Integrable complexity: Hofstadter Butterfly and Representation Theory

P. Wiegmann,

University of Chicago

Based on works with A. Zabrodin, A. Abanov, J. Talstra

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Almost Mathieu equation (aka Harper equation, Hofstadter problem)

$$\boxed{\psi_{n+1} + \psi_{n-1} + 2\lambda\cos(k + 2\pi n\Phi)\,\psi_n = E\psi_n}$$

One of the most celebrated problem of the spectral theory, with applications to localization theory, quasicrystals, chaos (kicked rotator), quantum Hall effect, etc.

Incomplete list of works:

Before 1990:

Zak 1964, Azbel 1964, Hofstadter 1976, Wannier 1978, Aubry-Andre 1980, Zak-Avron 1985, Bellissard-Simon 1980-1990, Thouless-Kohmoto 1982, Wilkinson 1987

After 1990:

Fadeev-Kashaev, Hatsugai-Kohmoto, Jitomirskaya, Last, Avila

- If the flux is a rational number $\Phi = P/Q$, the spectrum consists of Q bands (absolutely continuous): $E_m(k,k'),\ m = 1,\dots,Q$

$$\psi_{n+1}+\psi_{n-1}+2\lambda\cos(k+2\pi n\Phi)\,\psi_n = E\psi_n$$

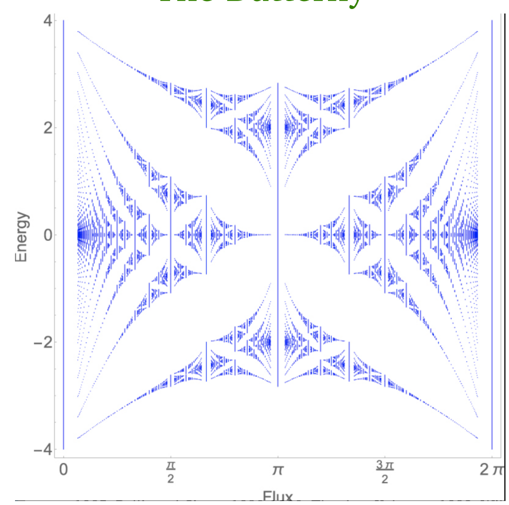
$$\text{Flux: } \Phi = \frac{P}{Q}$$

- $\psi_n = \psi_{n+Q}e^{iQk'}$ (k' is Flouquet parameter)

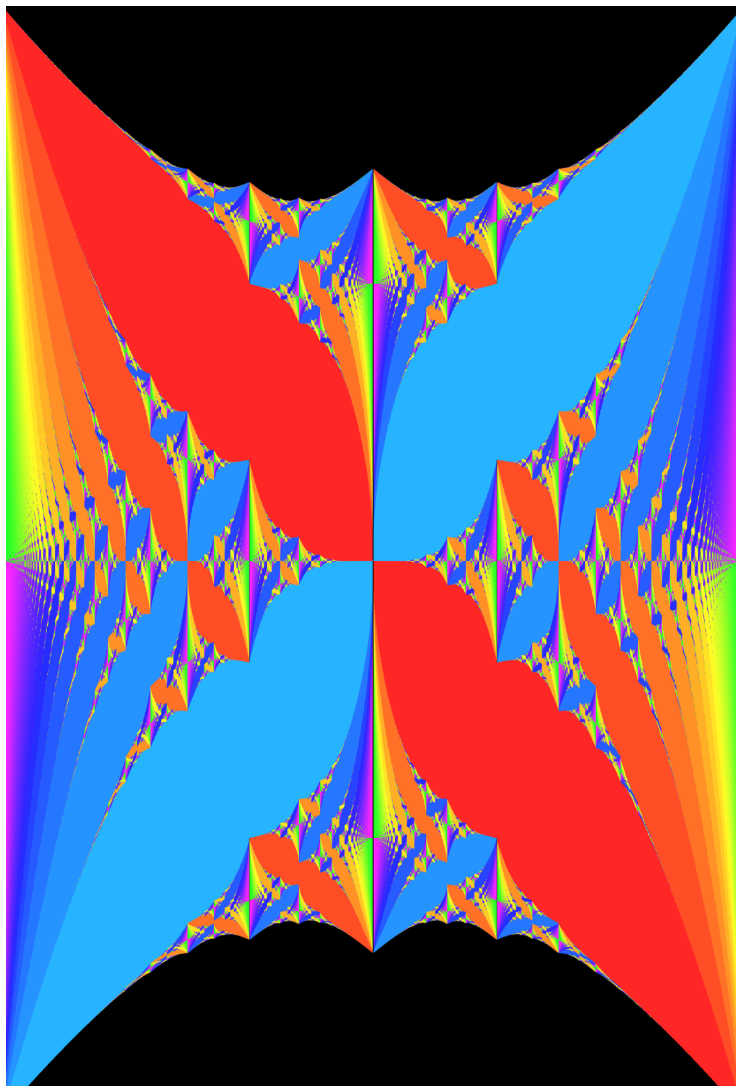
The spectrum is symmetric (Andre-Aubry)

