All correlation fn. on the sphere (plane too) are invariant under $SL(2,\mathbb{C})$.

$$\langle V_1(z_1,\bar z_1) \cdots V_n(z_n,\bar z_n)\rangle$$

$$= \langle h\circ V_1(z_1,\bar z_1) \cdots h\circ V_n(z_n,\bar z_n)\rangle$$

$$h(z) = \frac{az+b}{cz+d}, \quad ad-bc = 1$$

– 3 complex i.e. 6 real parameters.

$\Downarrow$

We can fix the dependence on 6 of the $2n$ variables $z_1, \cdots z_n$.

Consequences

If V_1, V_2, V_3 are primaries of weights $(\bar{h}_1, h_1)$, $(\bar{h}_2, h_2)$ & $(\bar{h}_3, h_3)$ then:

① $\langle V_1(z_1, \bar{z}_1) \rangle = 0$ unless $h_1 = 0$, $\bar{h}_1 = 0$

② $\langle V_1(z_1, \bar{z}_1) V_2(z_2, \bar{z}_2) \rangle = C_{12} \delta_{h_1, h_2} \delta_{\bar{h}_1 \bar{h}_2} \times (z_1 - z_2)^{-2h_1} (\bar{z}_1 - \bar{z}_2)^{-2\bar{h}_1}$

③ $\langle V_1(z_1, \bar{z}_1) V_2(z_2, \bar{z}_2) V_3(z_3, \bar{z}_3) \rangle$
$= C_{123} (z_1 - z_2)^{h_3 - h_1 - h_2} (z_1 - z_3)^{h_2 - h_1 - h_3} (z_2 - z_3)^{h_1 - h_2 - h_3}$
$\times$ anti-hol. part.

④ $\langle V_1(z_1,\bar{z}_1) \prod_{i=2}^{n} O_i(z_i,\bar{z}_i) \rangle \Rightarrow$ can be anything

primary of wt. $(h_1,\bar{h}_1)$

$$\xrightarrow[z_1 \to \infty]{} z_1^{-2h_1} \bar{z}_1^{-2\bar{h}_1} \times f(\{z_2, \dots z_n, \bar{z}_2, \dots \bar{z}_n\})$$

provided $z_2, \dots z_n \neq \infty$

General relations in a CFT.

— apply it to our CFT.

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Consequences

① $\langle b(z)\, c(w) \rangle = 0 \quad \Rightarrow \quad \lim_{z \to w} \frac{1}{z-w} \langle 1 \rangle$

b: $(0,2)$, c: $(0,-1)$

$\Rightarrow \quad \langle 1 \rangle = 0, \quad \langle 0 | 0 \rangle = 0$

② $\langle b(z)\, c(z_1)\, c(z_2) \rangle \xrightarrow[z \to \infty]{} z^{-4} f(z_1, z_2)$

$\Downarrow$

Has poles at $z = z_1$, $z = z_2$

$\Downarrow$ Ex

# of poles − # of zeros in the z-plane = 4.

Contradiction! – must vanish.

$z \to z_1 \Rightarrow \langle c(z_2) \rangle = 0$

$\langle b(z)\, c(z_1)\, c(z_2)\, c(z_3)\rangle$ vanishes by the same argument

- at most 3 poles & hence inconsistent with z^{-4} fall-off at ∞.

$z \to z_1$ limit $\Rightarrow$ $\langle c(z_2)\, c(z_3)\rangle = 0$.

$\langle b(z)\, c(z_1)\, c(z_2)\, c(z_3)\, c(z_4)\rangle$ can be non-zero since we have 4 poles.

