Small x-physics and Glasma dynamics in ultra-relativistic collisions

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Contents

1 Practicalities

Schedule

- We have 4 classes
- Mostly blackboard lectures
- Tutorial sessions, exercises

Evolving note and exercise problems in http://users.jyu.fi/~tulappi/ggi23

2 Literature

A large part of these lectures are based on courses lectured at the University of Jyväskylä: "High energy scattering in QCD" lectured in the spring of 2011 , http://users.jyu.fi/~tulappi/fysh560kl11/ (original handwritten notes for the blackboard lectures are partly in Finnish.) and later a part of the course "QCD" lectured in the spring of 2014, http://users.jyu.fi/~tulappi/fysh555kl14/ (slides and handwritten lecture notes in English), and lectures on "Early stages of Heavy-Ion Collisions" at the 2018 ECT* Doctoral Training Program https://indico.ectstar.eu/event/14/

Literature

Mostly these lectures are based on the following books:

[1] Y. V. Kovchegov and E. Levin, "Quantum chromodynamics at high energy," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **33** (2012).

[2] V. Barone and E. Predazzi, "High-Energy Particle Diffraction," Springer 2002

Potentially relevant are recent review articles:

[3] F. Gelis, "Initial state and thermalization in the Color Glass Condensate framework," Int. J. Mod. Phys. E **24** (2015) no.10, 1530008, [arXiv:1508.07974 [hep-ph]]

[4] J. L. Albacete and C. Marquet, "Gluon saturation and initial conditions for relativistic heavy ion collisions," Prog. Part. Nucl. Phys. **76** (2014) 1 [arXiv:1401.4866 [hep-ph]]

[5] S. Schlichting and D. Teaney, "The First fm/c of Heavy-Ion Collisions," Ann. Rev. Nucl. Part. Sci. **69** (2019), 447-476, [arXiv:1908.02113 [nucl-th]].

[6] J. Berges, M. P. Heller, A. Mazeliauskas and R. Venugopalan, "QCD thermalization: Ab initio approaches and interdisciplinary connections," Rev. Mod. Phys. **93** (2021) no.3, 035003 [arXiv:2005.12299 [hep-th]].

and other slightly less new reviews and others:

[7] E. Iancu and R. Venugopalan, "The Color glass condensate and high-energy scattering in QCD," In *Hwa, R.C. (ed.) et al.: Quark gluon plasma* 249-3363 [hep-ph/0303204]

[8] F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, "The Color Glass Condensate," Ann. Rev. Nucl. Part. Sci. **60** (2010) 463 [arXiv:1002.0333 [hep-ph]]

[9] T. Lappi, "Small x physics and RHIC data," Int. J. Mod. Phys. E **20** (2011) no.1, 1 [arXiv:1003.1852 [hep-ph]]

[10] H. Weigert, "Evolution at small x(bj): The Color glass condensate," Prog. Part. Nucl. Phys. **55** (2005) 461 [hep-ph/0501087]

[11] J. Jalilian-Marian and Y. V. Kovchegov, “Saturation physics and deuteron-Gold collisions at RHIC,” Prog. Part. Nucl. Phys. **56** (2006) 104 [hep-ph/0505052]

[12] F. Gelis, T. Lappi and R. Venugopalan, “High energy scattering in Quantum Chromodynamics,” Int. J. Mod. Phys. E **16** (2007) 2595 [arXiv:0708.0047 [hep-ph]]

[13] S. J. Brodsky, H. C. Pauli and S. S. Pinsky, “Quantum Chromodynamics and Other Field Theories on the Light Cone,” Phys. Rept. **301** (1998) 299 [arXiv:hep-ph/9705477].

[14] C. Marquet, “Chromodynamique quantique à haute énergie, théorie et phénoménologie appliqué aux collisions de hadrons,” PhD thesis, in French (!) http://tel.archives-ouvertes.fr/tel-00096416/fr/

[15] H. Hänninen, “Deep inelastic scattering in the dipole picture at next-to-leading order,” PhD thesis, [arXiv:2112.08818 [hep-ph]].