$$E(k,k') \underset{k\to k',\ \lambda\to\lambda^{-1}}{=} E(k',k)$$

The Butterfly



The Butterfly



$\boxed{\text{If } \Phi = \text{irrational}}$

- $\lambda > 1$: the spectrum is an infinite pure point set: an insulator: all bands reduce to isolated points - *Anderson localization*, an insulator;
- $\lambda < 1$: the spectrum consists of infinitely many bands (*absolutely continuous*), a metal;

$$\text{the total bandwidth}: \quad 4|\lambda - 1| \rightarrow \textit{zero}$$

- $\lambda = 1$: the spectrum is a peculiar Cantor-type set - *singular continuous*

  An uncountable set without isolated points but with zero measure. Neither a metal nor insulator

$$\boxed{\lambda = 1}$$

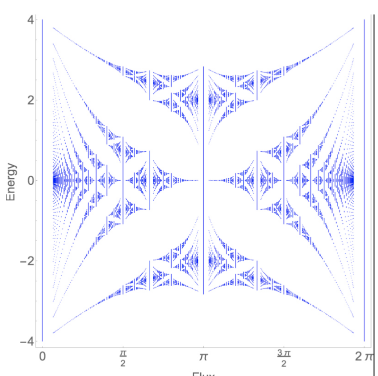
- Almost Mathieu equation is related to the cyclic representation of $U_q(sl(2))$ (a quantum deformation of $sl(2)$)
- Scaling hypothesis and Hierarchical structure of the spectrum (topology of the set)

- Generations: A specially chosen sequence of rational approximants P_j/Q_j with increasing Q_j to an irrational flux Φ so that

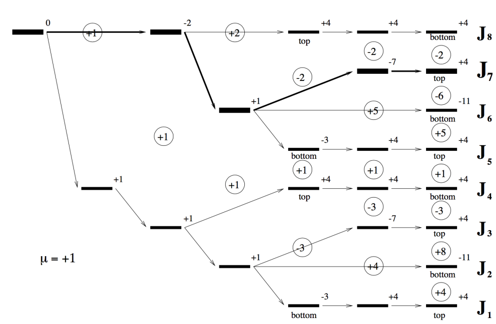
$$|\frac{P_j}{Q_j} - \Phi| < \text{const } Q_j^{-2}$$

- Parent and daughter bands: Connect the k-th band of the generation j (the daughter generation) to a certain band k' of a certain previous (parent) generation $j' < j$
- Spectrum of scaling dimensions The energies $E_k^{(\mathbf{J})}$ of a branch $\mathbf{J}$ of the tree form a sequence converging to the point $E^{(\mathbf{J})}$ of the spectrum in such a way that the sequence

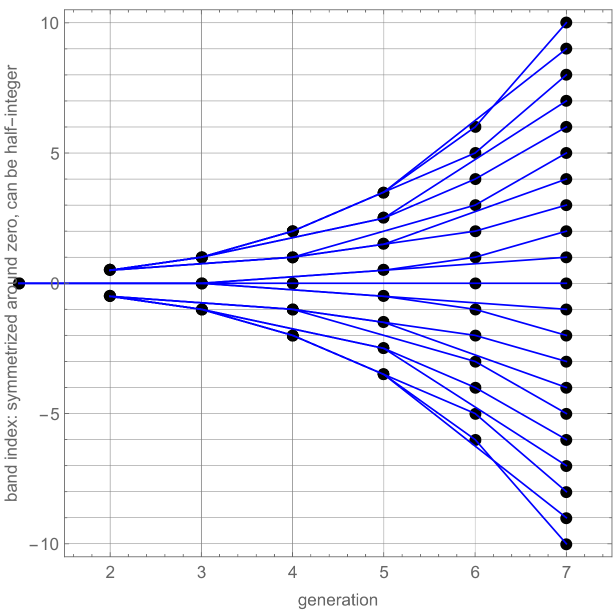
$$\boxed{|E_j^{(\mathbf{J})} - E^{(\mathbf{J})}| \sim Q_j^{-2+\epsilon_{\mathbf{J}}} \quad \text{is bounded but does not converge to zero}}$$



