = -m\int ds = -m\int d\xi \sqrt{-\frac{dX^{\mu}}{d\xi}\frac{dX^{\nu}}{d\xi}\eta_{\mu\nu}}$$

$ds \to$ proper time

$d\xi$: parameter along the world line

$(\xi^0, \xi^1) : \{\xi^\alpha\} \quad \alpha = 1, 2.$

Action (Nambu-Goto)

$$S = -T \int d^2\xi \sqrt{-\det h}, \quad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha} \qquad \alpha = 1, 2$$

$=$ "Area" of the world-sheet

$$T : \text{string tension} = \frac{1}{2\pi\alpha'}$$

We'll set $\hbar = 1$, $c = 1$, $\alpha' = 1$

We simplify the action by introducing an auxiliary set of variables $\gamma_{\alpha\beta}$

world-sheet metric

$$S_P = -\frac{1}{4\pi} \int \sqrt{-\det \gamma}\ \gamma^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\Downarrow$

Matrix Inverse of $\gamma_{\alpha\beta}$

Ex. $\gamma_{\alpha\beta}$ eq. of motion $\Rightarrow$ $\gamma_{\alpha\beta} \propto \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$

substitute into $S_P \Rightarrow S_{NG}$

Classically the two actions are equivalent

We quantize $S_P \rightarrow$ 2-D general coord. inv.

$\Rightarrow$ Gauge fix.

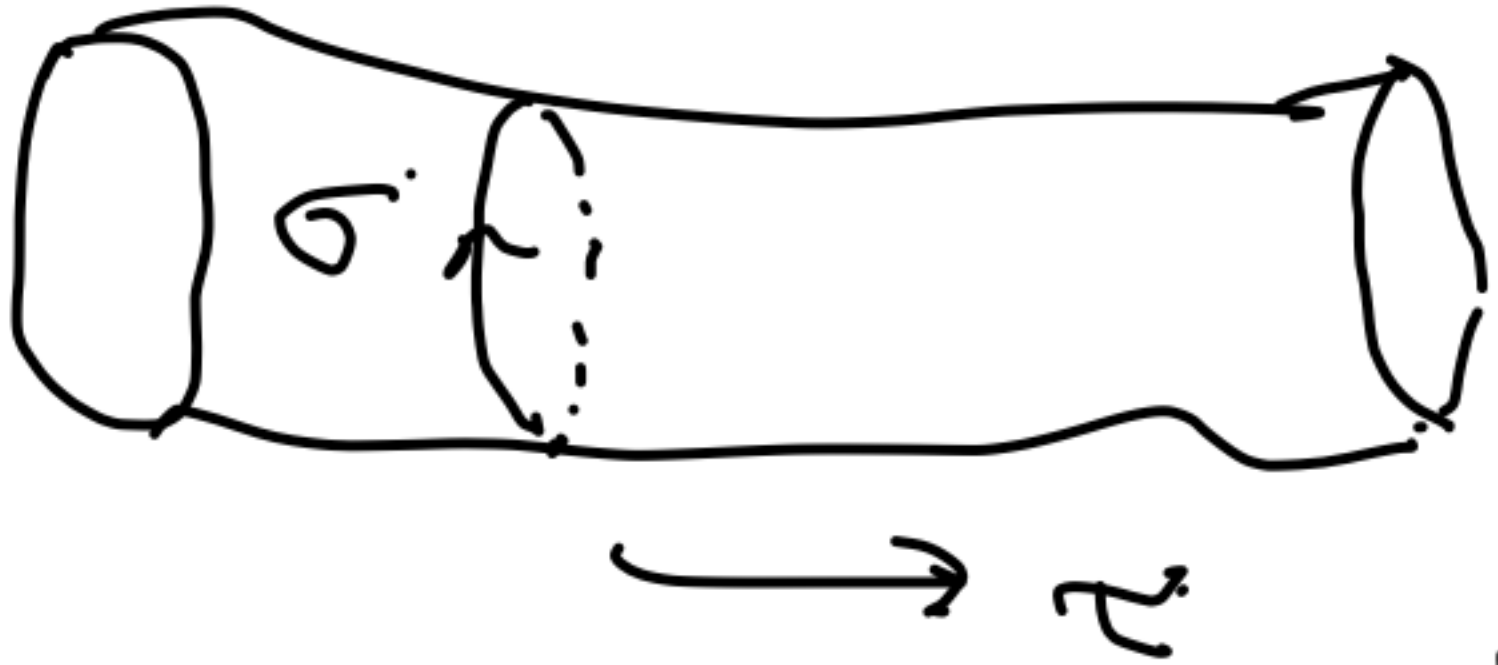
Choose gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$.

