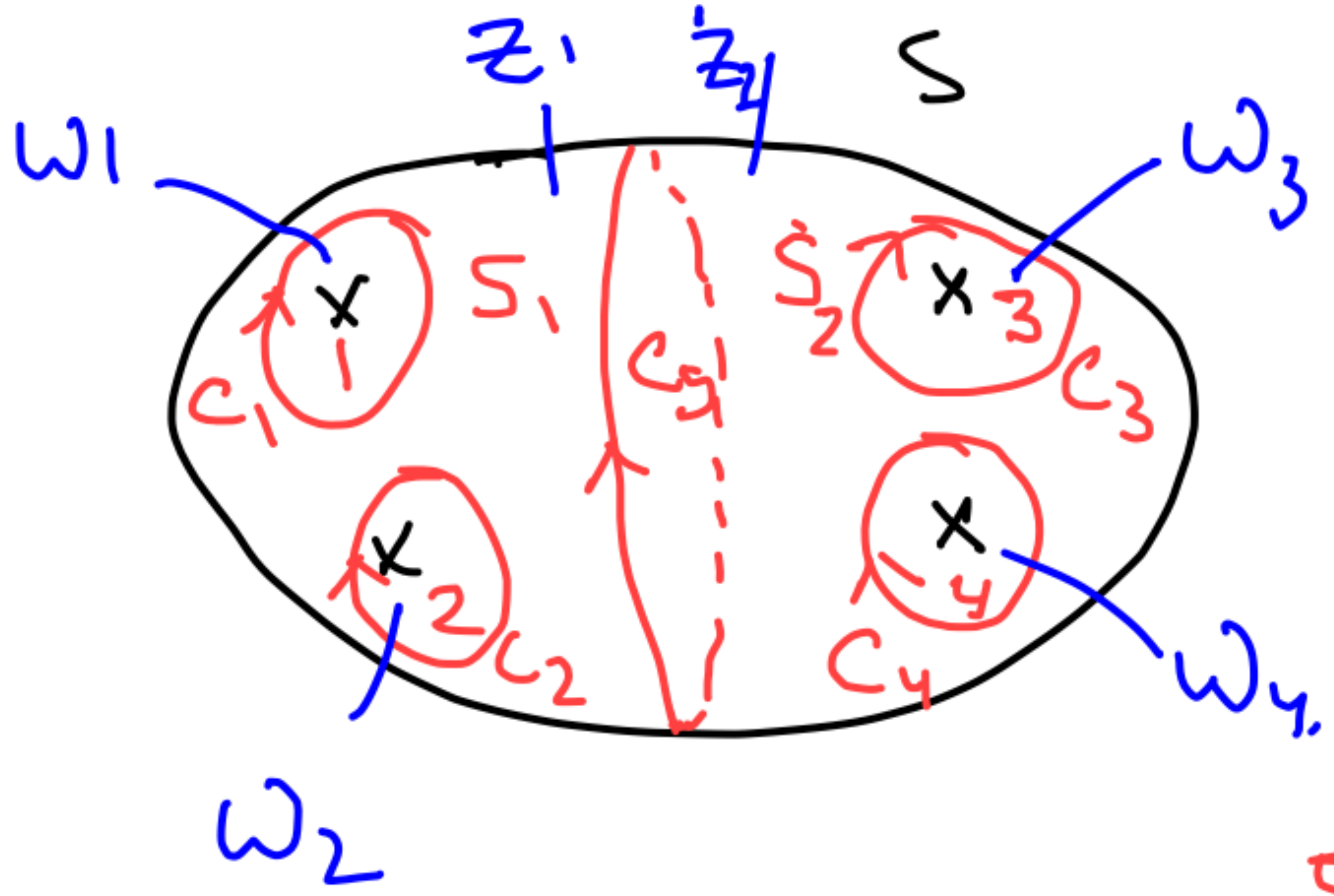
Application to 4 point amplitude of states $|V_1\rangle, \dots, |V_4\rangle$

$$\mathcal{M}_{0,4} = 2\text{-dimensional} \quad (6\overset{\overset{0}{\shortparallel}}{g} - 6 + 2\overset{\overset{4}{\shortparallel}}{n})$$

Task: Pick a section of $\mathcal{P}_{0,4}$

Any choice (almost) of the transition fn. $\{F_s\}$ parametrized by 2 parameters will do.