The Tree: $\Phi = \frac{1}{2}(\sqrt{13} - 3) = [3, 1, 3]$



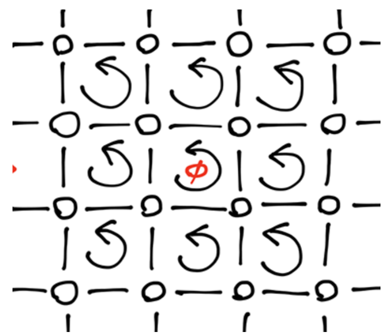
FIBONACCI TREE: $\Phi = \frac{1}{2}(\sqrt{5}-1)$



Known scaling dimensions of

$$\epsilon^{\text{uppermost}} = -0.374$$
$$\epsilon^{\text{central}} = +0.171$$

Bloch electrons in magnetic field: Group of magnetic translations



- Lattice electrons in magnetic field

$$\sum_{\mathbf{m}=\mathbf{n}\pm 1} t_{\mathbf{nm}}\psi_{\mathbf{m}} = E\psi_{\mathbf{n}}, \quad |t_{\mathbf{nm}}| = 1$$

$$\prod_{\text{plaquette}} t_{\mathbf{nm}} = e^{\mathrm{i}2\pi\Phi} := q^2$$

- Magnetic translation $\qquad T_{\mathbf{n}}T_{\mathbf{m}} = q^{-\mathbf{n}\times\mathbf{m}}T_{\mathbf{n}+\mathbf{m}}$

$$\boxed{H = T_x + T_x^{-1} + T_y + T_y^{-1}}$$

- Landau gauge: $\qquad T_x|\mathbf{n}\rangle = |\mathbf{n}+\mathbf{1_x}\rangle,\ T_y = q^2, \quad \psi_{\mathbf{n}} = e^{\mathrm{i}k' n_y}\psi_{n_x}(k)$

$$\boxed{\psi_{n+1} + \psi_{n-1} + 2\cos\left(k + 2\pi n\Phi\right)\psi_n = E\psi_n}$$

Hall conductance: Chern numbers

Hall conductance or the First Chern number - the topological characteristic of the spectrum

$$\sigma_m - \sigma_{m-1} = \frac{1}{2\pi i} \oint_{(k,k')} \psi_m^* d\psi_m$$

The Hall conductance σ_m of the m-th gap is the solution of the Diophantine equation (Thouless)

$$\boxed{P\sigma_m = m \pmod{Q}}$$

Example: $\frac{P}{Q} = \frac{4}{15}$,

$$\sigma_m = 4, -7, -3, 1, 5, -6, -2, 2, 6, -5, -1, 3, 7, 4$$

$$\boxed{\psi_{n+1} + \psi_{n-1} + 2\cos\left(k + 2\pi n\Phi\right)\psi_n = E\psi_n}$$

Equations for the mid band energy $k, k' = 0$,

$$q = e^{\mathrm{i}\pi\Phi}, \qquad \Phi = \frac{P}{Q}$$

$$E = 2\,(-1)^P \sin\left(\pi\Phi\right)\sum_{l=1}^{Q-1} z_l$$

Roots $z_1, \ldots z_{Q-1}$ obeys the Bethe Ansatz equations

$$\boxed{\frac{z_l^2+q}{q\,z_l^2+1} = (-1)^P\ \prod_{m\neq l}^{Q-1}\frac{q\,z_l - z_m}{z_l - q z_m}}$$

Polynomials

$$\Psi(z) = \prod_{l=1}^{Q-1}(z - z_l)$$

play a special role. Some of them have names: $\Psi_{E=0}(z)$ - is q-Legendre polynomial.

