Lecture 2

Conformal Bjorken + RTA

① $\tau \partial_\tau \log \epsilon = -\frac{4}{3} + \frac{\pi}{\epsilon}$

② $\partial_\tau \pi = \frac{4\eta}{3\tau\tau_\pi} - \beta_{\pi\pi}\frac{\pi}{\tau} - \frac{\pi}{\tau_\pi}$

To obtain the equation used to find the attractor we change variables

$$w \equiv \tau T$$

$$\varphi \equiv \tau \frac{\dot{w}}{w} = 1 + \frac{1}{4}\tau \partial_\tau \log \epsilon \qquad \dot{w} \equiv \partial_\tau w$$

Note that from ① one has $\frac{\pi}{\epsilon} = 4(\varphi - \frac{2}{3})$

This change of variables maps ① + ② to a single first-order ODE.

one can show (EXERCISE)

$$\frac{1}{\epsilon}\tau \partial_\tau \pi = 4 w \varphi \varphi'(w) + 16(\varphi - 1)(\varphi - \tfrac{2}{3})$$

and using this to rewrite ② one obtains $\boxed{\text{(EXERCISE)}}$

$C_\eta = \bar{\eta} = \frac{\eta}{s}$

$C_\tau = \tau_\pi T$

$C_{\eta/\pi} = \frac{C_\eta}{C_\tau}$

$$\boxed{\bar{w}\varphi\varphi' + 4\varphi^2 + [\bar{w} + (\beta_{\pi\pi} - \tfrac{20}{3})]\varphi - \tfrac{4}{9}C_{\eta/\pi} - \tfrac{2}{3}(\beta_{\pi\pi} - 4) - \frac{2\bar{w}}{3} = 0}$$

where $\bar{w} = \frac{w}{C_\pi} = \frac{\tau T}{C_\pi}$ [$C_\pi = 5\eta/s$ in RTA] is assumed to be held constant.

$$\varphi = \varphi(\bar{w}) \text{ and } \varphi'(\bar{w}) = \frac{d\varphi(\bar{w})}{d\bar{w}}$$

$\boxed{\text{Note: } \bar{w} \propto \text{"Knudsen \#"}}$ Knudsen # $\propto \frac{\lambda}{L} = \frac{\text{micro}}{\text{macro}}$

$\Rightarrow$ Solution only depends on $\bar{w} \equiv \frac{\tau T}{C_\pi}$, but details enter through $C_{n/\pi}$ and $\beta_{\pi\pi}$. $\hookrightarrow = \tau/\tau_{eq}$

$\Rightarrow$ In a given theory these are uniquely determined.

$\Rightarrow$ In RTA $\quad (p^\mu \partial_\mu f = -\frac{1}{\tau_{eq}}(f - f_{eq}))$

$$\tau_\pi = \tau_{eq} = \frac{5 C_n}{T} \quad \rightarrow \quad C_\pi = 5 C_n$$

$$C_{n/\pi} = \frac{C_n}{C_\pi} = \frac{1}{5}$$

$$\beta_{\pi\pi} = \frac{38}{21} \quad \text{(DNMR)}$$

$$\Rightarrow \quad \boxed{\bar{w}\,\varphi\varphi' + 4\varphi^2 + \left[\bar{w} - \frac{34}{7}\right]\varphi + \frac{48}{35} - \frac{2\bar{w}}{3} = 0}$$

Can specify boundary condition at $\bar{w} \rightarrow 0$ or $\bar{w} \rightarrow \infty$

Let's do the former ... we require that $\bar{w}\varphi\varphi' \rightarrow 0$ in this limit (no divergence in $\varphi\varphi'$)

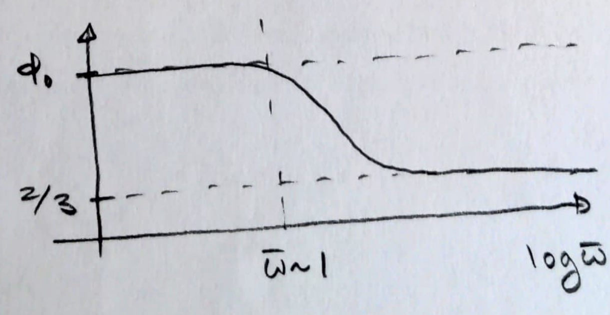
$$\lim_{\bar{w} \rightarrow 0} \quad 4\varphi^2 - \frac{34}{7}\varphi + \frac{48}{35} = 0$$

$$\rightarrow \varphi_0 = \frac{1}{140}\left(85 \pm \sqrt{505}\right)$$

"$-$" solution has singularity when used $\rightarrow$ ruled out

$$\therefore \quad \varphi_0 = \frac{1}{140}\left(85 + \sqrt{505}\right) = 0.767659$$

