Lecture 3

⇒ Is it possible to do better than 2^{nd} order viscous hydro?

⇒ we found that viscosity makes the dist. func anisotropic in momentum space, but it is linearized and breaks down at early times.

vHydro : $$f = f_{eq} + \delta f$$

"aHydro" ← anisotropic hydro : $$f = f_{eq}\left(\frac{\sqrt{p_\mu \Xi^{\mu\nu} p_\nu}}{\Lambda}\right)$$

⇒ $\Xi^{\mu\nu}$ replaces $\Pi^{\mu\nu}$; reduces to it in near equilibrium limit ; Λ sets scale

⇒ $f > 0$ is guaranteed (note $p_\mu \Xi^{\mu\nu} p_\nu > 0$)

Let's consider the simple case of conformal Bjorken hydro (BH)

$$\Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} \qquad \xi^\mu{}_\mu = 0 \qquad u^\mu \xi_{\mu\nu} = 0$$

By symmetry, for BH

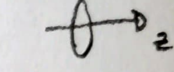
$$\xi^{\mu\nu}_{LRF} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma/2 & 0 & 0 \\ 0 & 0 & -\gamma/2 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$

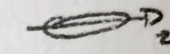
* Exercise

Show that the form for $\xi^{\mu\nu}_{LRF}$ can be written in the form below after a suitable rescaling/redefinition of Λ.

$$f(\vec{p}) = f_{eq}\left(\frac{\sqrt{p_T^2 + (1+\xi)\,p_z^2}}{\Lambda}\right)$$

$$\xi = \frac{1}{2}\frac{\langle p_T^2\rangle}{\langle p_z^2\rangle} - 1$$

$\xi > 0$ oblate 

$-1 < \xi < 0$ prolate 

can encode very larger momentum anisotropy with

$$P_L = \int dP\, p_z^2 f(\vec{p}) \xrightarrow[\xi \to \infty]{} 0$$

$$P_T = \frac{1}{2}\int dP\, p_T^2 f(\vec{p}) \xrightarrow[\xi = -1]{} 0$$

$\Rightarrow$ Focus on this Bjorken limit (Full 3+1D non-conformal treatment exists in the literature, but is dense.)

vHydro	aHydro
$\Pi(\tau)$	$\xi(\tau)$
$T(\tau)$	$\Lambda(\tau)$

$\Rightarrow$ Note that Λ can be written in terms of T

$$\epsilon = \int dP\, (p^0)^2 f(\vec{p})$$

$$= \int \frac{d^3p}{(2\pi)^3} |\vec{p}|\, f_{eq}\left(\sqrt{p_T^2 + (1+\xi)p_z^2}/\Lambda\right)$$

Rescale $\tilde{p}_z \equiv \sqrt{1+\xi}\, p_z \qquad d\tilde{p}_z = \sqrt{1+\xi}\, dp_z$

$$\epsilon = \frac{1}{(2\pi)^3}\frac{1}{\sqrt{1+\xi}}\int \frac{d^3p}{(2\pi)^3}\sqrt{p_T^2 + \frac{p_z^2}{1+\xi}}\, f_{eq}(p/\Lambda)$$

Evaluate in spherical coordinates + rescale $p \to \Lambda p$

$$\epsilon = \frac{1}{(2\pi)^2}\frac{\Lambda^4}{\sqrt{1+\xi}}\left[\int_{-1}^{1} du\sqrt{1-u^2+\frac{u^2}{1+\xi}}\right]\left[\int_0^\infty dp\, p^3 f_{eq}(p)\right]$$

Note in isotropic case $\xi = 0$

$$\epsilon_{eq} = \frac{1}{(2\pi)^2} T^4\left[\int_{-1}^{1} du\right]\left[\int_0^\infty dp\, p^3 f_{eq}(p)\right]$$

$$\therefore \frac{\epsilon}{\epsilon_{eq}} = \frac{\Lambda^4}{T^4}\cdot \underbrace{\frac{\int_{-1}^{1} du\sqrt{1-\xi u^2/(1+\xi)}}{2\sqrt{1+\xi}}}_{\equiv R(\xi)}$$

Requiring $\epsilon = \epsilon_{eq}$

"effective temperature" $\boxed{T^4 = R(\xi)\Lambda^4}$ $\qquad R(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\tan^{-1}\sqrt{\xi}}{\sqrt{\xi}}\right]$

Similar for $P_T + P_L$

Reminder: still conformal!

$$\boxed{\begin{aligned}\epsilon &= R(\xi)\,\epsilon_{eq}(T)\\ P_{T,L} &= R_{T,L}(\xi)\, P_{eq}(T)\end{aligned}}$$

Note: $P_{L,T} \geq 0$!