On C_1: $z_1 = F_1(w_1)$
on C_2: $z_1 = F_2(w_2)$
on C_3: $z_2 = F_3(w_3)$
on C_4: $z_2 = F_4(w_4)$

on C_5: $z_1 = -q/z_2$

q: complex parameter (2-real)

F_1, F_2, F_3, F_4 taken to be q independent (choice)

$$B_q = \sum_{s=1}^{5} \oint \frac{\partial F_s}{\partial q} b(\sigma_s)\, d\sigma_s + \sum_{s=1}^{5} \oint \frac{\partial \bar{F}_s}{\partial \bar{q}} \bar{b}(\bar{\sigma}_s)\, d\bar{\sigma}_s$$

$$= \oint_{c_5} \frac{\partial}{\partial q}\left(-\frac{q}{z_2}\right) b(z_1)\, dz_1 = \frac{1}{q} \oint_{c_5} z_1\, b(z_1)\, dz_1$$

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$$B_{\bar q} = \oint_{C_5} \bar q^{-1} \bar z_1 \bar b(\bar z_1)\, d\bar z_1$$

$$(-2\pi i)^{-(3g-3+n)}$$

$$A(V_1, V_2, V_3, V_4) = (-2\pi i)^{-1} g_s \int dq \wedge d\bar q$$

$$\Big\langle q^{-1} \oint_{C_5} b(z_1)\, z_1\, dz_1 \;\; \bar q^{-1} \oint_{C_5} \bar b(\bar z_1)\, \bar z_1\, d\bar z_1$$

$$F_1 \circ V_1(0)\;\; F_2 \circ V_2(0)\;\; \tilde F_3 \circ V_3(0)\;\; \tilde F_4 \circ V_4(0)\Big\rangle$$

$$\tilde F_3(\omega_3) = -q/F_3(\omega_3) \qquad \tilde F_4(\omega_4) = -q/F_4(\omega_4)$$

Consider on-shell states

$V_i = c\bar c W_i$ dim. (1,1) matter sector. primary.

$$A = (-2\pi i)^{-1} \times g_s \int dq \wedge d\bar{q} \; q^{-1} \bar{q}^{-1}$$

$$\left\langle \oint_{C_S} z_1 b(z_1)\, dz_1 \oint_{C_T} \bar{z}_1 \bar{b}(\bar{z}_1)\, d\bar{z}_1 \right.$$

$$c\bar{c} W_1(y_1) \quad c\bar{c} W_2(y_2) \quad c\bar{c} W_3\left(-\frac{q}{y_3}\right) c\bar{c} W_4\left(-\frac{q}{y_4}\right)$$

$$y_1 = F_1(0), \quad y_2 = F_2(0), \quad y_3 = F_3(0), \quad y_4 = F_4(0)$$

C_S encloses y_3 and y_4 in clockwise direction.

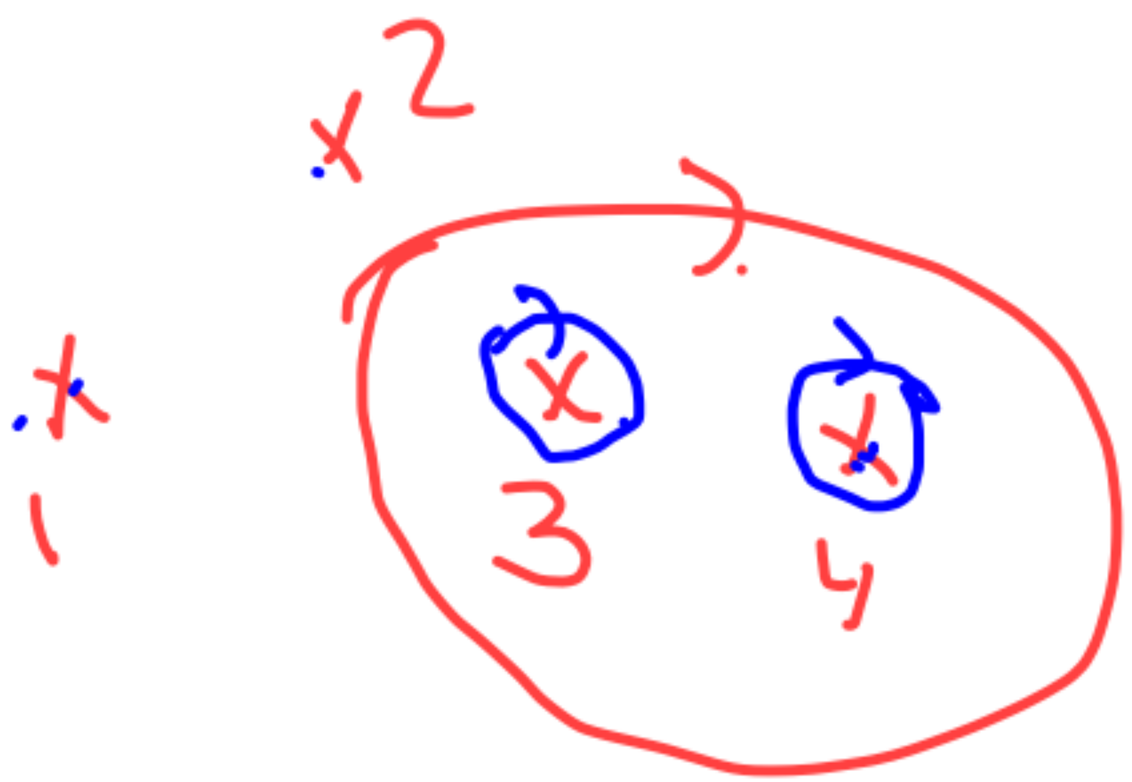
Simplify by taking $y_3 = \infty$, $y_4 = -1$