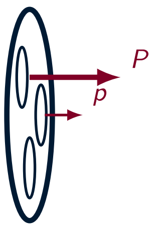
3 Plan for classes

Plan for classes

1. Eikonal scattering
2. “Hybrid formalism” of particle production in proton-nucleus collisions
3. Spacetime structure of the CGC field
4. The glasma: a collision of two color field configurations

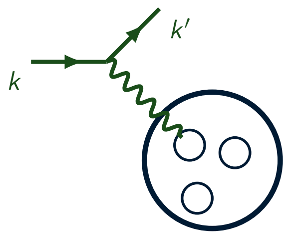
4 Preliminary stuff

4.1 Partons



J.D. Bjorken: At high energy hadrons consist of pointlike constituents: “partons”.

- Longit. momentum fraction x: $p = xP$
- Transverse scale p_T or Q



Measured in *D*eep *I*nelastic *S*cattering

$$\begin{aligned} Q^2 &= -(k-k')^2 = -q^2 \\ x &= \frac{Q^2}{2P \cdot (k-k')} \end{aligned}$$

Bj scaling:
Q^2-dependence is just like for a point-like particle.

Here the “scale” Q can roughly be thought of as the inverse transverse size of the parton. In textbook calculations in the parton model, the partons are treated as collinear, i.e. they only have longitudinal but not transverse momentum, $k_T = 0$. The resolution scale Q is determined by the physical scattering process that one studies, for example for DIS it is given by the virtuality of the process. In fact it is better to really think of the parton, when measured in such a process as having $k_T \sim Q$. The feature that the parton really has $k_T \sim Q$, but nevertheless one can calculate as if it was collinear, as long as the parton distribution that the parton comes from depends on Q, is the magic of collinear factorization! In this course, we will go beyond collinear factorization: when discussing the CGC and the physics of gluon saturation it is essential to actually think and calculate of small-x gluons as having an intrinsic transverse momentum.

- Textbooks: partons collinear
- Now: partons have k_T

When Bjorken introduced the term "parton," quarks and gluons were not yet known. Instead, Bjorken used the new word to describe some, so far unknown, elementary, pointlike constituents of the proton. Only later it became understood that partons are quarks and gluons. Today, one might consider the term "parton" to be a synonym of "quark and gluon". But usually the term "parton" is used in a slightly more limited sense, meaning roughly "quarks and gluons as constituents of a proton or nucleus, especially one accelerated to near the speed of light." If one is discussing e.g. the weak interaction, supersymmetry or the quark model for hadrons (i.e. with "constituent quarks"), one does not usually use the term "parton."

We shall be interested in these lectures at particles traveling practically at the speed of light, at very high energy. In this case it is convenient to think of the scattering problem in a coordinate system well adapted to things traveling at the speed of light in the z- or 3-direction. For this purpose, we introduce the kinematical variable of *rapidity* and *light cone coordinates*

4.2 Rapidity

Successive boosts in one direction with velocity v become easy if we define the *rapidity* ξ corresponding to the boost as $v = \tanh \xi$. Now for successive boosts $\xi = \xi_1 + \xi_2$. (exercise)

Rapidity
Momentum space rapidity y; $v^z = \tanh y$

$$E = \sqrt{m^2 + p_T^2} \cosh y = m_T \cosh y \qquad \mathbf{p}_T = (p^1, p^2) \tag{1}$$

$$p^3 = \sqrt{m^2 + p_T^2} \sinh y = m_T \sinh y \qquad p_T^2 = \mathbf{p}_T^2 \tag{2}$$

When boosting with a boost ξ this behaves additively. (exercise)

The pseudorapidity η is defined by taking $m = 0$ in Eq. (2):

$$p^3 = p_T \sinh \eta \tag{3}$$

+ Pseudorapidity just depends on the scattering angle (different components of the 3-momentum), so it easy to measure $\Longrightarrow$ it is the preferred quantity for experimentalists.
• Pseudorapidity is the same as rapidity y for massless particles.
- However, pseudorapidity does not Lorentz-transform easily for massive particles, it depends on the rest frame in a nontrivial way.