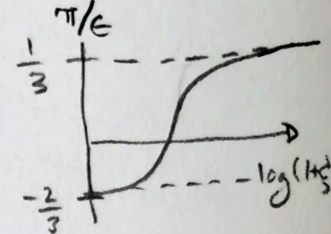
Note:

$$\frac{P_L}{P_T} = \frac{R_L(\xi)}{R_T(\xi)} \qquad \frac{P_{T,L}}{\epsilon} = \frac{R_{T,L}(\xi)}{3R(\xi)}$$

Conformal, so $R(\xi) = 2R_T(\xi) + R_L(\xi)$

can also relate π (not linearized) and ξ

$$\frac{\pi}{\epsilon} = \frac{P_{eq} - P_L}{\epsilon} = \frac{P_{eq}}{\epsilon_{eq}} - \frac{P_L}{\epsilon}$$



Bounded such that $P_{L,T} > 0$

Ⓐ

$$\to \boxed{\frac{\pi}{\epsilon} = \frac{1}{3}\left[1 - \frac{R_L(\xi)}{R(\xi)}\right]}$$

$-\frac{2}{3} \le \frac{\pi}{\epsilon} \le \frac{1}{3}$

$$Re_\pi^{-1} = \sqrt{\frac{3}{2}}\left| 1 - \frac{R_L(\xi)}{R(\xi)}\right|$$

$$Re_\pi^{-1} \le \sqrt{\frac{2}{3}} \simeq 0.816$$

* Re_π^{-1} smaller in oblate case relevant for HIC!

<u>Equations of motion in aHydro (CBH)</u>

Kinetic Theory $\to$

$$\left\{ \begin{array}{l} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu I^{\mu\nu\lambda} = -C_2^{\nu\lambda} \end{array} \right.$$

① $\to \boxed{\tau \partial_\tau \log\epsilon = -\frac{4}{3} + \frac{\pi}{\epsilon}}$ (same)

$\to$ In CBH project diagonals $\overbrace{X^i_\nu X^i_\lambda \partial_\mu I^{\mu\nu\lambda}}^{(EQ)^{ii}}$

with $i = 1, 2, 3$ then make linear combination

$$(EQ)^{zz} - \frac{1}{3}\sum_i (EQ)^{ii}$$

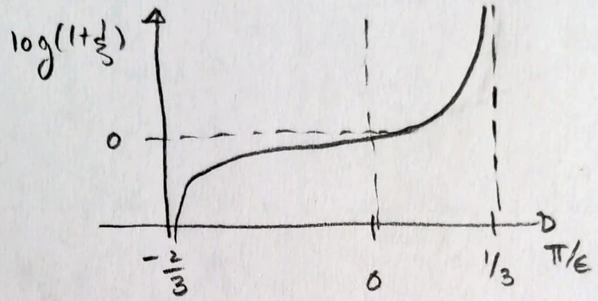
Makes traceless $\to$ similar to $\pi^{\mu\nu}$ eqs

Result in RTA

②

$$\boxed{\frac{1}{1+\xi}\partial_\tau \xi - \frac{2}{\tau} + \frac{R^{5/4}(\xi)}{\tau_{eq}}\xi\sqrt{1+\xi} = 0}$$

Using Ⓐ from previous page we can also determine $\xi(\pi)$



And change variables from $\xi \to \pi$

②

$$\Rightarrow \boxed{\frac{1}{\epsilon} d_\tau \pi + \frac{\pi}{\epsilon\tau}\left(\frac{4}{3} - \frac{\pi}{\epsilon}\right) - \left(\frac{2(1+\xi)}{\tau} - \frac{H(\xi)}{\tau_{eq}}\right)\bar{\pi}'(\xi) = 0}$$

$$\xi = \xi(\pi) \qquad \bar{\pi}'(\xi) = \frac{d}{d\xi}\left(\frac{\pi}{\epsilon}\right)\Big|_{\xi = \xi(\pi)}$$

$$H(\xi) = \xi(1+\xi)^{3/2} R^{3/4}(\xi)$$

$\Rightarrow$ Equations ① + ② can be solved numerically

$\Rightarrow$ $H(\xi)$ and $\bar{\pi}'(\xi)$ are all orders in π

$\Rightarrow$ <u>all orders resummation in Re_π^{-1}</u> !