(Ex. Final result does not depend on these choices)

$$A = (-2\pi i)^{-1} g_s \int dq \wedge d\bar q \; q^{-1} \bar q^{-1} \qquad \bar b(\bar z_1)\, \bar c(\bar q) = \frac{1}{\bar z_1 - \bar q}$$

$$\Big\langle \oint_{C_5} z_1\, b(z_1)\, dz_1 \oint_{C_5} \bar z_1\, \bar b(\bar z_1)\, d\bar z_1$$

$$c\bar c W_1(y_1)\; c\bar c W_2(y_2)\; c\bar c W_3(0)\; c\bar c W_4(q,\bar q)\Big\rangle$$



$$= (-2\pi i)^{-1} g_s \int dq \wedge d\bar q \; \cancel{q^{-1}}\, \cancel{\bar q^{-1}}$$

$$\Big\langle c\bar c W_1(y_1)\; c\bar c W_2(y_2)\; c\bar c W_3(0)$$

$$(-1)\, \cancel{q}\, \cancel{\bar q}\; W_4(q,\bar q)$$

$$= (2\pi i)^{-1} g_s \int dq \wedge d\bar q \Big\langle c\bar c W_1(y_1)\; c\bar c W_2(y_2)\; c\bar c W_3(0)$$

$$W_4(q,\bar q)\Big\rangle$$

General Structure of W:

$$e^{ik\cdot X} \times \text{pol. in } \partial^m X,\ \bar\partial^n X$$

$$\Downarrow$$

$$(z-z')^{-\text{integer}} \quad (\bar z-\bar z')^{-\text{integer}}$$

$$W_i(z)\ W_j(z')$$

$$\to (z-z')^{\frac{(k_i+k_j)^2}{4} - \frac{k_i^2}{4} - \frac{k_j^2}{4}} \ (\bar z-\bar z')^{\frac{(k_i+k_j)^2}{4} - \frac{k_i^2}{4} - \frac{k_j^2}{4}}.$$

$$e^{ik_i\cdot X(z)}\ e^{ik_j\cdot X(z')}$$

$$\sim e^{i(k+k')\cdot X(z')}$$

$$\sim |z-z'|^{k_i\cdot k_j}\ (z-z')^{-\text{integer}}\ (\bar z-\bar z')^{-\text{integer}}.$$

$\to$ blows up for $k_i\cdot k_j$ large and negative.

$\int d^2q \langle \cdots \rangle$ diverges for large -ve

$k_i \cdot k_j$

$\Rightarrow$ diverges for $q \to 0$ $(y_4 \to y_3)$

$\phantom{\Rightarrow \text{diverges for } q \to 0 \;} \overset{\shortparallel}{q} \quad \doteq 0$

Usual approach: Define the integral in the range of k_i's where it is well defined and then analytically continue to the rest of the region.

$\to$ works well in many cases.

e.g. 4-tachyon amplitude $(W_i = e^{ik_i \cdot X})$

$$A = \text{constant} \times \frac{\Gamma\left(-1-\frac{s}{4}\right)\Gamma\left(-1-\frac{t}{4}\right)\Gamma\left(-1-\frac{u}{4}\right)}{\Gamma\left(2+\frac{s}{4}\right)\Gamma\left(2+\frac{t}{4}\right)\Gamma\left(2+\frac{u}{4}\right)}$$

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_3)^2, \quad u = -(k_1+k_4)^2$$

However analytic continuation does not always work.

To understand when this happens we need to understand the origin of the divergences better.

General results: (1) Divergences in string theory amplitudes come from degenerate Riemann surfaces.

(2) All degenerate Riemann surfaces may be expressed as $q \to 0$ limit of some plumbing fixture variable.

We'll use this to classify degenerate Riemann surfaces in terms of Feynman diagrams.