4.3 Light cone variables

Light cone variables
For any 4-vector v^μ:

$$v^{\pm} = \frac{1}{\sqrt{2}} \left(v^0 \pm v^3\right) \tag{4}$$

Scalar product

$$x \cdot y = x^0 y^0 - x^3 y^3 - \mathbf{x}_\perp \cdot \mathbf{y}_\perp = x^+ y^- + x^- y^+ - \mathbf{x}_\perp \cdot \mathbf{y}_\perp \tag{5}$$

$$v^2 = (v^0)^2 - (v^3)^2 - {\mathbf{v}_\perp}^2 = 2v^+ v^- - {\mathbf{v}_\perp}^2 \tag{6}$$

Thus metric is

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu} \tag{7}$$

Raising and lowering indices is not just a sign:

$$v^+ = v_- \qquad v^- = v_+ \tag{8}$$

High energy particle traveling in $+$ direction

$$p^+ \approx \sqrt{2}E \quad \gg \quad p^- = \frac{m^2 + {\mathbf{p}_\perp}^2}{2p^+} \tag{9}$$

Lorentz-invariant phase space

$$\frac{\mathrm{d}^4 p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2) = \frac{\mathrm{d}p^+\,\mathrm{d}p^-\,\mathrm{d}^2\mathbf{p}_\perp}{(2\pi)^3}\delta(2p^+p^- - {\mathbf{p}_\perp}^2 - m^2) = \frac{\mathrm{d}^2\mathbf{p}_\perp}{(2\pi)^3}\frac{\mathrm{d}p^+}{2p^+} = \frac{\mathrm{d}^2\mathbf{p}_\perp}{2(2\pi)^3}\mathrm{d}y \tag{10}$$

Rapidity and light cone variables

$$k^+ = \frac{1}{\sqrt{2}}\sqrt{m^2 + k_T^2}e^{y} \tag{11}$$

$$k^- = \frac{1}{\sqrt{2}}\sqrt{m^2 + k_T^2}e^{-y} \tag{12}$$

Light cone quantization Using light cone coordinates is just a change of variables and does not affect the physics in any way. However, the coordinate system is particularly useful in the context of *light cone quantization* [13, 1]. In ordinary ("instant form") quantization one quantizes the theory by imposing equal time ($t = t'$) commutation relations between Heisenberg time-dependent operators, and studies the evolution of the system in time t generated by a Hamiltonian H, which is the 0-component of a Lorentz 4-vector. In light-cone (or light front) quantization one treats instead the coordinate x^+ as the time coordinate (light cone time), and imposes canonical commutation relations at equal x^+. The evolution of the Heisenberg picture operators is then generated by the light cone Hamiltonian $\hat{P}^-$, which is the $-$-component of a Lorentz 4-vector.

5 Eikonal scattering

5.1 Eikonal approximation

Scattering at high energy becomes much simpler with the *eikonal approximation*. Many textbooks, (e.g. Barone & Predazzi (Sec 2.3) start by describing the eikonal approximation for nonrelativistic scattering off a classical potential). A more formal field theoretic derivation for gauge theory (e.g. [12]) follows the important work of Bjorken et al [16] by starting from the operator definition of the scattering amplitude and looking at its properties when the incoming state is boosted to a very high energy. We will here follow an approach that is between these in the sense that we look just at solutions of wave equation, but inspect a situation that is

- relativistic and

- involves a vector potential (for the Abelian theory at first)

Klein-Gordon eq. in external vector potential:

$$\left[(i\partial_\mu + eA_\mu(x))(i\partial^\mu + eA^\mu(x)) - m^2\right]\phi(x) = 0 \tag{13}$$

We are now interested in the scattering problem where the boundary condition is that of a high energy particle approaching the target (represented by the gauge potential A_μ). Thus the boundary condition for the equation is a plane wave $e^{-ik\cdot x}$ at $x^+ \to -\infty$. The assumption that we are interested in high energy means that k^+ is much larger than any other momentum scale in the problem, in particular any gradients of the potential A_μ. It is natural to make this assumption about the momentum scales explicit and to separate out the rapidly oscillating behavior of the wave function from the smoother scales by using an ansatz where it is explicitly factorized. We can also assume that we are in Lorentz gauge $\partial_\mu A^\mu = 0$.

$$\phi(x) \to e^{-ik\cdot x}, \quad x^+ \to -\infty \tag{14}$$

Ansatz

$$\phi(x) = e^{-ik\cdot x}\varphi(x) \tag{15}$$

$$k^+ \gg \frac{\partial_\mu \varphi(x)}{\varphi(x)} \tag{16}$$

$$\partial_\mu A^\mu = 0 \tag{17}$$

Inserting the ansatz (15) into the original equation (13) leads to

$$e^{-ik\cdot x}\Big[\overbrace{k^2 - m^2}^{=0} + \underline{2ek^\mu A_\mu + 2ik^\mu\partial_\mu} + 2ieA^\mu\partial_\mu + e^2 A_\mu A^\mu\Big]\varphi(x) = 0 \tag{18}$$