$z \to z_1 \Rightarrow \langle c(z_2)\, c(z_3)\, c(z_4)\rangle$ can be $\neq 0$

$\propto (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)$

$\langle c(z_1)\, c(z_2)\, c(z_3)\, c(z_4) \rangle \to$ zeroes at $z_1 = z_2, z_3, z_4$

$\downarrow$

go as z_1^2 as $z_1 \to \infty$

# of poles − # of zeros $= -2$.

contradiction $\Rightarrow \langle c(z_1)\, c(z_2)\, c(z_3)\, c(z_4) \rangle = 0$

$\langle c(z_1)\, c(z_2)\, c(z_3)\, \bar{b}(\bar{z})\, \bar{c}(\bar{z}_1') \rangle$

$\swarrow$

$\bar{z}^{-4}$ as $\bar{z} \to \infty$ but only one

# of poles − # of zeros $= 4$,   pole at $\bar{z} = \bar{z}_1'$

$\Rightarrow$ must vanish, $\bar{z} \to \bar{z}_1'$ $\Rightarrow \langle c(z_1)\, c(z_2)\, c(z_3) \rangle \neq 0$.

Non-vanishing correlation fn.

$$\langle c(z_1)\, c(z_2)\, c(z_3)\, \bar{c}(\bar{z}_1)\, \bar{c}(\bar{z}_2)\, \bar{c}(\bar{z}_3) \rangle$$

$$\propto (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)(\bar{z}_1 - \bar{z}_2)(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_3)$$

Define ghost no: 1 to $c, \bar{c}$, -1 to $b, \bar{b}$

0 to matter (X)

Ex. A correln. fn. can be $\neq 0$ only if the operator have gh. no. 3 in the hol. sector and ghost no. 3 in the anti-hol. sector.

Normalization convention

Define $|k\rangle = e^{ik\cdot X(0)}|0\rangle$

Take $\langle k| c_{-1}\bar{c}_{-1}\, c_0\bar{c}_0\, c_1\bar{c}_1|k'\rangle$

$$= (2\pi)^{26}\,\delta^{(26)}(k+k')$$

$$\Downarrow$$

$$\langle c(z_1)\,\bar{c}(\bar{z}_1')\,c(z_2)\,\bar{c}(\bar{z}_2')\,c(z_3)\,\bar{c}(\bar{z}_3')\prod_{i=1}^{n} e^{ik_i\cdot X}(z_i'', \bar{z}_i'')\rangle$$

$$= (z_1-z_2)(z_1-z_3)(z_2-z_3)(\bar{z}_1'-\bar{z}_2')(\bar{z}_1'-\bar{z}_3')(\bar{z}_2'-\bar{z}_3')$$

$$\times \prod_{i<j}|z_i''-z_j''|^{k_i\cdot k_j}\,(2\pi)^{26}\,\delta^{(26)}\Big(\sum_{i=1}^{n}k_i\Big)$$

Final defn. BRST current and charge.

$$j_B(z) = c\,T^X + bc\partial c\,, \quad \bar{j}_B(\bar{z}) = \bar{c}\,\bar{T}^X + \bar{b}\bar{c}\bar{\partial}\bar{c}$$

$T^X = -\partial X^\mu \partial X^\nu \eta_{\mu\nu}$

$$Q_B = \oint_{|z|=\text{const}} j_B(z)\,dz + \oint_{|z|=\text{const}} \bar{j}_B(\bar{z})\,d\bar{z}$$

normalized so that $\oint \frac{dz}{z} = 1, \ \oint \frac{d\bar{z}}{\bar{z}} = 1$

Ex: $Q_B^2 = 0$ for $D=26$.

String theory Not all states in the CFT are physical states of the string.

Given a state $|V\rangle$ in the CFT, it is a physical state of the string, if:

$$b_0^- |V\rangle = 0, \quad L_0^- |V\rangle = 0, \quad Q_B |V\rangle = 0$$

& $|V\rangle$ has ghost no. 2.

$$b_0^\pm \equiv b_0 \pm \bar{b}_0, \quad L_0^\pm \equiv L_0 \pm \bar{L}_0$$

Equivalence relation: If $|V\rangle$ and $|\tilde{V}\rangle$ are physical states, they are equivalent if there is a state $|W\rangle$ such that:

$$|\tilde{V}\rangle - |V\rangle = Q_B |W\rangle, \quad b_0^- |W\rangle = 0, \quad L_0^- |W\rangle = 0.$$

Result: For non-zero k, we can take the representative physical states to be of the form $c\bar{c}\, Y(0)|0\rangle$

Here Y is a dimension $(1,1)$ primary in the matter sector → built from X.