① Take a (pair of) Riemann surface(s)

② Sew two punctures via $ww' = -q$ to generate a new Riemann surface.

$\Downarrow$

Parametrized by an extra complex parameter q

Goal: Express the integration measure over the moduli space of the new Riemann surface in terms of those on the original surfaces.

Extra factor: $dq \wedge d\bar{q} \;\; B_q B_{\bar{q}} \, (-2\pi i)^{-1}$

Recall that $\Omega_p^{(g,n)}$ has $(-2\pi i)^{-(3g-3+n)}$

Sewing: $(g_1, n_1) \oplus (g_2, n_2) \to (g_1+g_2, n_1+n_2-2)$

or $\quad (g, n) \to (g+1, n-2)$

Ex. In either case the Ω has an extra factor of $(-2\pi i)^{-1}$

$$\omega \omega' = -q \Rightarrow \omega = -\frac{q}{\omega'} \qquad \frac{\partial F_s}{\partial q} = -\frac{1}{\omega'} = \frac{\omega}{q}$$

$$F_s'''(\omega', q)$$

$$B_q = \oint_{|\omega| = |q|^{1/2}} \frac{\partial F_s}{\partial q} b(\omega)\, d\omega = \frac{1}{q} \oint_{|\omega| = |q|^{1/2}} \omega\, b(\omega)\, d\omega.$$

$$B_{\bar{q}} = \oint \bar{\omega}\, \bar{b}(\bar{\omega})\, d\bar{\omega}$$

Combine this with the plumbing fixture relation.

$$\frac{dq \wedge d\bar{q}}{(-2\pi i)} \phi_r(0)\, \phi_s(0) \langle \phi_r^c | \; | \phi_s^c \rangle \, q^{h_s} \bar{q}^{\bar{h}_s}$$

$B_q \; B_{\bar{q}}$

$$b(w) = \sum_n b_n w^{-n-2}$$

$$B_q = \frac{1}{q} \oint w\, b(w)\, dw = q^{-1} b_0$$

$$B_{\bar{q}} = \frac{1}{\bar{q}} \oint \bar{w}\, \bar{b}(\bar{w})\, d\bar{w} = \bar{q}^{-1} \bar{b}_0$$

$$q = r e^{i\theta}$$

$$\left(-\frac{1}{2\pi i}\right) (-2i)\, r\, dr\, d\theta \; r^{-2} \; \phi_r(0)\, \phi_s(0)$$

$$\langle \phi_r^c | b_0 \bar{b}_0 | \phi_s^c \rangle \; r^{h_s + \bar{h}_s} \; e^{i\theta(h_s - \bar{h}_s)}$$

Our goal: Understand the divergence from the $q \to 0$ regim. $q = re^{i\theta}$

$\Downarrow$

$r \to 0$ $\quad 0 \le r \le a$ $\quad |q| \le a$

$\searrow$ small no.

Recall: $\omega \omega' = -q$

$$\frac{\omega}{\sqrt{a}} \frac{\omega'}{\sqrt{a}} = -\frac{q}{a} \Rightarrow \text{new } q \quad 0 \le |q| \le a.$$

$\Downarrow$ new ω $\quad$ = new ω'

$\Rightarrow$ Called adding stubs.