Can be solved numerically with this BC



* Exercise: Show that one can take the late time limit $\varphi \to 2/3$ (corresponds to $\Pi \to 0$) and obtain the same solution.

$\Rightarrow$ Once $\varphi(\bar{w})$ is known $\frac{\Pi}{\epsilon} = 4\left(\varphi - \frac{2}{3}\right)$ and

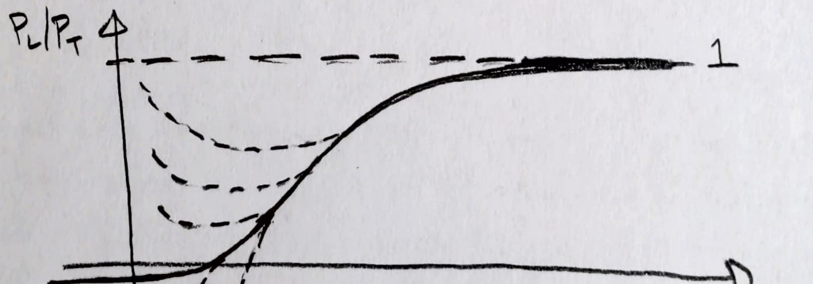
$$\frac{P_L}{P_T} = \frac{P_{ideal} - \Pi}{P_{ideal} + \Pi/2} = \frac{3-\varphi}{2\varphi - 1}$$

* Exercise: Show that this is because $P_L < 0$

$\Rightarrow$ If $\varphi > 3/4$ then this ratio is negative!

$\Rightarrow$ But $\varphi_0^{DNMR} \simeq 0.768$ $\to$ DNMR breaks down @ early times but is better than NS

$\Rightarrow$ If we numerically solve the original DNMR equations (① + ② page 1) then all solutions approach the attractor at late times



and when plotted as a function of $\bar{w}$ they do so irrespective of the value of η assumed in the equations!

* Exercise: Show that in the Navier-Stokes limit one has and O(1) Conformal Bjorken exp

$$\varphi_{NS} = \frac{2}{3} + \frac{4}{9}\frac{C_\eta/\pi}{\bar{w}}$$

Can we do better than DNMR?

DNMR based on Kinetic theory, so let's review how to obtain dissipative hydro from KT

⇒ Assume on-shell quasiparticles

⇒ To keep it simple let's work in the conformal limit

⇒ Boltzmann equation

$$p^\mu \partial_\mu f = -C[f]$$

collisional Kernel

$f = f(\vec{p})$ conformal so $E_p = |\vec{p}|$

⇒ Can take integral moments of this equation by multiplying by an operator of the form

$$\hat{I}^{\nu\sigma\ldots\lambda}[g] \equiv \underbrace{\int dP}_{\int \frac{d^3p}{(2\pi)^3}\frac{1}{E_p}} p^\nu p^\sigma \cdots p^\lambda g$$

⇒ Zeroth moment

$$\hat{I}[g] = \int dP \, g$$

Apply this to Boltzmann Eq

$$\underbrace{\int dP \, p^\mu \partial_\mu f}_{\partial_\mu \underbrace{\int dP \, p^\mu f}_{j^\mu}} = -\int dP \, C[f]$$

$$= -C_0$$

In number conserving theories $C_0 = 0$

$$\Rightarrow \partial_\mu j^\mu = 0 \quad \text{(continuity EQ)}$$

Introduce basis

$$\text{orthogonal}\begin{cases} (X_0^\mu)_{LRF} = (1,0,0,0) = u^\mu_{LRF} \\ (X_1^\mu)_{LRF} = (0,1,0,0) = X^\mu_{LRF} \\ (X_2^\mu)_{LRF} = (0,0,1,0) = Y^\mu_{LRF} \\ (X_3)_{LRF} = (0,0,0,1) = Z^\mu_{LRF} \end{cases}$$

$$X_0^\mu X_{0,\mu} = 1$$

and $X_i^\mu X_{i,\mu} = -1$

$$g^{\mu\nu} = X_0^\mu X_0^\nu - \sum_i X_i^\mu X_i^\nu$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - X_0^\mu X_0^\nu = -\sum_i X_i^\mu X_i^\nu$$

$$j^\mu = n u^\mu + \sum_i c_i X_i^\mu$$

$$= n u^\mu + \underbrace{V^\mu}_{\text{diffusion current: net flow of charge transverse to } u^\mu}$$

