Lecture 1

Goal: Derivation of ideal hydro equations of motion and MIS

Energy-momentum tensor

$$T^{\mu\nu} = \int dP \, p^\mu p^\nu f(\vec{p}) \qquad \text{(kinetic theory)}$$

$$dP = \frac{1}{p\cdot u}\frac{d^3\vec{p}}{(2\pi)^3}$$ u^μ = four velocity of LRF

$$\to \frac{1}{p^0}\frac{d^3\vec{p}}{(2\pi)^3} \text{ in LRF}$$ $u^\mu u_\mu = +1$, $g^{\mu\nu} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$

$\Rightarrow T^{\mu\nu} = T^{\nu\mu} \longrightarrow 16 \to 10$ independent components

$\Rightarrow$ If there is rotational symmetry, then only building blocks for $T^{\mu\nu}$ are u^μ and $g^{\mu\nu}$

$\Rightarrow$ Therefore, it must expressible as

$$T^{\mu\nu} = A g^{\mu\nu} + B u^\mu u^\nu$$

where A, B are Lorentz scalars

$\Rightarrow$ Since $u_\mu u_\nu T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3}(p\cdot u) f \equiv \epsilon$

and $u_\mu u_\nu T^{\mu\nu} = A + B \quad \Rightarrow \quad \epsilon = A + B$

$\Rightarrow$ Can introduce a transverse projector which projects components on a 4-vector orthogonal to u^μ

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$$

properties

$$u_\mu \Delta^{\mu\nu} = \Delta^{\mu\nu} u_\nu = 0$$

$$\Delta^{\mu\lambda}\Delta_\lambda{}^\nu = \Delta^{\mu\nu}$$

$$\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}T_{\mu\nu} = A\,\underbrace{\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}\,g_{\mu\nu}}_{\Delta_{\mu\alpha}\Delta^{\mu}{}_{\beta} = \Delta_{\alpha\beta}} = A\,\Delta_{\alpha\beta}$$

To fix A (Lorentz scalar), we can evaluate this in the LRF

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu} = \begin{pmatrix} 0 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\therefore \quad \Delta^{\mu}_{i}\Delta^{\nu}_{j}T^{\mu\nu} = -\delta^{ij}A \qquad i,j \in \{1,2,3\}$$

From kinetic theory in LRF

$$\Delta^{\mu}_{i}\Delta^{\nu}_{j}T^{\mu\nu} = \int dP\; \vec{p}_i\,\vec{p}_j\, f(\vec{p})$$

In equilibrium (ideal hydro) $f(\vec{p}) \to f(|\vec{p}|)$ due to isotropy and one has

$$\Delta^{\mu}_{i}\Delta^{\nu}_{j}T^{\mu\nu} = \delta^{ij}\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{p_i^2}{p^0}f}_{\text{pressure } P} = \delta^{ij}P$$

(iso so $P_x = P_y = P_z$)

Equating the two, we obtain

$$A = -P$$

using $\epsilon = A + B \quad \to \quad B = \epsilon + P$

$\Rightarrow$ Plugging these into our original decomposition

$$T^{\mu\nu} = -P\,g^{\mu\nu} + (\epsilon + P)\,u^{\mu}u^{\nu}$$

$$\boxed{T^{\mu\nu}_{ideal} = \epsilon\,u^{\mu}u^{\nu} - \Delta^{\mu\nu}P}$$

In LRF $\left(T^{\mu\nu}_{ideal}\right)^{LRF} = \begin{pmatrix} \epsilon & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{pmatrix}$

$$T^{\mu\nu}_{\text{ideal}} = (\epsilon + P) u^\mu u^\nu - P g^{\mu\nu}$$

Conservation of energy

$$\partial_\mu T^{\mu\nu}_{\text{ideal}} = 0$$

$$\partial_\mu T^{\mu\nu}_{\text{ideal}} = [\partial_\mu(\epsilon + P)] u^\mu u^\nu + (\epsilon+P)[(\partial_\mu u^\mu) u^\nu + u^\mu \partial_\mu u^\nu]$$
$$- \partial^\nu P$$

① Project with u^ν.