Now comes the crux of the eikonal approximation. We are assuming that k^+ is a large variable, thus at leading order we only need to keep the terms proportional to k^+ in equation (18). These only appear in the terms that are underlined in the equation. Thus, and recalling that now the initial condition is $\varphi(x) \to 1, \quad x^+ \to -\infty$, and factoring out the $2ik^\mu$ we end up with

$$\left[\partial_+ - ieA^-(x)\right]\varphi(x) = 0 \implies \varphi(x) = \exp\left\{ie\int_{-\infty}^{x^+} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x}_\perp)\right\} \tag{19}$$

Note that at this point only the A^- component of the target gauge potential matters, due to the large k^+ momentum of the probe. In the far future after the target, $x^+ \to \infty$, the outgoing wave is written in terms of the incoming one in terms of the

Eikonal phase

$$\chi(\mathbf{x}_\perp) = e\int_{-\infty}^{\infty} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x}_\perp) \tag{20}$$

$$\phi(x)\big|_{x^+\to\infty} \approx e^{-ik\cdot x}\left[1 - \left(1 - e^{i\chi(\mathbf{x}_\perp)}\right)\right] \tag{21}$$

At this point we will just state that the part of the expression corresponding to the outgoing wave is just the scattering amplitude. We need to Fourier-transform it to momentum space to measure scattering differentially in tranverse momentum. We get an elastic cross section for the eikonal scattering of charged scalar particles off a classical gauge field as:

$$\frac{\mathrm{d}\sigma_{\text{el.}}}{\mathrm{d}^2\mathbf{q}_\perp} = \left|\frac{-i}{2\pi}\int \mathrm{d}^2\mathbf{x}_\perp e^{-i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}\left(1 - e^{i\chi(\mathbf{x}_\perp)}\right)\right|^2 \tag{22}$$

We will do a more careful calculation of the cross section later, when we have introduced a bit more notation in Light Cone Perturbation theory. But for now, we can also try to just directly calculate the cross section starting from the outgoing wavefunction (21) in terms of the definition of the cross section. In the outgoing wavefunction (21) we clearly see which wave is the incoming wave and which one the scattered one:

$$\phi(x)|_{x^+\to\infty} = \phi_{\text{in}} + \phi_{\text{scat}} \tag{23}$$

with

$$\phi_{\text{in}} = e^{-ik\cdot x} \tag{24}$$
$$\phi_{\text{scat}} = e^{-ik\cdot x}\left(1 - e^{i\chi(\mathbf{x}_\perp)}\right) \tag{25}$$

This derivation follows the way a cross section is calculated from a scattering amplitude in nonrelativistic quantum mechanics. A cross section is defined as

$$\mathrm{d}\sigma = \frac{\mathrm{d}(\text{ number of particle in phase space})/\text{time}}{\text{flux in}} \tag{26}$$

Now the square of our wavefunction in coordinate space is related to the probability density of particles. Let us calculate the flux:

$$\text{flux} = \frac{\text{number}}{\text{time} \times \text{area}} = \frac{\text{number}}{\text{volume}}\frac{\text{length}}{\text{time}} = |\phi_{\text{in}}|^2 \times \frac{k}{E(k)} = |e^{-ik\cdot x}|^2 \times \frac{k}{E(k)} = \frac{k}{E(k)} \tag{27}$$

where the length ($=$ volume$/$area) is some distance in the direction of the incoming particles which, divided by the unit of time that it takes the particles to travel this distance, gives us the velocity. The velocity of a relativistic particle is its momentum divided by energy. Now, for the outgoing particles, we do not want to measure the density as a function of transverse coordinate, but rather transverse momentum. Thus we need to Fourier-transform the wavefunction to transverse momentum, to get the number of outgoing particles. Define the mixed space wavefunction as

$$\phi(\mathbf{q}_\perp, z, t) = \int \mathrm{d}^2\mathbf{x}_\perp e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}\phi(\mathbf{x}_\perp, z, t) \tag{28}$$

Now the square of this quantity gives the number of particles per unit length and unit transverse momentum, with a $(2\pi)^2$ that always goes with the Fourier transform[1]

$$|\phi(\mathbf{q}_\perp, z, t)|^2 = (2\pi)^2\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{q}_\perp\,\mathrm{d}z} \tag{29}$$