Wick rotate the world-sheet and set

$\gamma_{\alpha\beta} = \delta_{\alpha\beta}$.

$\xi^0 = i\tau$ $\quad \xi^1 : \sigma \quad \sigma \equiv \sigma + 2\pi$

$\rightarrow$ Euclidean world-sheet time.



Complex coordinate

$w = \tau + i\sigma$

$z = e^{w} = e^{\tau + i\sigma}$

$\tau \to -\infty \Rightarrow z = 0$, $\tau \to \infty \Rightarrow z = \infty$

Maps the cylinder to a plane.

Add $z = \infty$ point to the complex plane $\rightarrow$ sphere

τ ordering $\leftrightarrow$ $|z|$ ordering (radial ordering)

Gauge fixed action

$$S = -\frac{1}{2\pi}\int d^2w\,[\partial X^\mu \bar\partial X^\nu \eta_{\mu\nu} + b\bar\partial c + \bar c\,\partial \bar b]$$

$\partial \equiv \frac{\partial}{\partial w}$, $\bar\partial = \frac{\partial}{\partial \bar w}$, $b, c, \bar b, \bar c$: ghosts (anti-commuting)

Eq. of motion:

$$\partial\bar\partial X^\mu = 0, \quad \bar\partial c = 0, \quad \bar\partial b = 0, \quad \partial \bar c = 0, \quad \partial \bar b = 0$$

$\Rightarrow$ ∂X^μ, b, c : holomorphic

$\bar\partial X^\mu$, $\bar b$, $\bar c$: anti-holomorphic

Mode expansions

$$X^\mu = -\frac{i}{\sqrt{2}}\left[\sum_{n\neq 0} \alpha^\mu_n e^{-nw} + \sum_{n\neq 0} \bar{\alpha}^\mu_n e^{-n\bar{w}}\right] + x^\mu + i p^\mu \tau$$

$$b = \sum_n b_n e^{-nw}, \quad c = \sum_n c_n e^{-nw}$$

$$\bar{b} = \sum_n \bar{b}_n e^{-n\bar{w}}, \quad \bar{c} = \sum_n \bar{c}_n e^{-n\bar{w}}$$

Equal time (anti-) commutator.

Ex. $[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$, $\{b_m, c_n\} = \delta_{m+n,0}$.

similarly for the anti-holomorphic ones.

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}.$$

Response of the theory to deformations of $\gamma_{\alpha\beta}$:

$\rightarrow$ captured in the energy-momentum tensor $T_{\alpha\beta}$.

Result: $T_{\omega\omega} \equiv T = -\eta_{\mu\nu}\partial X^\mu \partial X^\nu - 2b\partial c + c\partial b$

$T_{\bar\omega\bar\omega} \equiv \bar T = -\eta_{\mu\nu}\bar\partial X^\mu \bar\partial X^\nu - 2\bar b\bar\partial\bar c + \bar c\bar\partial\bar b$

$T_{\omega\bar\omega} = 0 \Rightarrow$ Traceless $\Rightarrow$ Conformally inv. theory.

Conservation law $\partial^\alpha T_{\alpha\beta} = 0$.

$\underbrace{\partial_{\bar\omega} T_{\omega\omega}}_{\rightarrow \text{hol.}} + \underbrace{\partial_\omega T_{\bar\omega\omega}}_{=0} = 0, \quad \partial_\omega T_{\bar\omega\bar\omega} + \partial_{\bar\omega} T_{\omega\bar\omega} = 0.$

Suppose that $K(\omega)$ is a holomorphic field.

$$\partial_{\bar{\omega}} \left(f(\omega) K(\omega) \right) = 0$$

$\hookrightarrow$ any holomorphic fn..

$$\partial_{\bar{\omega}} \underbrace{J_\omega}_{\equiv f(\omega) K(\omega)} + \partial_\omega \underbrace{J_{\bar{\omega}}}_{\equiv 0} = 0 \Rightarrow Q_\tau = \oint_{\mathrm{Re}\,\omega = \tau} J_\omega \, d\omega \propto \oint J^\tau d\sigma \to \text{Conserved}$$

since $\partial_\alpha J^\alpha = 0$.

$$\Rightarrow Q_\tau = \oint_{\mathrm{Re}\,\omega = \tau} J_\omega \, d\omega \text{ is } \tau \text{ independent.}$$