$$u_\nu \partial_\mu T^{\mu\nu}_{\text{ideal}} = D(\epsilon + P) + (\epsilon+P)\Theta + (\epsilon+P) u^\mu \underbrace{u_\nu \partial_\mu u^\nu}_{0}$$
$$- DP$$

where $\boxed{D \equiv u_\mu \partial^\mu}$ $\boxed{\Theta \equiv \partial_\mu u^\mu}$

$$\text{2D} \quad \boxed{\boxed{D\epsilon + (\epsilon + P)\Theta = 0}}$$

② Project with $\Delta_{\alpha\nu}$

$$0 + (\epsilon+P)\Delta_{\alpha\nu} D u^\nu - \nabla_\alpha P = 0$$

$$\boxed{\nabla_\alpha \equiv \Delta_{\alpha\nu} \partial^\nu}$$

$$\Delta_{\alpha\nu} D u^\nu = - u^\nu (D \Delta_{\alpha\nu}) \qquad \text{since } D(\Delta_{\alpha\nu} u^\nu) = 0$$

Expanding gives

$$\Delta_{\alpha\nu} D u^\nu = + D u_\alpha$$

$$\text{2D} \quad \boxed{\boxed{(\epsilon + P) D u_i - \nabla_i P = 0}} \qquad i = 1, 2, 3$$

Four equations. How many unknowns?

$$\epsilon, P, u_i \rightarrow 5$$

Need something else to complete the equations

2D EQUATION OF STATE $\quad T^{\mu}_{\mu} = \underbrace{\mathcal{I}(T)}_{\text{interaction measure}}$

aka trace anomaly

$$\text{2D} \quad \epsilon - 3P = \mathcal{I}(T)$$

For conformal systems $\epsilon = 3P$

$$\text{2D} \quad \boxed{\begin{aligned} D\epsilon + \tfrac{4}{3}\epsilon\Theta &= 0 \\ 4\epsilon D u_i - D_i \epsilon &= 0 \end{aligned}}$$

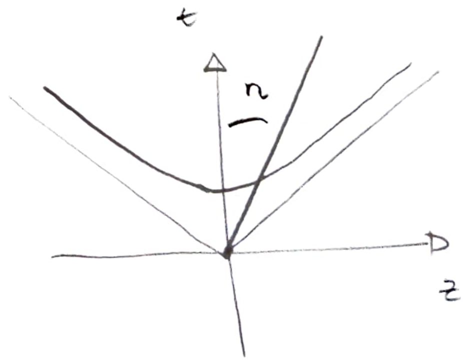
Bjorken Limit

Assume system is transversally homogeneous

$$\partial_{x,y}\epsilon = 0$$

and boost invariant $\quad \partial_\eta \epsilon = 0$

Milne Coordinates $(\hat{x})$



$$t = \tau \cosh\eta$$
$$z = \tau \sinh\eta$$

$$\hat{g}^{\mu\nu}_{\text{milne}} = \frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu} g_{\alpha\beta}$$

$$= \text{diag}(1, -1, -1, -1/\tau^2)$$

we assume variables only depend on propertime τ
$\therefore$ there can be no flow in the x, y or η directions

In t, z coordinates we have (Minkowski)

$$u^\mu = (\cosh\eta, 0, 0, \sinh\eta) = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

This is just a boosted static flow $\Lambda^{\mu\nu}_{\text{boost}}(\eta)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$$\Theta = \partial_\mu u^\mu = \frac{1}{\tau}$$

$$\hat{u}^\mu = \frac{\partial x^\alpha}{\partial \hat{x}^\mu} u_\mu = (1, 0, 0, 0)$$
$\tau \hat{u}_\eta$

$$2D \quad D = \hat{u}^\mu \hat{\partial}_\mu = \partial_\tau$$

$\therefore$ Hydro equations in ideal limit become

$$① \quad \partial_\tau \epsilon + \frac{4}{3}\frac{\epsilon}{\tau} = 0$$

$$② \quad 0 = 0$$

no flow, $\perp$ homogenous
+ boost invariant
$\partial_\eta \epsilon = 0$

Solution to ① is

$$\boxed{\epsilon = \epsilon_0 \left(\frac{\tau_0}{\tau}\right)^{4/3}}$$

In conformal system $\epsilon \propto T^4$

$$\therefore \quad \boxed{T = T_0 \left(\frac{\tau_0}{\tau}\right)^{1/3}}$$

* <u>EXERCISE</u>: what happens if $\frac{dP}{d\epsilon} = c_s^2 = \text{const.}$ in this limit?

what happens if we aren't in perfect thermal equilibrium?