Examples ① $Y = e^{ik\cdot X}$ $\dim\left(\frac{k^2}{4}, \frac{k^2}{4}\right)$.

We need $k^2 = 4$

Our convention for $\eta_{\mu\nu}$: $(-1, +1, +1, \cdots)$

$\Rightarrow -(k^0)^2 + \vec{k}^2 = 4 \quad (k^0)^2 = \vec{k}^2 - 4$

Compare with $(k^0)^2 = \vec{k}^2 + \underbrace{m^2}_{\text{mass}}$

$\Rightarrow$ We have a state with $m^2 = -4$

— tachyon! $\Rightarrow$ Bosonic string theory is not fully consistent.

- Will be rectified in the heterotic and superstring theories.

For now on we proceed with the bosonic string theory → good toy model.

Example 2. $\mathcal{V} = S_{\mu\nu}\, \partial X^{\mu} \bar{\partial} X^{\nu} e^{ik\cdot X}$

$$\underbrace{k^2 = 0}_{\text{dim}(1,1)}, \quad \underbrace{k^{\mu} S_{\mu\nu} = 0, \quad k^{\nu} S_{\mu\nu} = 0.}_{\text{primary Ex.}}$$

(a) Symmetric part of $S_{\mu\nu}$: graviton.
(traceless)

(b) Trace of $S_{\mu\nu}$: massless scalar
(dilaton)

(c) Anti-symmetric part of $S_{\mu\nu}$:
2-form field $B_{\mu\nu}$.

Equivalence relations

Take $W = \xi_\mu c \partial X^\mu e^{ik\cdot X} + \bar{\xi}_\mu \bar{c} \bar{\partial} X^\mu e^{ik\cdot X}$

$k^2 = 0, \quad k^\mu \xi_\mu = 0, \quad k^\mu \bar{\xi}_\mu = 0$

Ex. $Q_B |W\rangle \propto i(k_\nu \bar{\xi}_\mu - k_\mu \xi_\nu) c \bar{c} \partial X^\mu \bar{\partial} X^\nu$
$e^{ik\cdot X(0)} |0\rangle$

$\Rightarrow$ Equivalence relation
$\delta \zeta_{\mu\nu} \propto i(k_\nu \xi_\mu - k_\mu \bar{\xi}_\nu)$ / $\bar{\xi}_\mu = -\xi_\mu \Rightarrow$ gauge trs. of graviton
$\delta \zeta_{\mu\nu} \propto i(k_\nu \xi_\mu + k_\mu \xi_\nu) \rightarrow$ linearized gen. coord. trs.
$\bar{\xi}_\mu = \xi_\mu \Rightarrow \delta B_{\mu\nu} \propto i(k_\mu \xi_\nu - k_\nu \xi_\mu)$

This show linearized general coord. tr.
To establish full non-linear general coordinate inv. we need to show that these "pure gauge states" decouple from the S-matrix.
$\Rightarrow$ Will be discussed later.

Besides these states we also have ∞ tower of massive states.

Mass scale is set by the string scale.

$\sim 1/\sqrt{\alpha'}$ (in $\hbar = 1$, $c = 1$ unit).

– can be made unobservable (today) by taking the string scale to be sufficiently large.

Goal: Compute string S-matrix.

For this we need to return to CFT correlators:

$$\langle V_1(z_1,\bar{z}_1) \cdots V_n(z_n,\bar{z}_n)\rangle$$

Punctures: Locations of vertex ops.

at $z = z_1, z_2, \cdots z_n$

Around each puncture we introduce a local complex coordinate system w_i

$i = 1, \cdots n$.

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Convention: i-th puncture is located at $w_i = 0$.

There is a functional relationship of the form $z = f_i(w_i)$ $\forall i$.

$$z_i = f_i(0). \quad \text{e.g. } f_i(w_i) = w_i + z_i$$

$$\text{or } f_i(w_i) = w_i + z_i + 2w_i^2$$

We'll consider a correlation fn. where V_i is inserted at the i-th puncture using w_i coordinate system.