If $f(\vec{p}) = f(-\vec{p})$ $\quad V^\mu = 0$

$$\Rightarrow j^\mu = n u^\mu$$

$$\partial_\mu j^\mu = u^\mu \partial_\mu n + n \partial_\mu u^\mu$$

$$= Dn + n\Theta$$

$$\boxed{\partial_\mu j^\mu = Dn + n\Theta}$$

1st moment

$$\underbrace{\int dP\, p^\nu p^\mu \partial_\mu f}_{\partial_\mu \underbrace{\int dP\, p^\mu p^\nu f}_{T^{\mu\nu}}} = -\underbrace{\int dP\, p^\nu C[f]}_{=0 \text{ in energy conserving theories}}$$

$$\Rightarrow \boxed{\partial_\mu T^{\mu\nu} = 0}$$

This enough for ideal hydro. To go beyond this, we need the 2nd moment (and beyond)

2nd moment

$$\boxed{\partial_\mu I^{\mu\nu\lambda} = -C_2^{\nu\lambda}}$$

$$I^{\mu\nu\lambda} = \int dP\, p^\mu p^\nu p^\lambda f$$

$$C_2^{\nu\lambda} = \int dP\, p^\nu p^\lambda C[f]$$

$I^{\mu\nu\lambda}$ is completely symmetry under interchange of indices. If conformal, traceless in any two indices.

Tensor bases for $T^{\mu\nu}$ and $I^{\mu\nu\lambda}$

$$T^{\mu\nu} = \underbrace{\epsilon}_{1} u^\mu u^\nu + \sum_i \Big(\underbrace{P_i}_{3} X_i^\mu X_i^\nu + \sum_{j<i} \underbrace{P_{ij}}_{3} X_i^\mu X_j^\nu\Big)$$

$$7 + 3 \text{ from } u^\mu \Rightarrow 10 \text{ DOF } \checkmark$$

$$\begin{aligned} I^{\mu\nu\lambda} = {} & I_{uuu}\, u^\mu u^\nu u^\lambda \\ & + I_{uux}(u^\mu u^\nu x^\lambda + u^\mu x^\nu u^\lambda + x^\mu u^\nu u^\lambda) \\ & + \ldots \end{aligned}$$

$$I_{uuu} \equiv u_\mu u_\nu u_\lambda I^{\mu\nu\lambda} = \int dP\, E_p^3 f$$

$$I_{uux} = u_\mu u_\nu X_\lambda I^{\mu\nu\lambda} = \int dP\, E_p^2\, p_x f = 0$$

if $f(\vec{p}) = f(-\vec{p})$

only I structure fncs with two spacelike basis vectors survive if $f(\vec{p}) = f(-\vec{p})$

Can expand $\partial_\mu I^{\mu\nu\lambda} = -C_2^{\nu\lambda}$ out in terms of tensor basis and then project with

$$\Delta^{\alpha\beta}_{\nu\lambda}\, \partial_\mu I^{\mu\nu\lambda} \longrightarrow \text{EQ FOR } \pi^{\alpha\beta}$$

$$\Delta_{\nu\lambda}\, \partial_\mu I^{\mu\nu\lambda} \longrightarrow \text{EQ FOR } \Pi$$

<u>Small deviations around equilibrium</u>

$$f = f_{eq} + \delta f$$

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu f$$

$$= \underbrace{\int dP\, p^\mu p^\nu f_{eq}}_{T^{\mu\nu}_{ideal}} + \underbrace{\int dP\, p^\mu p^\nu \delta f}_{\Pi^{\mu\nu}}$$

$$\boxed{\pi^{\mu\nu} = \int dP\, p^{\langle\mu} p^{\nu\rangle} \delta f}$$

$$\boxed{\Pi = -\tfrac{1}{3}\Delta_{\alpha\beta} \int dP\, p^\alpha p^\beta \delta f}$$

$\Rightarrow \Pi^{\mu\nu}$ counts as first order in gradients in the MIS/DNMR scheme so only terms $\propto (\delta f)^2$ remain in the end. This linearization is what causes 2nd-order hydro to break at early times.