q integral $\ni$

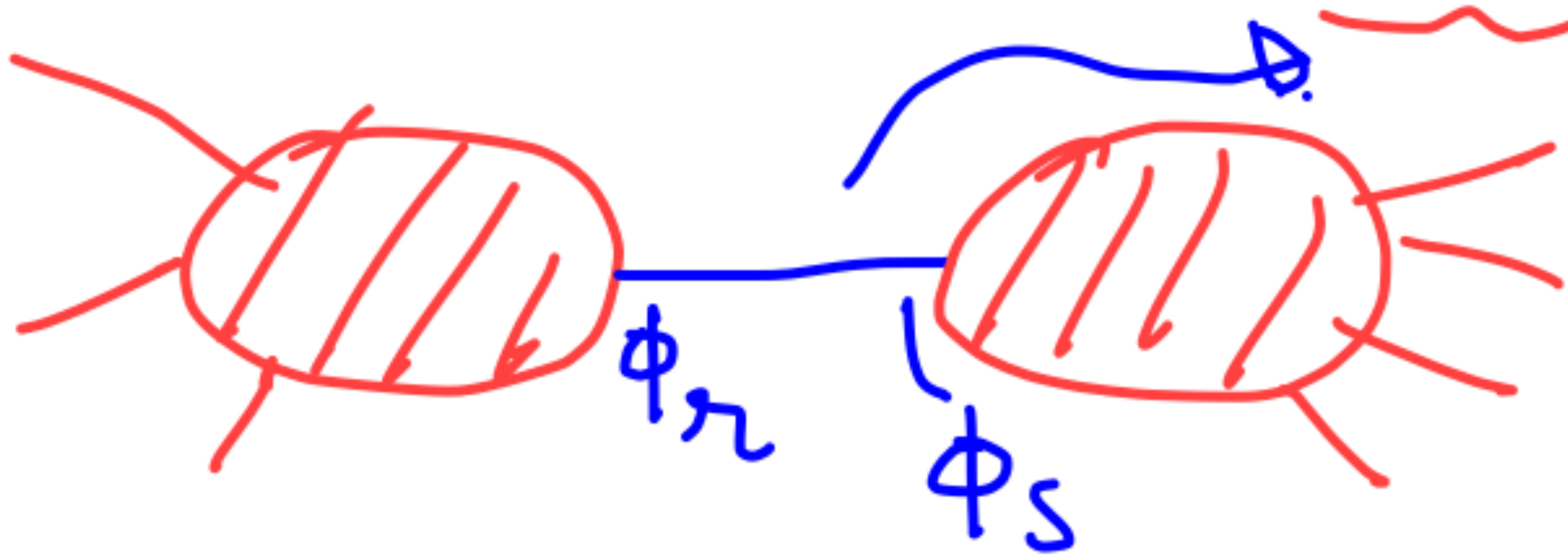
$$\frac{1}{2\pi}\times 2\int_0^1 dr \int_0^{2\pi} d\theta \; r^{h_s+\bar h_s+1-2}\, e^{i\theta(h_s-\bar h_s)}$$

$(h_s,\bar h_s)$: $(L_0,\bar L_0)$ e.v. of $|\phi_s\rangle$ & of $|\phi_s^c\rangle$

$$= \delta_{h_s,\bar h_s}\,\frac{2}{h_s+\bar h_s} \quad \text{for } h_s+\bar h_s>0.$$

Effect of plumbing fixture is to insert:

$$\underbrace{\phi_r(0)\,\phi_s(0)\,\langle\phi_r^c|\, b_0\bar b_0\, \frac{2}{L_0+\bar L_0}\,\delta_{L_0,\bar L_0}\,|\phi_s^c\rangle}_{\text{propagator}}$$



$$\frac{2}{\frac{k^2}{4}+\frac{k^2}{4}+N} = \frac{4}{k^2+\underbrace{2N}_{m^2}}$$

Origin of the divergence

$q = r e^{i\theta}$ $\quad q \to 0$ corresponds to $r = 0$

$$\int_0^1 dr\, r^{h_s + \bar{h}_s - 1} = \frac{1}{h_s + \bar{h}_s}$$

$r = e^{-t} \to$ Schwinger parameter representing the propagator.