$\Rightarrow f_i \circ V_i(0)$ in the z-coordinate system.

We'll consider correlation fn.s of the form

$$\left\langle \prod_{i=1}^{n} f_i \circ V_i(0) \right\rangle$$

$\Rightarrow V_i$ is inserted in the w_i coordinate system.

$SL(2,\mathbb{C})$ invariance:

$$h(z) = \frac{az+b}{cz+d}$$

$$= \left\langle \prod_{i=1}^{n} h \circ f_i \circ V_i(0) \right\rangle$$

$$= \left\langle \prod_{i=1}^{n} \hat{f}_i \circ V_i(0) \right\rangle$$

$$\hat{f}_i(z) = h(f_i(z))$$

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Introduce two sets of basis states

$\{|\phi_r\rangle\}$ and $\{|\phi_r^c\rangle\}$ in the CFT s.t.

$$\langle \phi_r^c | \phi_s \rangle = \delta_{rs} \Rightarrow \langle \phi_s | \phi_r^c \rangle = (-1)^{\phi_s} \delta_{rs}$$

$\Downarrow$

1 if ϕ_s is grassmann even (even ghost no.)

-1 if ϕ_s is grassmann odd (odd ghost no.)

$(I(z) = -\frac{1}{z})$

$\Downarrow$

$$\langle I \circ \phi_r^c(0)\, \phi_s(0) \rangle$$

$$\Downarrow I$$

$$\langle \phi_r^c(0)\, I \circ \phi_s(0) \rangle$$

$$= (-1)^{\phi_s} \langle I \circ \phi_s(0)\, \phi_r^c(0) \rangle$$

Completeness relation:

Ex. $\sum_{r} |\phi_r\rangle\langle\phi_r^c| = \mathbb{1}$, $\sum_{r} |\phi_r^c\rangle\langle\phi_r| (-1)^{\phi_r} = \mathbb{1}$.

Consider:

$$A = \langle f_1 \circ V_1(0) \cdots f_n \circ V_n(0) \; g_1 \circ \tilde{V}_1(0) \cdots g_m \circ \tilde{V}_m(0) \rangle$$

$|f_i(0)| > |g_j(0)|$ for all pair i, j.

$$A = \langle 0 | R(f_1 \circ V_1(0) \cdots f_n \circ V_n(0)) | \phi_r \rangle \langle \phi^c_r | \phi^c_s \rangle$$

$$\langle \phi_s | (-1)^{h_s} R(g_1 \circ \tilde{V}_1(0) \cdots g_m \circ \tilde{V}_m(0)) | 0 \rangle$$

Sum over r, s implied

$\Downarrow$

$$\langle I \circ \phi_s(0) (-1)^{\phi_s} \quad g_1 \circ \tilde{V}_1(0) \cdots g_m \circ \tilde{V}_m(0) \rangle$$

$f_q(z) = qz$

$\Downarrow$ $SL(2, \mathbb{C})$ by $f_q \circ I$

$$\langle f_q \circ \phi_s(0) (-1)^{\phi_s} \prod_{i=1}^{m} f_q \circ I \circ g_i \circ \tilde{V}_i(0) \rangle \Rightarrow \langle \prod_{i=1}^{m} \tilde{f}_i \circ \tilde{V}_i(0) \; q^{h_s} \bar{q}^{h_s} \phi_s(0) \rangle$$