The wave function

$$\psi_n = \sum_{m=1}^{Q-1} c_{nm} \Psi(z)\Big|_{z=q^m}$$

The coefficients are *quantum di-logarithms*

$$c_{nm} = q^{2nm+\frac{m}{2}} \prod_{j=0}^{m-1} \frac{1+q^{-j-\frac{1}{2}}}{1-q^{j+\frac{1}{2}}}$$

- The Bethe Ansatz is equivalent to a Heisenberg spin chain on only two sites but with large spin equal to the number of bands $Q-1$

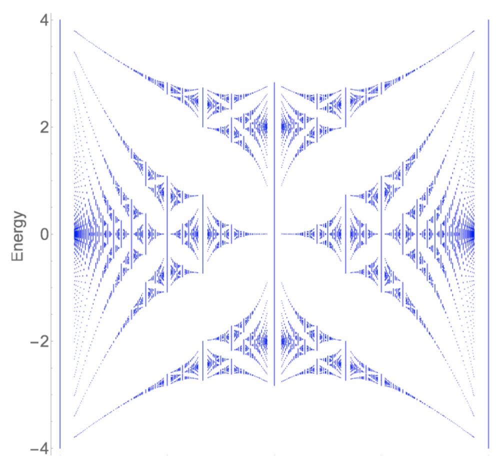
$$\boxed{\frac{z_l^2+q}{qz_l^2+1} = (-1)^P \prod_{m\neq l}^{Q-1} \frac{qz_l-z_m}{z_l-qz_m}}$$

✓ How to obtain these equations?

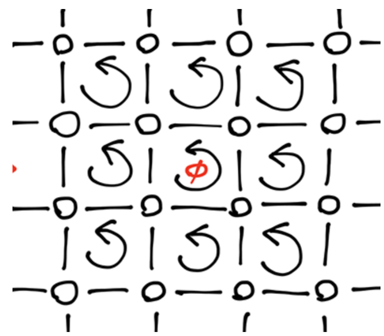
- How to solve them in the limit $Q \to \infty$, $P \to \infty$, $\Phi(=P/Q) \to$ irrational

✓ How to construct the hierarchical tree?

How to compute the dimensions ϵ^{J}? - Analytically unclear, limited numerical results



Lattice electrons in magnetic field



- $$\sum_{\mathbf{m}=\mathbf{n}\pm 1} t_{\mathbf{nm}}\psi_{\mathbf{m}} = E\psi_{\mathbf{n}}, \quad |t_{\mathbf{nm}}| = 1$$
$$\prod_{\text{plaquette}} t_{\mathbf{nm}} = e^{\mathrm{i}2\pi\Phi} := q^2$$
- Magnetic translation $T_{\mathbf{n}}T_{\mathbf{m}} = q^{-\mathbf{n}\times\mathbf{m}}T_{\mathbf{n}+\mathbf{m}}$

$$\boxed{H = T_x + T_x^{-1} + T_y + T_y^{-1}}$$

- Landau gauge: $T_x|\mathbf{n}\rangle = |\mathbf{n}+\mathbf{1_x}\rangle,\ T_y = q^2, \quad \psi_{\mathbf{n}} = e^{ik' n_y}\psi_{n_x}(k)$

$$\boxed{e^{ik'}\psi_{n+1} + e^{-ik'}\psi_{n-1} + 2\cos\left(k + 2\pi n\Phi\right)\psi_n = E\psi_n}$$

- Chiral gauge $\quad t_{\mathbf{n},\mathbf{n}+\mathbf{1_x}} = e^{-\mathrm{i}\frac{\Phi}{2}n_+}, \quad t_{\mathbf{n},\mathbf{n}+\mathbf{1_y}} = e^{+\mathrm{i}\frac{\Phi}{2}(n_++1)}, \quad n_+ = n_x + n_y, \quad \Psi_{\mathbf{n}} = e^{\mathrm{i}\mathbf{k}\mathbf{n}}\Psi_{n_+}$

$$\boxed{\mathrm{i}q^{-1/2}(1+q^{2n+1})\Psi_{n+1} - \mathrm{i}q^{1/2}(1+q^{-2n+1})\Psi_{n-1} = E\Psi_n} \quad (k,k') = (\tfrac{\pi}{2},\tfrac{\pi}{2}),\ q^2 = e^{i\Phi}.$$

- What is the advantage this gauge?