$\Rightarrow$ when linearized around equilibrium

$$\frac{\pi}{\epsilon} = \frac{8}{45}\xi\left(1 - \frac{13}{21}\xi + O(\xi^2)\right)$$

$$H = \xi + \frac{2}{3}\xi^2 + O(\xi^3)$$

$\Rightarrow$ Inverting one obtains

$$\xi = \frac{45}{8}\frac{\pi}{\epsilon}\left(1 + \frac{195}{8}\frac{\pi}{\epsilon} + O(\pi^2)\right)$$

$$\bar{\pi}' = \frac{8}{45} - \frac{26}{21}\frac{\pi}{\epsilon} + \frac{1061}{392}\left(\frac{\pi}{\epsilon}\right)^2 + O\left(\left(\frac{\pi}{\epsilon}\right)^3\right)$$

to linear order in π ② becomes

$$\partial_\tau \pi = \frac{4\eta}{3\tau\tau_\pi} - \frac{38}{21}\frac{\pi}{\tau} - \frac{\pi}{\tau_\pi} + \mathcal{O}((\pi/\epsilon)^2)$$

This is second-order DNMR result!

$\Rightarrow$ In practice, one just solves ② numerically which includes $\frac{\pi}{\epsilon}$ to ALL ORDERS

* Exercise: Show that ② on the previous page is correct.

Attractor EQ

Following a similar method as with vHydro we obtain the attractor equation

③ $$\boxed{\bar{w}\,\varphi\,\varphi'(\bar{w}) = \left[\tfrac{1}{2}(1+\xi) - \tfrac{\bar{w}}{4}H(\xi)\right]\bar{\pi}'(\xi)}$$

$$\xi = \xi(\bar{\pi}) = \xi(4\varphi - 8/3)$$

Boundary condition for φ in limit $\bar{w} \to 0$

$$\lim_{\bar{w}\to 0} \varphi \equiv \varphi_0 = 3/4 \quad \to \quad P_L = 0$$

Conclusions

$\Rightarrow$ aHydro gives all-order resummation in Re^{-1}_π

$\Rightarrow$ If we go beyond CBH, also Re^{-1}_π!

Question

$\Rightarrow$ Is it better?

In order to compare different frameworks, we need an exact solution to judge them against.

Exact solution of CB Kinetic theory in RTA

$$p_\mu \partial^\mu f = -\frac{p\cdot u}{\tau_{eq}}\left(f - f_{eq}\right) \qquad \tau_{eq} = \frac{5\bar{\eta}}{T}$$

To keep it simple (can go beyond) we consider conformal + classical statistics

$$f_{eq} = e^{-p\cdot u/T}$$

$\Rightarrow$ Introduce new boost-invariant variables

$0 \le w \le \infty$

$$\boxed{\begin{aligned} w &= t p_z - z E \\ v &= E t - p_z z = \sqrt{w^2 + p_T^2 \tau^2} \end{aligned}}$$

(different w!)

$$\Rightarrow \quad E = \frac{vt + wz}{\tau^2} \qquad p_z = \frac{wt + vz}{\tau^2}$$

$$dP = \frac{dp_z}{(2\pi)^3 E}\, d^2p_T$$

$$\tau^2 dp_z = t\,dw + z\,dv$$

$$dv = \frac{w\,dw}{v}$$

$$\tau^2 dp_z = \frac{dw}{v}\underbrace{\left(vt + wz\right)}_{E\tau^2} \Rightarrow dp_z = \frac{E\,dw}{v}$$