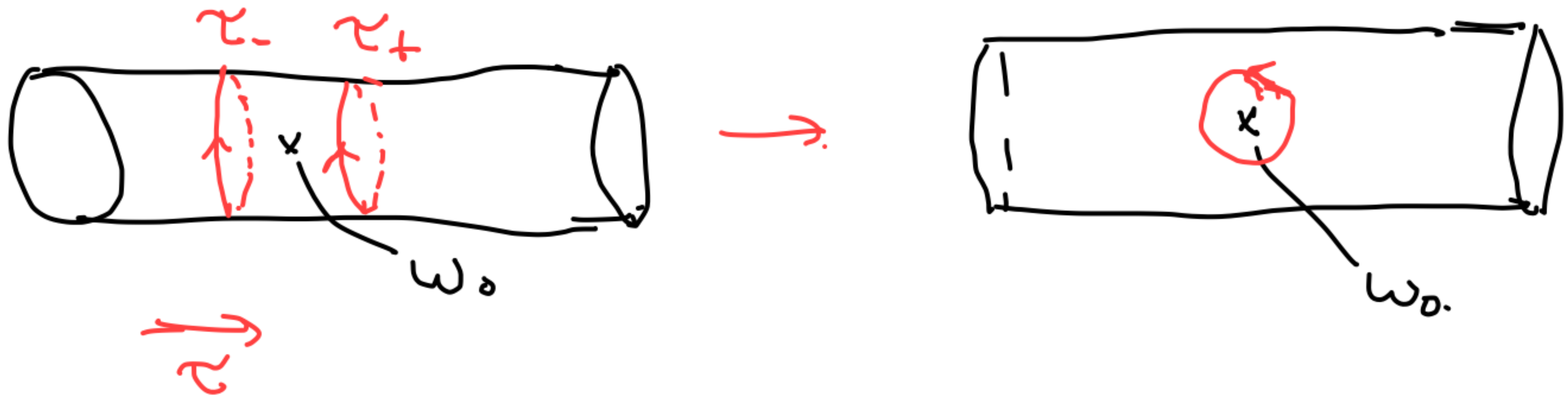
contains $\frac{1}{2\pi i}$ factor.

suppose $\phi(w,\bar{w})$ is another operator.

$$[Q, \phi(w_0,\bar{w}_0)] = Q_{\tau_+}\,\phi(w_0,\bar{w}_0) - \phi(w_0,\bar{w}_0)\,Q_{\tau_-}$$

$\tau_+ > \mathrm{Re}\,w_0$, $\tau_- < \mathrm{Re}\,w_0$

$$= \oint_{\mathrm{Re}\,w=\tau_+} T(J_w(w)\,\phi(w_0,\bar{w}_0)) - \oint_{\mathrm{Re}\,w=\tau_-} T(J_w(w)\,\phi(w_0,\bar{w}_0))$$



$$[Q, \phi(w_0, \bar{w}_0)] = \oint_{w_0} T(J_w(w)\, \phi(w_0, \bar{w}_0))$$

$\Rightarrow$ the coefficient of $\frac{1}{w-w_0}$ in the product $T(J_w(w)\, \phi(w_0, \bar{w}_0))$ gives $[Q, \phi(w_0, \bar{w}_0)]$.

Converse: Given a commutator we can determine singular terms in the operator product.

Ex. $b(\omega_1)\, c(\omega_2) \simeq \frac{1}{\omega_1-\omega_2}$, $\partial X^\mu(\omega_1)\, \partial X^\nu(\omega_2)$

$\simeq -\frac{1}{2}\frac{\eta^{\mu\nu}}{(\omega_1-\omega_2)^2}$

$\bar{b}(\bar{\omega}_1)\, \bar{c}(\bar{\omega}_2) \simeq \frac{1}{\bar{\omega}_1-\bar{\omega}_2}$

$\bar{\partial} X^\mu(\bar{\omega}_1)\, \bar{\partial} X^\nu(\bar{\omega}_2) \simeq -\frac{1}{2}\frac{\eta^{\mu\nu}}{(\bar{\omega}_1-\bar{\omega}_2)^2}$

Convention: All products are inside time ordering.

Singularities in the operator product are the same in all coordinates (local properties)

$T_{w\bar{w}} = 0 \Rightarrow$ Theory is conformally invariant

Conformal trs: $w \to f(w)$

locally holomorphic

A field $\phi(w,\bar{w})$ is a primary of dimension (weight) $(\bar{h}, h)$ if

$$\phi_w(w,\bar{w}) = \left(f'(w)\right)^h \left(\bar{f}'(\bar{w})\right)^{\bar{h}} \phi_{\cancel{f(w)}}\left(f(w), \overline{f(w)}\right)$$

$\phi_{f(w)}$: ϕ is measured in $f(w)$ coordinate system.