  Consider a difference equation, such that $\Psi_n = \Psi(z)\Big|_{z=q^n}$

$$i(z^{-1}+qz)\Psi(qz) - i(z^{-1}+q^{-1}z)\Psi(q^{-1}z) = E\Psi(z)$$

- A set of solutions of this difference equation are polynomials

$$\Psi(z) = \prod_{l=1}^{Q-1}(z-z_l)$$

- Comparing singularities we obtain equations for the roots

$$\frac{z_l^2+q}{qz_l^2+1} = (-1)^P \prod_{m\neq l}^{Q-1} \frac{qz_l - z_m}{z_l - qz_m}$$

- Q: When a class of solutions of the 2nd order ODE are polynomials?

$$H\Psi = \left[a(z)\frac{d^2}{dz^2} + b(z)\frac{d}{dz} + c(z)\right]\Psi(z) = E\Psi(z)$$

A: If the operator is equivalent to the Euler top

$$H = \sum_{i,j=1,2,3} \alpha_{ij} S_i S_j + \sum_{i=1,2,3} \beta_i S_i$$

where

$$S_3 = z\frac{d}{dz} - j,\ \ S_+ = z\left(2j - z\frac{d}{dz}\right),\ \ S_- = \frac{d}{dz}$$

are finite dimension representation of $SL(2)$ (A. Turbiner 1988)

- Q: When a class of solutions of the difference equation are polynomials ?

$$a(z)\Psi(q^2z)+d(z)\Psi(q^{-2}z)+\mathrm{v}(z)\Psi(z)=E\Psi(z)$$

$$\Psi(z)=\prod_l(z-z_l)$$

Setting $z=q^n$ we obtain solvable discrete equation

$$a_n\psi_{n+1}+d_n\psi_{n-1}+\mathrm{v}_n\psi_n=E\psi_n$$

- Lie group → quantum deformation $SL(2)\to U_q(SL(2))$ (A. Zabrodin & P.W.)

$$\{\mathbf{1},\,S_+,\,S_-,\,S_3\}\to\{A,\,B,\,C,\,D\}$$

$$\begin{aligned}[S_3,\,S_\pm]&=\pm S_\pm,\\ [S_+,\,S_-]&=S_3\end{aligned}\qquad\Longrightarrow\qquad\begin{aligned}&AB=qBA,\ BD=qDB,\\ &DC=qCD,\ CA=qAC,\\ &AD=1,\ [B,C]=\frac{A^2-D^2}{q-q^{-1}}\end{aligned}$$

$U_q(SL(2))$

- Universal R-matrix, obeying Yang-Baxter equation

$$R(u) = \begin{bmatrix} \frac{uA - u^{-1}D}{q - q^{-1}} & C \\ B & \frac{uD - u^{-1}A}{q - q^{-1}} \end{bmatrix}$$

Magnetic translations embedded into $U_q(SL(2))$

- Hamiltonian happens to be equal

$$H = T_x + T_{-x} + T_y + T_{-y} = B + C$$

- Embedding

$$\boxed{T_{\mathbf{n}} T_{\mathbf{m}} = q^{-\mathbf{n}\times\mathbf{m}} T_{\mathbf{n}+\mathbf{m}}}$$

$$\begin{aligned} &AB = qBA,\ BD = qDB,\\ &DC = qCD,\ CA = qAC,\\ &AD = 1,\ [B,C] = \frac{A^2 - D^2}{q - q^{-1}} \end{aligned}$$

$$\begin{aligned} T_{-x} + T_{-y} &= B, & T_x + T_y &= C,\\ T_{-y} T_x &= q^{-1} A^2, & T_{-x} T_y &= q D^2 \end{aligned} \tag{1}$$