$$\therefore \boxed{dP = \frac{d^2p_T\, dw}{(2\pi)^3 v}}$$

* Exercise: Show that for Boost invariant Bjorken flow

$$p^\mu \partial_\mu f = \frac{v}{\tau} \partial_\tau f$$

$$p \cdot u = \frac{v}{\tau}$$

In these coordinates, boost invariant / tranversally homogeneous flow gives

$$\boxed{\partial_\tau f = -\frac{1}{\tau_{eq}} (f - f_{eq})}$$

Note: $\tau_{eq} = \tau_{eq}(\tau)$

Solution

① $$\boxed{f(\tau) = D(\tau, \tau_0) f_0 + \int_{\tau_0}^{\tau} d\tau' \frac{D(\tau, \tau')}{\tau_{eq}(\tau')} f_{eq}(\tau')}$$

$$D(\tau_2, \tau_1) = e^{-\int_{\tau_1}^{\tau_2} d\tau'' \frac{1}{\tau_{eq}(\tau'')}} \quad ; \quad D(0,0) = 1$$

$$\frac{\partial}{\partial \tau_2} D(\tau_2, \tau_1) = -\frac{1}{\tau_{eq}(\tau_2)} D(\tau_1, \tau_2)$$

* Exercise: show that ① solves $\partial_\tau f = -\frac{1}{\tau_{eq}(\tau)} (f - f_{eq})$

Above $f(\tau)$, $f_{eq}(\tau)$, and f_0 also depend on w and p_T

$$f_{eq}(\tau') = e^{-p \cdot u / T(\tau')} = e^{-v/T\tau'} = e^{-\frac{\sqrt{w^2 + p_T^2 \tau'^2}}{\tau' T(\tau')}}$$

Defining $z^\mu = (\overbrace{\sinh \eta}^{z/\tau}, 0, 0, \overbrace{\cosh \eta}^{t/\tau})$ Boosted $z^\mu_{LRF} = (0,0,0,1)$

$$u \cdot z = 0$$

$$\boxed{p \cdot z = -\frac{w}{\tau}}$$

$$\therefore \quad f_{eq} = e^{-\sqrt{p_T^2 + p_z^2 (\tau/\tau')^2} / T(\tau')}$$

Note: This is in aHydro form!

we can pick $f_0(w, p_T)$. Example

$$f_0(w,p_T) = e^{-\sqrt{w^2(1+\xi_0)+p_T^2\tau_0^2}/\tau_0\Lambda_0}$$

(spheroidal deformation of aHydro type)

Multiplying ① by $\int dP\,(p\cdot u)^2$ on left and right gives (EXERCISE)

② $$\boxed{T^4(\tau) = D(\tau,\tau_0)T_0^4\,\frac{H^{20}\left(\frac{\alpha_0\tau_0}{\tau}\right)}{H^{20}(\alpha_0)} + \int_{\tau_0}^{\tau}\frac{d\tau'}{2\tau_{eq}(\tau')}D(\tau,\tau')\,T^4(\tau')\,H^{20}\left(\frac{\tau'}{\tau}\right)}$$

⇒ This integral equation can be solved iteratively

guess $T^4(\tau)$, plug into RHS → get new $T^4(\tau)$

⇒ Once $T(\tau)$ is known, can plug this into ① on the previous page to get $f(\tau, w, p_T)$!

Can also compute general moments of

$$\boxed{M^{nm}[f] \equiv \int dP\,(p\cdot u)^n(p\cdot z)^{2m} f}$$

$\epsilon = M^{20}$ $\quad n = M^{01}$ $\quad P_L = M^{10}$

Result is below. Solve ①, obtain $T(\tau)$ plug into below EQ

$$\boxed{M^{nm}(\tau) = \frac{n(n+2m+2)}{(2\pi)^2}\left[D(\tau,\tau_0)\,2^{(n+2m+2)/4}\,\frac{T_0^{n+2m+2}\,H^{nm}\left(\frac{\alpha_0\tau_0}{\tau}\right)}{\left[H^{20}(\alpha_0)\right]^{(n+2m+2)/4}} + \int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{eq}(\tau')}D(\tau,\tau')\,T^{n+2m+2}(\tau')\,H^{nm}\left(\frac{\tau'}{\tau}\right)\right]}$$

with $H^{nm}(y) = \dfrac{2y^{2m+1}}{2m+1}\,{}_2F_1\left(\frac{1}{2}+m, \frac{1-n}{2}; \frac{3}{2}+m; 1-y^2\right)$