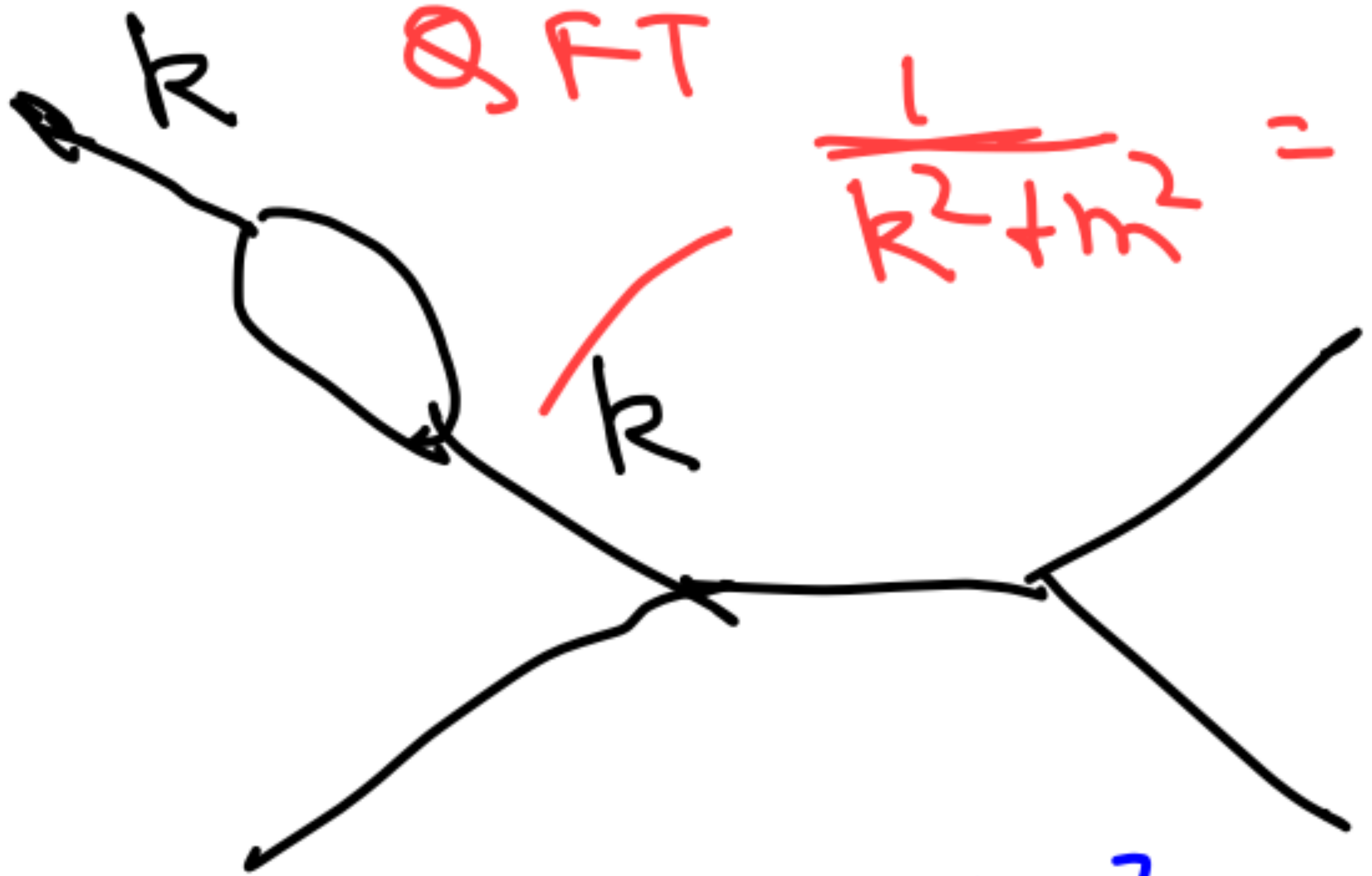
- identity if $h_s + \bar{h}_s > 0$ (both sides are finite)
- if $h_s + \bar{h}_s < 0$, r.h.s. is well defined by the left hand side diverges.

Using r.h.s. $\equiv$ analytic continuation of lhs
$\to$ use the Feynman diagram interpretation.

$$\int_0^1 dr\, r^{h_s+\bar{h}_s-1} = \frac{1}{h_s+\bar{h}_s} \sim \frac{4}{k^2+M^2}$$

For $h_s+\bar{h}_s=0$ both sides diverge.
$\rightarrow$ analytic continuation not useful.
From the Feynman diagram perspective
$h_s+\bar{h}_s=0 \Rightarrow$ on-shell internal propagator.
We need to invoke QFT insight
to deal with these cases.

QFT



$\frac{1}{k^2+m^2} = \frac{1}{0}$ by on-shell condition.

String theory

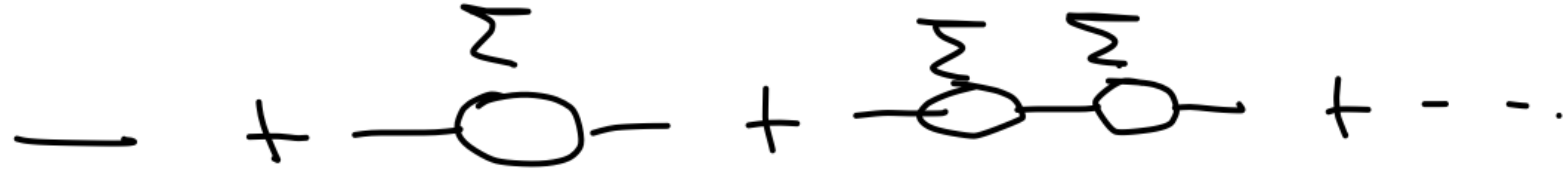
$q \to 0$ limit produces a divergence that cannot be tamed by analytic continuation

$$\int \frac{d^2q}{|q|^2} |q|^{(k^2+m^2)} \cdots = I$$

$= \infty$ for $k^2+m^2=0$

& cannot be analytically continued.

Remedy in QFT:

① Resum: 

$$= \frac{1}{k^2+m^2} + \frac{\Sigma(k)}{(k^2+m^2)^2} + \cdots$$

$$= \frac{1}{k^2+m^2-\Sigma(k)}$$

② Look for pole position: Mass renormalization.

③ Apply LSZ to calculate S-matrix.

Step 1: Organize the off-shell amplitudes into sum over Feynman diagrams. → to be described in detail

Step 2 Follow the usual QFT techniques to deal with on-shell internal propagators.