Hydro formulation (1st order)

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}$$

10 DOF total, 4 (u_i, ϵ) in $T^{\mu\nu}_{\text{ideal}}$

$\hookrightarrow$ 6 independent components in $\Pi^{\mu\nu}$

Landau Frame $\quad u_\mu T^{\mu\nu} = \epsilon u^\nu$

$\therefore \quad u_\mu \Pi^{\mu\nu} = 0 \rightarrow$ 4 eqs

$\Pi^{\mu\nu} = \Pi^{\nu\mu} \rightsquigarrow 10 - 4 = 6$ DOF ✓

In LRF

$$\Pi^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \Pi^{ij} & \\ 0 & & & \end{pmatrix}$$

$\Pi^{ij} = \Pi^{ji} \rightarrow$ 6 independent components ✓

Shear/Bulk decomposition

$$\Pi^{\mu\nu} = \underbrace{\pi^{\mu\nu}}_{\text{shear tensor}} + \Delta^{\mu\nu}\underbrace{\Pi}_{\text{bulk viscous pressure}} \qquad \pi^{\mu}{}_{\mu} = 0$$

General EOMs become (EXERCISE)

$$D\epsilon + (\epsilon + P)\Theta - \pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$$

$$(\epsilon + P)Du^i - \nabla^i P + \Delta^i{}_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

Notation

$A_{(\mu}B_{\nu)} \equiv \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$

Constitutive relations (1st order) $\boxed{\text{"NAVIER-STOKES"}}$

$\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$

$\pi^{\mu\nu} = \eta \nabla^{\langle\mu}u^{\nu\rangle} = \eta\sigma^{\mu\nu}$	η = shear viscosity
$\Pi = \zeta \nabla_\mu u^\mu = \zeta\Theta$	ζ = bulk viscosity

$$\nabla^{\langle\mu} u^{\nu\rangle} = 2\nabla^{(\mu} u^{\nu)} - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha$$

- Symmetric and traceless
  - 1st is obvious
  - 2nd : $\nabla^{\langle\mu} u_{\mu\rangle} = 2\overbrace{\nabla^{(\mu} u_{\mu)}}^{\nabla^\mu u_\alpha = \theta} - \frac{2}{3}\underbrace{\Delta^\mu{}_\mu}_{3}\underbrace{\nabla_\alpha u^\alpha}_{\theta}$

$$= 2\theta - 2\theta = 0 \quad \checkmark$$

Can be made formal using four-index projector

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \Delta^{(\mu}_{\alpha}\Delta^{\nu)}_{\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$$

Resulting equations are parabolic $\longrightarrow$ ACAUSAL

$\Rightarrow$ Need an evolution equation for $\pi^{\mu\nu}$

Before that,
let's look in the simple case of conformal
Bjorken flow

- $\Pi = 0$
- By symmetry $\pi^{xx} = \pi^{yy} \equiv \pi/2$, $\pi^{ij} = 0 \quad i \neq j$
- By $\pi^\mu{}_\mu = 0$ $\quad \pi^{xx} + \pi^{yy} + \pi^{zz} = 0$

$$\therefore \pi^{zz} = -\pi$$

in LRF $\quad \pi^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \pi/2 & 0 & 0 \\ 0 & 0 & \pi/2 & 0 \\ 0 & 0 & 0 & -\pi \end{pmatrix}$

Longitudinal and transverse pressures

$$P_L = T^{zz}_{eq} + \pi^{zz} = P_{eq} - \pi$$

$$P_T = T^{xx}_{eq} + \pi^{xx} = P_{eq} + \pi/2 \quad \text{(same for yy)}$$

$$\frac{P_L}{P_T} = \frac{P_{eq} - \pi}{P_{eq} + \pi/2}$$

For Bjorken flow and 1st order

$$u^\mu = (\cosh\eta, 0, 0, \sinh\eta) = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

$$\eta = \operatorname{arctanh}\left(\frac{z}{t}\right) \qquad \tau = \sqrt{t^2 - z^2}$$

$$\pi^{\mu\nu} = \eta \nabla^{\langle\mu} u^{\nu\rangle} \qquad \nabla^{\langle\mu} u^{\nu\rangle} = 2\nabla^{(\mu}u^{\nu)} - \frac{2}{3}\Delta^{\mu\nu}\theta$$

$$\pi^{xx} = \eta\Big(2\nabla^{(x}u^{x)} - \frac{2}{3}\underbrace{\Delta^{xx}}_{-1}\underbrace{\theta}_{1/\tau}\Big) = \frac{2\eta}{3\tau} = \pi^{yy}$$

$$\pi^{zz} = -(\pi^{xx} + \pi^{yy}) = -4\eta/3\tau$$

$$\therefore \quad \pi = 4\eta/3\tau$$

$$\Rightarrow \quad \frac{P_L}{P_T} = \frac{P_{eq} - 4\eta/3\tau}{P_{eq} + 2\eta/3\tau} \qquad \lim_{\tau\to\infty} \frac{P_L}{P_T} \to 1$$

as τ decreases, $\frac{P_L}{P_T}$ decreases $\Rightarrow$ PRESSURE ANISOTROPY!