Again we need to change this from a number of particles per unit length to a number of particles per unit time. This involves the velocity of the particle. Here, we need to again evoke the fact that we are working in the high energy limit, where the scattering angle is very small. Thus, even if the particle does acquire some transverse momentum $\mathbf{q}_\perp$ in the scattering, we consider that the longitudinal momentum k is still much larger, $k \gg |\mathbf{q}_\perp|$, and the velocity (in the z-direction) is given by $k/E(k)$. Then we get the elastic cross section as

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\text{el.}}}{\mathrm{d}^2\mathbf{q}_\perp} = \frac{\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{q}_\perp\,\mathrm{d}z}\times\frac{k}{E(k)}}{\text{flux}} = \frac{1}{(2\pi)^2}|\phi_{\text{scat}}(\mathbf{q}_\perp, z, t)|^2 &= \frac{1}{(2\pi)^2}\left|\int \mathrm{d}^2\mathbf{x}_\perp e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}e^{ik\cdot x}\left(1 - e^{i\chi(\mathbf{x}_\perp)}\right)\right|^2 \\ &= \left|\frac{-i}{2\pi}\int \mathrm{d}^2\mathbf{x}_\perp e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}\left(1 - e^{i\chi(\mathbf{x}_\perp)}\right)\right|^2,\end{aligned} \tag{30}$$

using the fact that $\mathbf{k}_\perp = 0$ and thus $e^{ik\cdot x}$ just cancels in the absolute value. The final $-i$ of course inside the integral just gives one in the absolute value. We scale it out so that we recover the conventional complex phase of the scattering amplitude.

[1] You can check this by checking that the total number of particles is

$$N = \int \mathrm{d}^2\mathbf{x}_\perp\,\mathrm{d}z|\phi(\mathbf{x}_\perp, z, t)|^2 = \int \frac{\mathrm{d}^2\mathbf{q}_\perp}{(2\pi)^2}\,\mathrm{d}z|\phi(\mathbf{q}_\perp, z, t)|^2.$$

5.1.1 Summary

This was a sketchy derivation. Some points of interest:

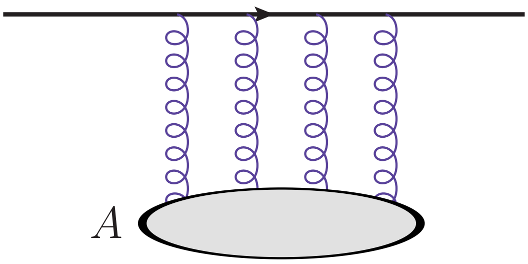
- We haven't specified anything about x^-: we will come back to this later
- The outgoing wave does not appear to expand spherically as was the case for the optical problem (see e.g. Barone & Predazzi [2]). This is mainly an artefact of the coordinate system; and a result of the eikonal approximation that assumes small angle scattering.
- If we do expand to leading order in the eikonal phase χ, we see that Γ is imaginary, i.e. the scattering amplitude real. But when expanded to second order, the scattering amplitude develops an imaginary phase. The eikonal approximation builds in a relation between the real and complex parts of the scattering amplitude, i.e. between elastic and inelastic (absorptive) scattering, that is consistent with unitarity of the scattering S-matrix.

5.2 Eikonal scattering off target of glue

So far we have only been discussing Abelian theory. Let us now move to QCD in the high energy limit. We will, however still stay within the eikonal picture of scattering. In particular, we will assume that QCD scattering is dominated by scattering off the gluon field of the target. This is in fact not just an assumption, but a consequence of the spin of the gluons: the coupling strength of a high energy particle to the gluon field is proportional to the momentum of the particle. The coupling to the fermion field of the target, on the other hand, is proportional to the square root of the high energy, because the fermion field has spin half.

We also know from experiment, and can show in perturbative QCD (BFKL evolution), that the number of gluons in the proton or nucleus grows at higher energy (i.e. smaller x). We may therefore assume that there are many of these gluons, in fact so many that eventually at some point the gluon field becomes nonperturbatively strong.