Hierarchical tree

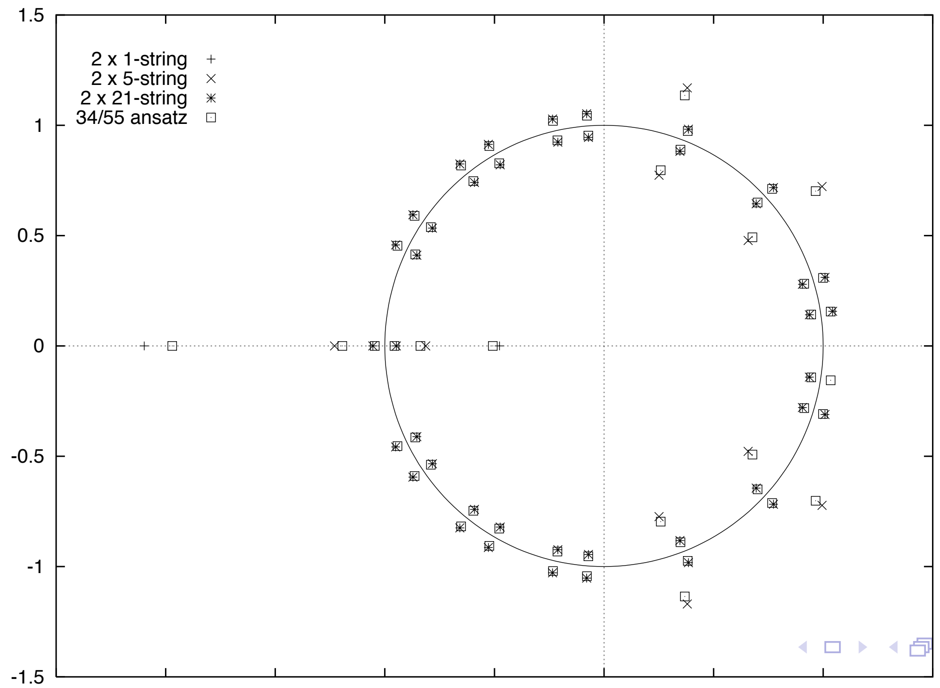
Is it possible to solve the Bethe Ansatz equations

$$\boxed{\frac{z_l^2+q}{qz_l^2+1} = (-1)^P \prod_{m\neq l}^{Q-1} \frac{qz_l - z_m}{z_l - qz_m}}$$

- At large Q solutions consist of collections of *strings*

A string of spin l centered at x_l is a set roots of unity $z^{(l)} = x_l \times \{e^{i\pi k/l}\}, \quad k = 1, \ldots, l$

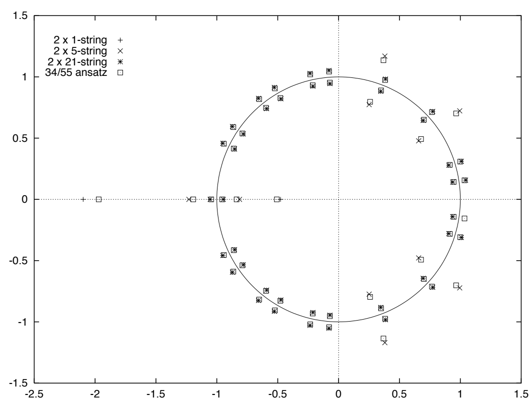
$$\Phi = 34/55$$



- The length of the longest string of a given band is the **Hall conductance** of the band:

$$(2l+1)_{\max} = |\text{Chern number}| = |\sigma(m)|$$

$$P\sigma_m = m \ (\text{mod } Q), \quad \sigma(m) = \sigma_m - \sigma_{m-1}$$

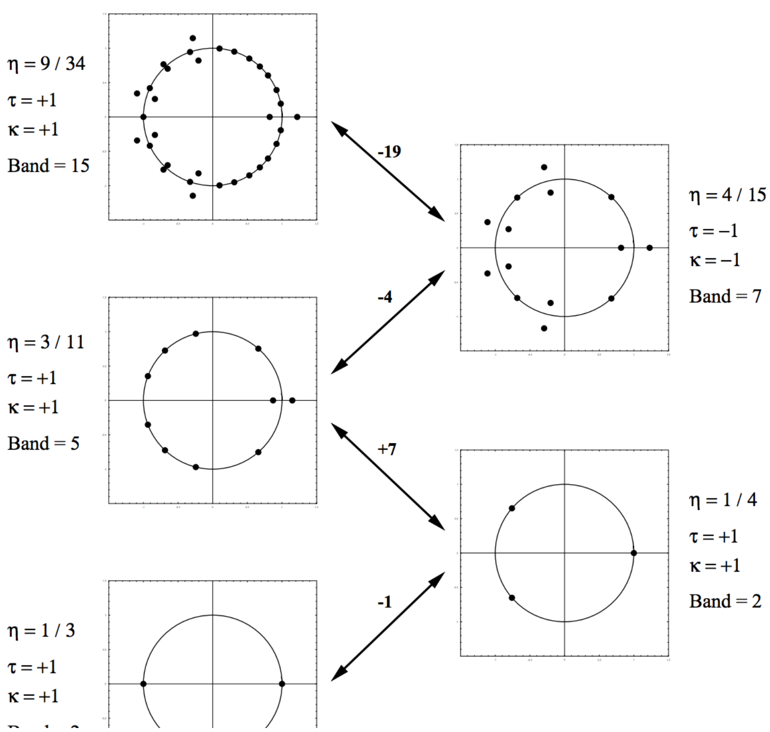


- Each solution is labeled by a content of strings $\{l_j, l_{j-1}, \ldots\}$
- The length of the longest string of a given band is the **Hall conductance** of the band:

$$(2l+1)_{\max} = |\text{Chern number of the band}| = |\sigma(k)|$$

- The remaining roots of the state is a solution of the Bethe equation for the parent state

$$\Psi^{\text{daugther}}(z) \approx \prod_{m=-l}^{l} (z - x_l q_l^m) \Psi^{\text{parent}}(z)$$



Fibonacci Tree: Example

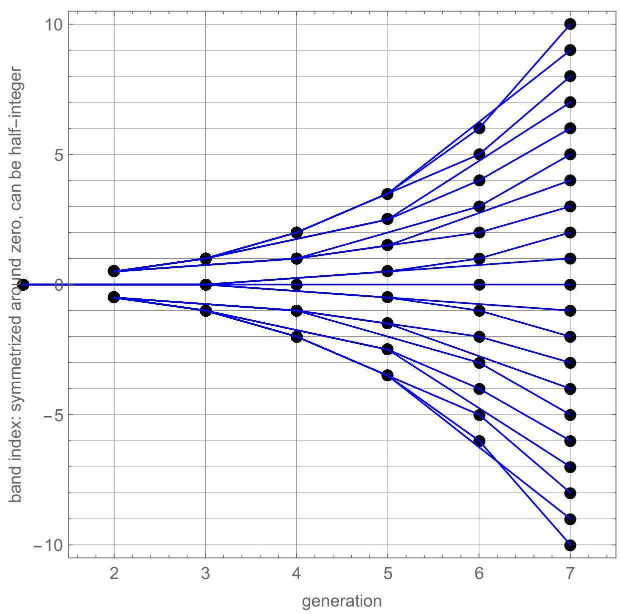
- Golden mean $\Phi = \frac{\sqrt{5}-1}{2}$
- The sequence of rational approximants is given by ratios of subsequent Fibonacci numbers

$$\Phi_i = \frac{F_{i-1}}{F_i} : \qquad F_i = F_{i-2} + F_{i-1} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

- The set of Hall conductances (lengths of strings) are again Fibonacci numbers: F_{k-1}. The wave function of this state is

$$\Psi\left(z|\Phi_k = \frac{F_{3k-1}}{F_{3k}}\right) \approx \prod_{n=0}^{k-1} \prod_{j=-\frac{1}{2}(F_{3n}-1)}^{\frac{1}{2}(F_{3n}-1)} \left(z - e^{i\pi \frac{F_{3n-1}}{F_{3n}} j}\right)^2$$

FIBONACCI TREE



$$\Phi_n = \frac{F_{n-1}}{F_n} \to \frac{1}{2}(\sqrt{5}-1)$$

Known scaling dimensions of

$$\epsilon^{\text{uppermost}} = -0.374$$
$$\epsilon^{\text{central}} = +0.171$$

V. Nabokov "Gift" Chapter 4

Truth bends her head to fingers curved cupwise;
And with a smile and care
Examines something she is holding there
Concealed by her from our eyes.

Увы! Что б ни сказал потомок просвещенный,
все так же на ветру, в одежде оживленной,
к своим же Истина склоняется перстам,

с улыбкой женскою и детскою заботой,
как будто в пригоршне рассматривая что-то,
из-за плеча ее невидимое нам.