$\phi(w) \equiv \phi_w(w)$, $\phi(z) \equiv \phi_z(z)$

Infinitesimal conformal trs: $w \to w + \epsilon(w)$
$\bar{w} \to \bar{w} + \bar{\epsilon}(\bar{w})$

is generated by:

$$\oint \epsilon(w) T(w)\, dw + \oint \bar{\epsilon}(\bar{w}) \bar{T}(\bar{w})\, d\bar{w}$$

$\epsilon(w) T(w) \quad \phi(w_0, \bar{w}_0) \to$ singularities

in this product knows about

$$\left[\oint \epsilon(w) T(w)\, dw, \phi(w_0, \bar{w}_0) \right]$$

$\hookrightarrow$ conformal trs. of ϕ

Ex. $$T(w)\,\phi(w',\bar{w}') \simeq \frac{h}{(w-w')^2}\,\phi(w',\bar{w}') + \frac{1}{w-w'}\,\partial_{w'}\phi(w',\bar{w}')$$

$$\bar{T}(\bar{w})\,\phi(w',\bar{w}') \simeq \frac{\bar{h}}{(\bar{w}-\bar{w}')^2}\,\phi(w',\bar{w}') + \frac{1}{\bar{w}-\bar{w}'}\,\partial_{\bar{w}'}\phi(w',\bar{w}')$$

Ex. $$T(w)\,b(w') \simeq \frac{2}{(w-w')^2}\,b(w') + \frac{1}{w-w'}\,\partial_{w'}b(w')$$

?

$-2b\partial c + c\partial b$ $\Rightarrow$ b is a primary of weight $(0,2)$.

Ex. c is a primary of weight $(0,-1)$

$\bar{b}$ " " " " $(2,0)$

$\bar{c}$ " " " " $(-1,0)$

∂X^μ " " " " $(0,1)$

$\bar{\partial} X^\mu$ " " " " $(1,0)$

$(\bar{h},h)$ Using these results we can convert the mode expansions to z coord.

$$z = e^{w}$$

Ex: $b(w) = \sum_n b_n e^{-nw} \Rightarrow b(z) = \sum_n b_n z^{-n-2}$

$$\bar{b}(\bar{z}) = \sum_n \bar{b}_n \bar{z}^{-n-2}$$

$$c(z) = \sum_n c_n z^{-n+1}$$

$$\bar{c}(\bar{z}) = \sum_n \bar{c}_n \bar{z}^{-n+1}$$

Not all operators are primaries.

Ex. If ϕ is a primary of dim $(\bar{h}, h)$ then $\partial\phi$ is not a primary.

Even for non-primaries we can assign conformal dimensions.

If an operator transforms as a primary of weight $(\bar{h}, h)$ under $w \to \lambda w$, then its weight is $(\bar{h}, h)$.

Ex. If ϕ is a primary of weight $(\bar{h}, h)$ then $\partial\phi$ has weight $(\bar{h}, h+1)$

Ex. $T(\omega_1)\,T(\omega_2) \simeq \dfrac{D-26}{2(\omega_1-\omega_2)^4} + \dfrac{2}{(\omega_1-\omega_2)^2}T(\omega_2)$

$+ \dfrac{1}{\omega_1-\omega_2}\partial_{\omega_2}T(\omega_2)$

use expression in terms of b, c, X

D: # of values taken by μ.
(space-time dimension)

Similar for $\bar{T}\bar{T}$ product

$T^X = -\eta^{\mu\nu}\partial X^\mu \partial X^\nu$

D: from T^X

-26: " b, c part.

$\Rightarrow$ For $D=26$ T is a primary of weight $(0,2)$

$\bar{T}$ has weight $(2,0)$

$D=26$ is needed for consistency.
We work with $D=26$ from now on.
(We'll see later how to connect this to 4D space-time)
$D-26$ is known as "central charge"

$$T(w) = \sum L_n e^{-nw} \Rightarrow T(z) = \sum L_n z^{-n-2}$$

$$\bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}$$

From now on we work in z coordinate system but the Hilbert space interpretation will still be on the cylinder.

<u>Defn:</u> $|0\rangle$ is a special state such that $\phi(z,\bar{z})|0\rangle$ is finite for any finite z and any local operator ϕ.

e.g. $b(z)|0\rangle = \sum_n b_n z^{-n-2}|0\rangle$

Finiteness at $z=0 \Rightarrow b_n|0\rangle = 0$ for $n \geq -1$

Ex. Similarly $\bar{b}_n|0\rangle = 0$ for $n \geq -1$

$c_n, \bar{c}_n|0\rangle = 0$ for $n \geq 2$

$\alpha^\mu_n, \bar{\alpha}^\mu_n|0\rangle = 0$ for $n \geq 1$, $p^\mu|0\rangle = 0$

$X^\mu: i p^\mu \tau + \cdots$

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Ex. $L_n, \bar{L}_n |0\rangle = 0$ for $n \geq -1$.

follows trivially from mode expansion of $T, \bar{T}$ but also by expressing $T, \bar{T}$ in terms of $X^\mu, b, c, \bar{b}, \bar{c}$ and then using the mode expansions of these fields.