→ will not be discussed since the procedure is the same as in QFT.

Note: Since all divergences in string theory come from $q \to 0$ limit of some plumbing fixture variable

$\updownarrow$

On-shell propagator.

$\Rightarrow$ there are no UV divergences of the kind we encounter in QFT.

Step 1. Begin by specifying off-shell 3 point fn.

$\mathcal{M}_{0,3}$: 0-dimensional $\quad 6g-6+2n$

Section of $\mathcal{P}_{0,3}$ $\Rightarrow$ choose some local coordinate at the 3 punctures.

$$z = f_i(w_i) \qquad i = 1, 2, 3.$$

$$A(V_1, V_2, V_3) = \langle f_1 \circ V_1(0)\; f_2 \circ V_2(0)\; f_3 \circ V_3(0) \rangle$$

Constraint: Choose the f_i's such that $A(V_1, V_2, V_3)$ is invariant (up to sign) under permutations of V_1, V_2, V_3.

Two ways to implement this.

① Given any permutation P of $1, 2, 3$ there should be an $SL(2, \mathbb{C})$ trs. h s.t. $h(f_i(z)) = f_{P(i)}(z)$

→ may be too restrictive.

② Take

$$A(V_1, V_2, V_3) = \frac{1}{3!}\Big[\langle f_1 \circ V_1(0)\ f_2 \circ V_2(0)\ f_3 \circ V_3(0) \rangle$$

$$+ \text{permutations of } f_1, f_2, f_3 \Big]$$

$P_{0,3}$

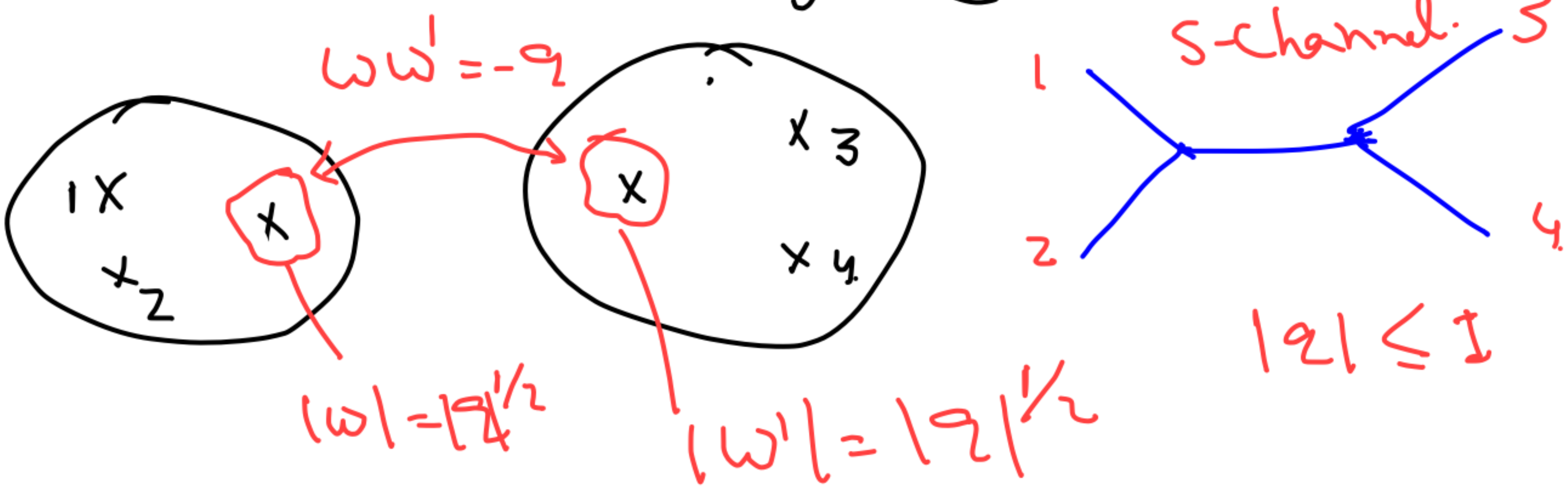
S_1
S_2
$\vdots$
S_6

$$\frac{1}{6}(S_1 + S_2 + \cdots S_6)$$

$\parallel$

Generalized section.