$\Downarrow$

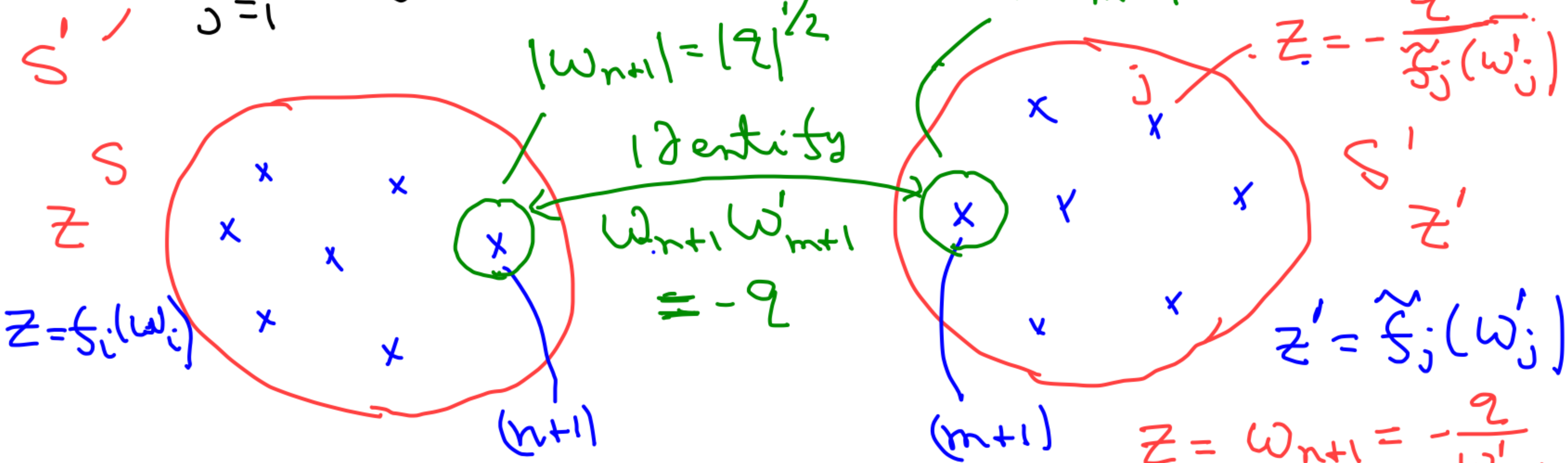
$q^{h_s} \bar{q}^{h_s} \phi_s(0)$

$$\tilde{f}_i = f_q \circ I \circ g_i = -q/g_i$$

$$\left\langle \prod_{i=1}^{n} f_i \circ V_i(0) \prod_{j=1}^{m} g_j \circ \tilde{V}_j(0) \right\rangle$$

$$= \underbrace{\left\langle \prod_{i=1}^{n} f_i \circ V_i(0)\ \phi_r(0) \right\rangle}_{S} \left\langle \phi_r^c \middle| \phi_s^c \right\rangle q^{h_s} \bar{q}^{\bar{h}_s}$$

$$\underbrace{\left\langle \prod_{j=1}^{m} \tilde{f}_j \circ \tilde{V}_j(0)\ \phi_s(0) \right\rangle}_{S'}$$



$$z = w_{n+1} = -\frac{q}{w'_{m+1}} = -\frac{q}{z'}$$

More general result:

$$\left\langle \prod_{i=1}^{n} f_i \circ V_i(0)\, f_{n+1} \circ \phi_r(0) \right\rangle \langle \phi_r^c | \phi_s^c \rangle q^{h_s} \bar{q}^{h_s}$$

$$\left\langle \prod_{j=1}^{m} \tilde{f}_j \circ \tilde{V}_j(0)\, \tilde{f}_{m+1} \circ \phi_s(0) \right\rangle$$

$$= \left\langle \prod_{i=1}^{n} \hat{f}_i \circ V_i(0) \prod_{j=1}^{m} \hat{\tilde{f}}_j \circ \tilde{V}_j(0) \right\rangle$$

$\swarrow$

Obtained by sewing the two original spheres by $w_{n+1} w'_{m+1} = -q$.