$\Rightarrow$ can also invert this relationship

In 14-moment approximation (DNMR), Classical gas

$$\delta f = \delta f_{shear} + \delta f_{bulk} \qquad \epsilon, P = \epsilon_{eq}, P_{eq}$$

$$\delta f_{shear} = f_{eq} \frac{p_\mu p_\nu \pi^{\mu\nu}}{2(\epsilon+P)T^2}$$

$$\delta f_{bulk} = -f_{eq} \frac{1}{\beta_\Pi T}\left[\frac{m^2}{3 p\cdot u} - \left(\frac{1}{3} - c_s^2\right) p\cdot u\right]\Pi$$

In small m/T limit $\qquad c_s^2 = \frac{dP}{d\epsilon}$

$$\beta_\Pi = \zeta = \frac{5}{432}\left(\frac{m}{T}\right)^4 (\epsilon + P)$$

$\Rightarrow$ Restrict attention to conformal ($m=0$) case

$$f = f_{eq}\left(1 + \frac{p_\mu p_\nu \pi^{\mu\nu}}{8PT^2}\right)$$

$\Rightarrow$ If $\pi^{\mu\nu}$ components are big correction $\gg 1$

$\Rightarrow$ can even go negative and $\gg 1$ $\Rightarrow$ BAD

<u>Controlling the approximations</u>

Inverse Reynold's number

$$(Re^{-1})_\pi \equiv \frac{\sqrt{\pi_{\mu\nu}\pi^{\mu\nu}}}{P}$$

For Bjorken flow

$$(Re^{-1})_\pi = 3\sqrt{\frac{3}{2}}\frac{|\pi|}{\epsilon}$$

when $(Re^{-1})_\pi > 1 \Rightarrow$ large corrections relative to eq. P

In Bjorken case

$$\pi_{xx} = \pi_{yy} = +\frac{\pi}{2} \qquad \pi_{zz} = -\pi$$

one has

$$p_\mu p_\nu \pi^{\mu\nu} = \frac{\pi}{2}\left(p_x^2 + p_y^2 - p_z^2\right)$$

$$\rightarrow f = f_{eq}\left(1 + \left(\frac{p_x^2 + p_y^2 - p_z^2}{8T^2}\right)\underbrace{\frac{\pi}{P}}_{\propto \sqrt{\frac{2}{3}} Re_\pi^{-1}}\right)$$

viscous corrected distribution fnc is anisotropic in momentum space and @ large p_z^2 goes negative. Problem becomes worse if there is a large Re_π^{-1}. (Bulk has similar problem.) isotropic $P < 0$!

From the attractor solution we found that

$$\frac{\pi}{\epsilon} = 4\left(\varphi - \frac{2}{3}\right)$$

boundary condition $\varphi_0 = 0.768 \rightarrow \left(\frac{\pi}{\epsilon}\right)_0 \simeq 0.405$

$\Rightarrow (Re_\pi^{-1})_0 \simeq 1.5$ correction larger than isotropic pressure!

$\Rightarrow$ negative P_L

⇒ Conclusion: At early times in a HIC the pressure + one-particle distribution functions are highly anisotropic

$$P_L \ll P_T$$

$$\langle p_z^2 \rangle \ll \frac{1}{2} \langle p_T^2 \rangle$$

⇒ Although it's possible to solve the 2nd-order viscous hydro equations in this limit, there are signs that breaking down.

⇒ How can we know if the conclusions we have reached are sound? They were all based on hydro.

⇒ Can look at exact semi-analytic or analytic solutions using AdS/CFT or RTA kinetic theory

⇒ Conclusion is the same in both

$$\lim_{\tau \to 0} P_L \ll P_T$$

⇒ Can also use numerical solutions of QCD effective kinetic theory.

⇒ Next lecture:
- An alternative hydro scheme
- An exact solution in RTA that can be use to test various models and more.