Once we have chosen f_1, f_2, f_3, we construct a family of 4-punctured sphere by plumbing fixture

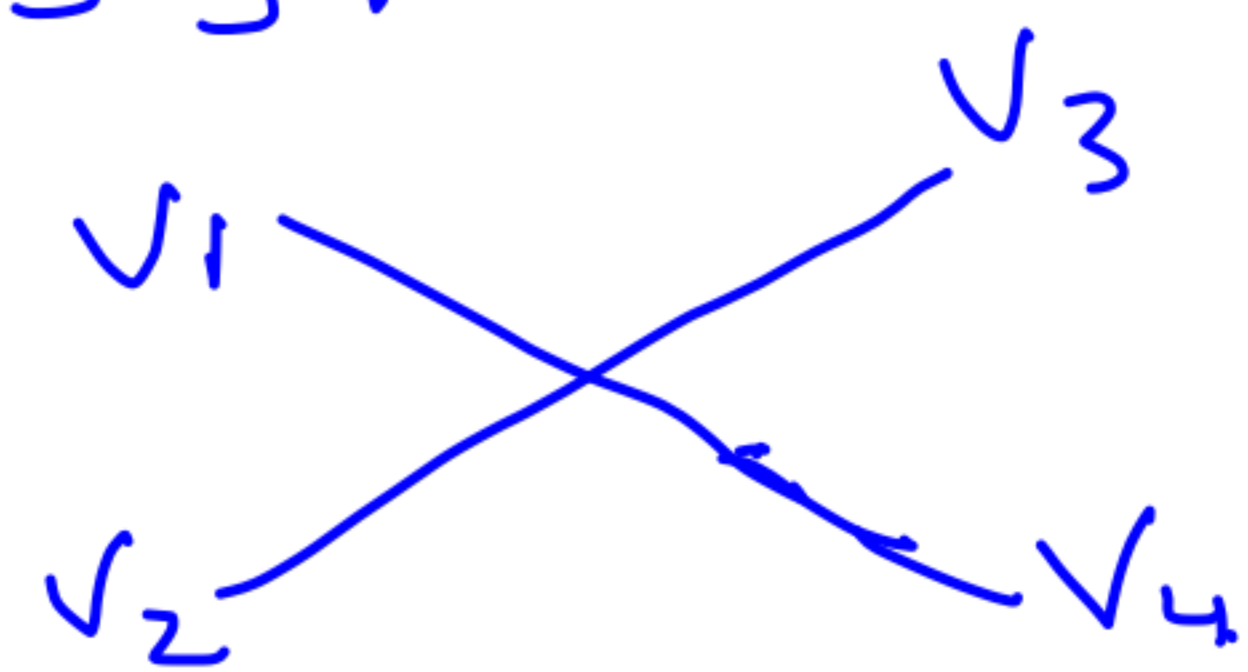


$\Rightarrow$ a family of 4-punctured spheres together with local coordinates at the punctures

$\rightarrow$ segment of a section in $\mathcal{P}_{0,4}$

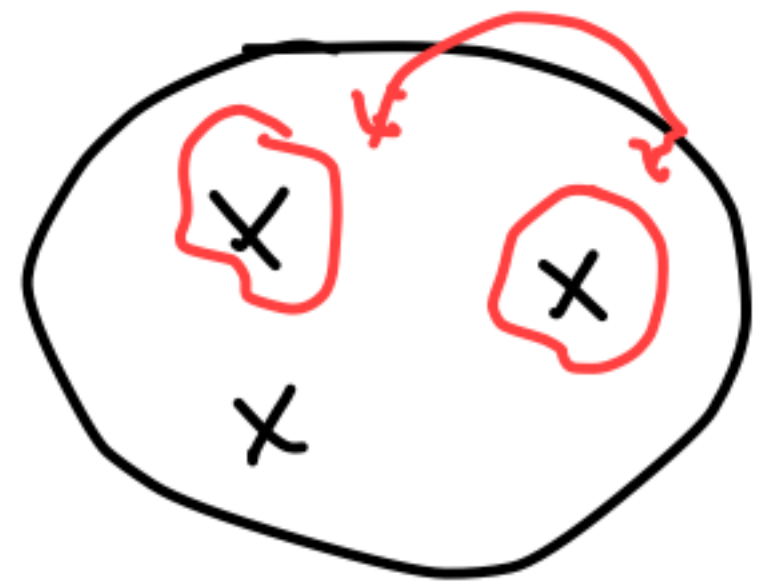
$$\longrightarrow \mathcal{M}_{0,4}$$

$S'_{0,4}$ is actually connected 2-d. segment of the section.



$$\to \; g_s \int_{S'_{0,4}} \Omega^{(0,4)}_2 (V_1, V_2, V_3, V_4)$$

$S'_{0,4} \to$ choose s.t. it is symmetric under exchanges of V_1, V_2, V_3, V_4.



 $\rightarrow$ give the rest.

One loop "counterterm" $\Rightarrow$ needed for gauge invariance and not UV finiteness.