State operator correspondence

For every local operator $V(z,\bar{z})$ there is a state $|V\rangle = V(0)|0\rangle$ and vice versa..

e.g. $c(z) \leftrightarrow c_1|0\rangle$, $\partial c(z) \leftrightarrow c_0|0\rangle$

$\partial X^\mu(z) \leftrightarrow \frac{i}{\sqrt{2}}\alpha^\mu_{-1}|0\rangle$ etc.

Ex. $e^{ik\cdot X}$ is a primary of wt. $\left(\frac{k^2}{4}, \frac{k^2}{4}\right)$

Compute OPE with $T(z) = -\eta_{\mu\nu}\partial X^\mu \partial X^\nu$.

$$\partial X^\mu(z)\, X^\nu(w) \simeq -\frac{1}{2(z-w)}\eta^{\mu\nu}$$

Define $|k\rangle = e^{ik\cdot X(0)}|0\rangle = e^{ik\cdot x}|0\rangle$

k: space-time momentum.

In the CFT the complete basis of states is obtained from $b_n, c_n, \alpha^\mu_n, \bar{b}_n, \bar{c}_n, \bar{\alpha}^\mu_n$ acting on $|k\rangle$

$\Leftrightarrow$ complete basis of operators is obtained from products of (derivatives of) $b, c, \partial X^\mu, \bar{b}, \bar{c}, \bar{\partial} X^\mu$ and $e^{ik\cdot X}$

Def: For any local operator $V(z, \bar z)$ we define $f \circ V$ as the conformal transform of V under $z \to f(z)$

e.g. if V is primary of wt $(\bar h, h)$ then

$$f \circ V(z) = \left(f'(z)\right)^h \left(\overline{f'(z)}\right)^{\bar h} V\left(f(z), \overline{f(z)}\right)$$

Def: $I(z) \equiv -\frac{1}{z}$

Define $\langle 0|$ such that $\langle 0| I \circ V(z, \bar z)$ is finite for all finite z and all local operators V.

e.g. $\langle 0| I \circ b(z) = \langle 0| \sum_n b_n z^{n-2} (-1)^n$

$\Rightarrow \langle 0| b_n = 0$ for $n \leq 1$

$$\left.\begin{array}{l}\text{Similarly } \langle 0| \bar{b}_n = 0 \text{ for } n \leq 1 \\ \langle 0| c_n, \bar{c}_n = 0 \text{ for } n \leq -2.\end{array}\right\} \text{Ex.}$$

Given a state $|V\rangle = V(0)|0\rangle$, we define BPZ conjugate state $\langle V|$ as $\langle V| = \langle 0| I \circ V(0)$.

Ex. Find the BPZ conjugates of $c_1|0\rangle$, $c_0|0\rangle$

Ex. $\langle 0| L_n, \bar{L}_n = 0$ for $n \leq 1$

Defn. of correlation fn.

$$\langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle$$

$$= \langle 0| R\left(V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n)\right) |0\rangle$$

$\langle 0| L_{\pm 1, 0} = 0, \quad L_{\pm 1, 0}|0\rangle$ & similary for $\bar{L}_{\pm 1, 0}$.

$$\Rightarrow \langle 0| \left[L_{\pm 1, 0}, R\left(V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n)\right)\right] |0\rangle$$

$$= 0$$

$L_{\pm 1}, L_0$. generate infinitesimal conformal trs.

$$z \to z + \epsilon_{-1} + \epsilon_0 z + \epsilon_1 z^2$$

$\Rightarrow \langle V_1(z_1,\bar{z}_1) \cdots V_n(z_n,\bar{z}_n) \rangle$ is invariant under this conformal trs.

Ex. Finite version of this trs.

$$z \to \frac{az+b}{cz+d}, \quad a,b,c,d \text{ complex}$$

$ad-bc=1$

$SL(2,\mathbb{C})$ trs.

$z \to \frac{az+b}{cz+d}$

maps the $\underbrace{\text{complex plane} + \infty}_{\text{Sphere}}$

to complex plane $+ \infty$ in a one to one fashion

(Maximal possible trs. with this property)

Reason why this is a symmetry of the correlation fr.