Color Glass Condensate (CGC)
Target has many gluons: $A_\mu \gg 1/g \implies$ described as classical gluon field



To see how a colored particle interacts with such a target color field, we can follow the same procedure as previously, by solving the Dirac equation of a colored fermion in this field. Since the argument is essentially the same as previously in the abelian theory, we will skip the details here. The only difference with respect to eq. (19) is that now the field is a matrix in color space, and does not commute with the field at a different coordinate x^+. Thus, when multiplying or exponentiating matrices, one naturally ends up with a path ordered exponential.

Quark in color field, Dirac:

$$(i\partial\!\!\!/ - gA\!\!\!/)\psi(x) = 0 \tag{31}$$

Ansatz $\psi(x) = V(x)e^{-ip\cdot x}u(p)$ + eikonal approx.

$$\partial_+ V(x^+, x^-, \mathbf{x}_\perp) = -igA^-(x^+, x^-, \mathbf{x}_\perp)V(x^+, x^-, \mathbf{x}_\perp) \tag{32}$$

Wilson line = path-ordered exponential

$$V(x^+, x^-, \mathbf{x}_\perp) = \mathbb{P}\exp\left\{-ig\int^{x^+} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x}_\perp)\right\} \tag{33}$$

$$\mathbb{P}\left[A(y^+)A(x^+)\right] \equiv \theta(y^+ - x^+)A(y^+)A(x^+) + \theta(x^+ - y^+)A(x^+)A(y^+) \tag{34}$$

These path ordered exponentials are the same quantities that one uses to formulate lattice field theory (the link matrices). The difference is that now, we are following a light-like path, whereas in lattice field theory the links are in the spatial or imaginary time (also $\sim$ spatial) direction. In our case we are interested in the propagation of a very energetic particle moving at the speed of light in stead of a static equilibrium system.

Recap

- Eikonal approx: $k^+ \gg$ anything else
- Nonperturbative, resum $(A_\mu)^n$
- Only A^- matters at $k^+ \to \infty$

- We used an eikonal approximation: assume that longitudinal momentum is larger than other momenta involved
- Our result is not perturbative, i.e. it sums a series of different powers of the external field A_μ
- Only one component of the vector field A^- matters. This is related to the high energy approximation: the current of charged particles with a large momentum k^+ has a J^+-component that is larger than all others, and this component of the current only couples to the A^--component of the gauge field. Incidentally, this feature is independent of the spin of the probe, and is thus the same also if we replace our incoming charged scalar particles by fermions or charged vector bosons. On the other hand, this feature does very much depend on the fact that we are scattering off a spin-1 field (a vector field), it is this feature that causes the strength of the interaction to be proportional to the momentum of the incoming particle. The graviton field would couple to the momentum squared, whereas a scalar field does not couple to the momentum. The same feature can easily be seen by calculating some tree level Feynman diagrams for such scattering [exercise]. In terms of Feynman diagrams one can state the eikonal approximation in terms of the "eikonal vertex". The fact that only A^- matters indicates that there is a gauge dependence in our result. It should be possible to do a gauge transformation that changes A^-, but leaves the cross section invariant. Thus there are gauges where obtaining the correct result requires taking into account other components of the gauge field, which in some gauge might be large enough to compensate the lack of an explicit enhancement by the large component k^+.

Digression: Probe and target light cone gauges

If we wanted to count the photons or gluons of the target in a rigorous well-defined quantum mechanical way, we would want to go to the light-cone gauge of the leftmoving target, $A^- = 0$. Then our interaction would look very different from what we have now. Thus, even the perturbative limit (power series expansion in A_μ) of our result is a perturbation series in the interaction with a Coulomb field of the target, not really with a specific number of actual radiation quanta (photons) in it.

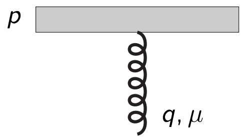
5.3 Eikonal vertex

The same eikonal approximation for the kinematics that we have here discussed in terms of scattering off a classical field, can of course also be used in a perturbative Feynman diagram calculation. The corresponding approximation in Feynman diagrams is known as the *eikonal vertex*.

We are interested in the interactions of spin-1 gauge bosons (gluons, also photons) with different kinds of matter. Very generically, a spin-1 particle is represented by a vector field A_μ, which couples in the Lagrangian to some 4-current carrying the appropriate charge: $J^\mu A_\mu$. For an elementary vertex particle $\to$ particle + gauge boson in perturbation theory, the vector A_μ becomes the polarization vector of the gauge boson, i.e. the Lorentz-index in the gauge boson propagator. This polarization index then couples to some 4-vector, and the question is: which one?