$\Rightarrow$ Longitudinal pressure will go negative as $\tau \to 0$, <u>break down of hydro</u>!

$\Rightarrow$ Let's <u>estimate</u> when this happens

$$\frac{4\eta}{3\tau} > P_{eq} \qquad \underbrace{P_{eq} + \epsilon_{eq}}_{4P_{eq}} = TS_{eq}$$

$$\frac{16\eta}{3\tau} > S_{eq}T$$

$$\boxed{\bar\eta \equiv \frac{\eta}{S}} \to \boxed{\tau < \frac{16\bar\eta}{3T}} \to \boxed{w \equiv \tau T < \frac{16\bar\eta}{3}}$$

breaks when

as $\bar\eta \to \infty$ or $T \to 0$ $\rightsquigarrow$ $\tau < \infty$ always broken!

as $\bar\eta \to 0$ or $T \to \infty$ $\rightsquigarrow$ $\tau < 0$ always OK

$\Rightarrow$ Foreshadowing in RTA $\quad \bar w \equiv \frac{\tau}{\tau_{eq}} = \frac{\tau T}{5\bar\eta} < \frac{16}{15}$

How do we go beyond 1^{st}-order Navier-Stokes?

$\Rightarrow$ Need evolution equation for $\pi^{\mu\nu}$ because NS form has instantaneous response to u^μ

$\Rightarrow$ Simplest model is to introduce a relaxation time equation by hand:

$$\tau_\pi D\pi^{\mu\nu} + \pi^{\mu\nu} = \eta D^{<\mu} u^{\nu>} = \eta \sigma^{\mu\nu}$$

$$\tau_\Pi D\Pi + \Pi = \zeta D_\mu u^\mu = \zeta\theta$$

$\Rightarrow$ Two new terms are 2^{nd}-order in gradients hence $\rightarrow$ 2nd order viscous hydro

$\Rightarrow$ Simple form above makes equations hyperbolic $\rightarrow$ restores causality, but is ad-hoc. [Still some restrictions on τ_{eq} though.]

$\Rightarrow$ In a systematic approach we need to collect all possible terms that could appear at 2nd order.

$\Rightarrow$ Can do this systematically using kinetic theory or symmetries (conformal + Weyl etc.

As an example (without derivation) in relaxation time approximation ($\partial_\mu f^\mu = -\frac{p \cdot u}{\tau_{eq}}(f - f_{eq})$) the resulting equations have the form

① $$D\pi^{<\mu\nu>} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\ (\mu}\omega^{\nu)\gamma} - \tau_{\pi\pi}\pi_\gamma^{\ <\mu}\sigma^{\nu>\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}$$

② $$D\Pi = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}$$

$\Rightarrow$ Above $\beta_\pi \equiv \eta/2$ and $\beta_\Pi = \zeta$. Other coefficients are new "transport coefficients"

$\Rightarrow$ First two terms on RHS are 1st order, rest are 2nd order in gradients

$\Rightarrow$ New structure called "vorticity tensor" appears

$$\omega^{\mu\nu} \equiv \frac{1}{2}\left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)$$

$\Rightarrow$ Note that beyond RTA other transport coeffs/terms can appear.

Bjorken Limit (RTA 2nd order hydro + conformal)

$$\partial_\mu T^{\mu\nu} \longrightarrow \quad \tau\,\partial_\tau \log \epsilon = -\frac{4}{3} + \frac{\pi}{\epsilon}$$

$$D\pi^{\langle\mu\nu\rangle} \longrightarrow \quad \partial_\tau \pi = \frac{4\eta}{3\tau\tau_\pi} - \beta_{\pi\pi}\frac{\pi}{\tau} - \frac{\pi}{\tau_\pi}$$

In RTA one has (correct "DNMR" result)

$$\tau_\pi = \tau_{eq} = 5\bar{\eta}/T$$

$$\beta_{\pi\pi} = \frac{38}{21}$$

* incomplete MIS has $\tau_\pi = \frac{6}{5}\tau_{eq}$ and $\beta_{\pi\pi} = \frac{4}{3}$

<u>Note:</u> Can solve these numerically.

Still find that P_L can go negative at early times despite having restored causility*

* still constraints on relation of τ_π + η