Global coordinate on the resulting sphere is z'' (not z or z'). $\longrightarrow$ $z'' = \hat{f}_i(w_i)$

$z'' = \hat{\tilde{f}}_j(w'_j)$

Further generalization

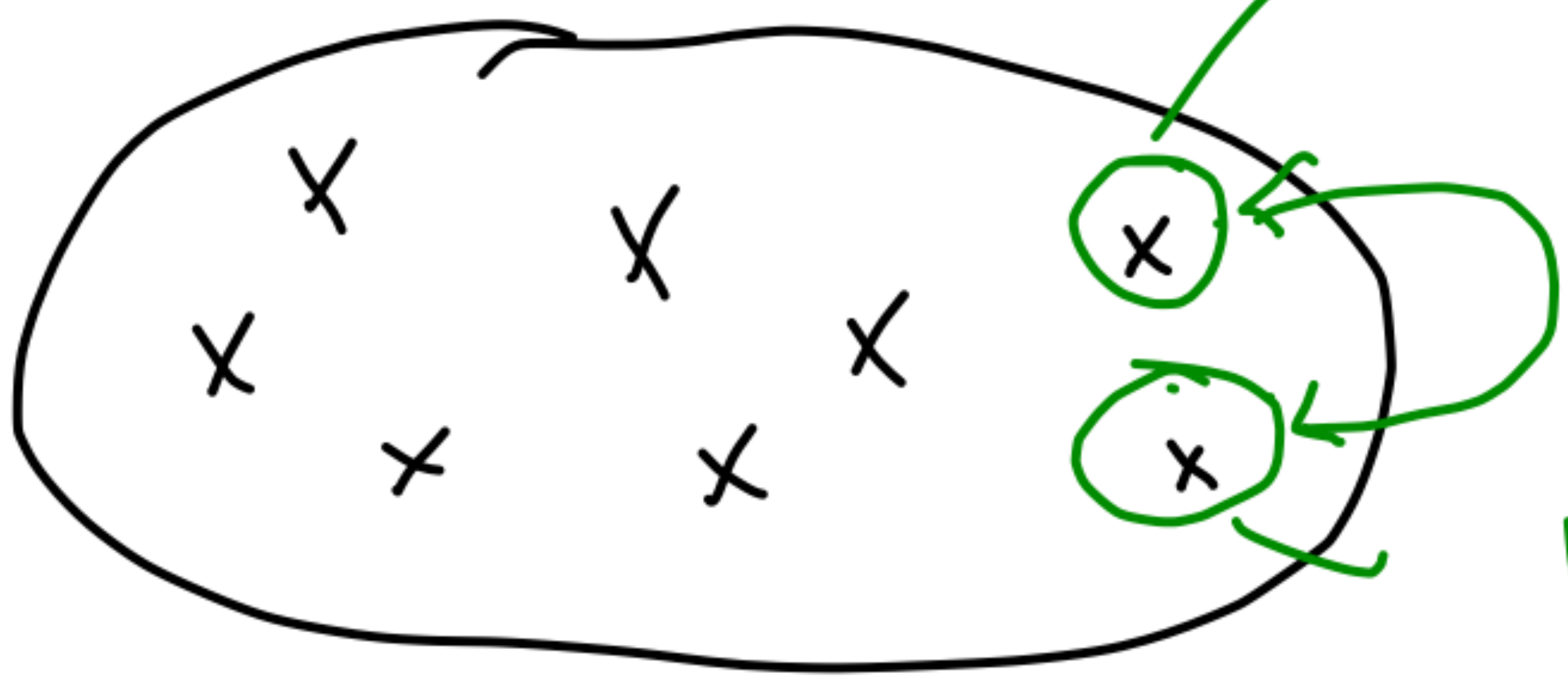
$$\left\langle \prod_{i=1}^{n} f_i \circ V_i(0)\; f_{n+1}\circ \phi_r(0)\; f_{n+2}\circ \phi_s(0) \right\rangle$$

$$\langle \phi_r^c | \phi_s^c \rangle\, q^{h_s}\, \bar{q}^{\bar{h}_s}$$

$$\ni \quad \omega_{n+1}\,\omega_{n+2} = -q$$

$|\omega_{n+1}| = |q|^{1/2}$

n punctures on a torus.



$\omega_{n+1}\,\omega_{n+2} = -q$

$|\omega_{n+2}| = |q|^{1/2}$