We have basically two options. The 4-vector that the gauge boson couples to can be made up from 4-vectors of particle momenta, and from spins. Without any general proof, I claim that at in the high energy limit (when the momentum of the matter particle is much larger than that of the gauge boson), the spin does not matter. Thus, apart from the color factor, the interaction of a gauge boson with a high energy particle is always the same.

At high energy: spin does not matter



$\approx 2igp^{\mu} \times$[color]

While I have not here given any general proof, the eikonal structure is not just a conjecture. Let us derive it explicitly for some known types of particles: scalars, fermions and gauge bosons.

- Interaction with a scalar field. The standard vertex (with momenta as above) is

$$ig(p^{\mu} + (p-q)^{\mu}) \approx 2igp^{\mu}, \quad p \gg q \tag{35}$$

- Interaction with a fermion: the vertex is

$$ig\bar{u}_{\lambda'}(p-q)\gamma^{\mu}u_{\lambda}(p) \tag{36}$$

Here λ', λ are the quark spins, and we are not writing out the color factor). How does one treat this? One can always use the explicit forms, but a more straightforward option is to use the "Gordon identity"

$$ig\bar{u}_{\lambda'}(p-q)\gamma^{\mu}u_{\lambda}(p) = ig\bar{u}_{\lambda'}(p-q)\left[(p+(p-q))^{\mu} + i\sigma^{\mu\nu}(p-(p-q))_{\nu}\right]u_{\lambda}(p). \tag{37}$$

Now we see that explicitly the spin-dependent part ($\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$ is the generator of Lorentz-tranformations, including rotations, and thus related to the spin) is proportional to the small momentum q. Neglecting terms proportional to q and using the spinor normalization

$$\bar{u}_{\lambda'}(p)u_{\lambda}(p) = \delta_{\lambda'\lambda} \tag{38}$$

we arrive at the eikonal vertex. At this point the reader might wonder why we say that the vertex has the spinors $\bar{u}$ and u which, in the usual Feynman rules, are only attached to external lines. To see why also internal quark lines entering a vertex can be thought of as contributing a similar factor, one can write the quark propagator as

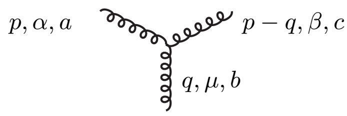
$$\frac{i}{\not{p} - m + i\varepsilon} = i\frac{\not{p}+m}{p^2 - m^2 + i\varepsilon} = i\sum_{\sigma}\frac{u_{\sigma}(p)\bar{u}_{\sigma}(p)}{p^2-m^2+i\varepsilon}. \tag{39}$$

The first u is then associated with the vertex at the beginning of the propagator (following the fermion line), and the $\bar{u}$ with the one at the end. Usually in Feynman diagram calculations one uses the relation

$$\sum_{\sigma} u_{\sigma}(p)\bar{u}_{\sigma}(p) = \not{p} + m \tag{40}$$

to get rid of the u's in calculating the square of the amplitude. But we can also use it in the other direction, to change all the $\not{p}$'s from the propagators into u's (or v's). This leads to more explicit expressions for all the fermion-gauge boson vertices in terms of the helicities of the particles.

- Gluon: we start from the elementary vertex (note that now p is incoming, $p-q$ and q outgoing momenta)



$$gf^{abc}\left[g^{\alpha\mu}(p+q)^{\beta} + g^{\mu\beta}(-q+(p-q))^{\alpha} + g^{\beta\alpha}(-(p-q)-p)^{\mu}\right] \tag{41}$$

We first neglect the small momentum components $q \ll p$, and this becomes

$$gf^{abc}\left[g^{\alpha\mu}p^{\beta} + g^{\mu\beta}p^{\alpha} - 2g^{\beta\alpha}p^{\mu}\right] \tag{42}$$

Now we must, in a way that is similar to the insertion of the spinors needed for the fermionic vertex, discuss the polarization states of the gluons. If the incoming (p) and outgoing ($p-q$) hard gluons are external, on-shell particles, the vertex must be contracted with their polarization vectors $\varepsilon_\alpha(p)$ and $\varepsilon^*_\beta(p-q) \approx \varepsilon^*_\beta(p)$. These polarization vectors are transverse, i.e. $k^\mu \varepsilon_\mu(k) = 0$. Thus, in this case at least, the terms $\sim p^\alpha$ and $\sim p^\beta$ do not contribute. Also if these gluons are internal lines, we can do the calculation assuming that they are transverse (i.e. taking the propagators in Landau gauge); we know that the result for a scattering amplitude is gauge invariant. Finally, relating the structure constant to the generator of the adjoint representation of SU(3) $(T^c)_{ba} = if_{abc}$ we can write the vertex as

$$2igp^\mu g^{\beta\alpha}(T^c)_{ba} \tag{43}$$

(signs $\sim \ldots$). The way to read this is that the eikonal 3-gluon vertex has the universal eikonal coupling of a color charged particle to gluons, $2igp^\mu$. This is multiplied by a color factor $(T^c)_{ba}$ (ordering of the color indices starting from the end of the diagram) corresponding to the adjoint representation color charge. There is also a polarization delta function $g^{\beta\alpha}$ which ensures that the polarization of the fast particle does not change, just like the spin of a quark does not change in the eikonal interaction.

Notes:

- This eikonal vertex should only be used in covariant gauge, where the polarization vector of the gauge particle $\varepsilon^\mu(q)$ has in principle all 4 components, but only the one projected out by the large momentum p^μ matters. Later in this course we will also work in light cone gauge, where one chooses the gauge in such a way that $\varepsilon^\mu(q)p_\mu = 0$, i.e. so that the component of the polarization vector that couples to the eikonal vertex vanishes by a gauge choice. In this case the interaction cannot be taken to be given by the same eikonal vertex, and one needs to take into account the other components.
- The eikonal interaction conserves the spin/helicity/polarization of the "matter" particle.
- Note that the two approximations: neglecting the momentum of the gauge particle from the point of view of the matter particle $p^+ \approx p^+ - q^+$ and the interaction not depending on the spin of the matter particle together mean that we are neglecting the *quantum* nature of the radiation. The eikonal approximation or eikonal vertex corresponds to the classical interaction with charged matter with a gauge field: acceleration or deceleration of the matter due to the Lorentz force, and potential classical radiation due to this acceleration or deceleration. Thus physics in the leading high energy limit can, of one does it carefully, be represented by classical (color) charges interacting with classical gauge fields.

6 Hybrid formalism for pA collisions

6.1 Forward rapidity in proton-nucleus collisions

Now we have developed the concept of measuring the gluon field in a dense target by measuring the scattering amplitude of a simple elementary colored probe going through this target. What would be a possible scattering experiments for doing this? It will turn out in the end that, in many ways, the cleanest such a process is deep inelastic scattering. However, it is easier for us to start with another process that looks even simpler at first, which is single inclusive hadron production in proton-nucleus collisions at forward rapidity. For the purposes of our discussion here this is indeed the simplest process. When doing actual phenomenology or higher order calculations it has complications that DIS does not have (collinear divergences, parton distributions, fragmentation functions, real-virtual cancellations, ...), but we will not encounter them in this first, introductory discussion.

Experimental data that we might want to address could be something like the nuclear modification ratio R_{pA}. Some experimental measurements for this quantity from ALICE [17] and LHCb [18, 19] are shown in in Fig.1.

The nuclear modification ratio R_{pA} is the ratio of a cross section for something in proton-nucleus collisions, divided by the mass number A times the corresponding cross section in proton-proton collisions. We expect saturation effects to be more important in nuclei than in protons and more important at small x than at large x. Thus a suitable place to look for them is in a collision system where a large x parton, preferably a quark, collides with small-x degrees of freedom. This, very generically, happens at forward rapidity (i.e. in the proton-going direction) in a proton-nucleus collision. To see this let us take a quick look at the kinematics. We consider a proton moving in the positive z direction, with a large momentum p^+ and

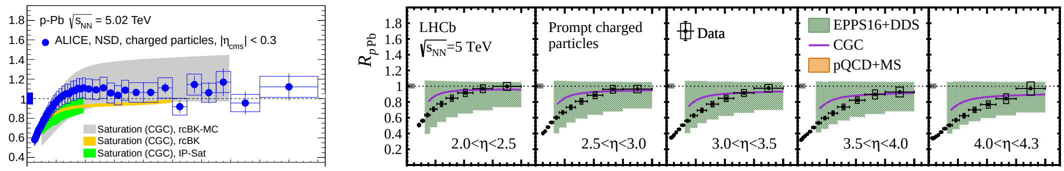
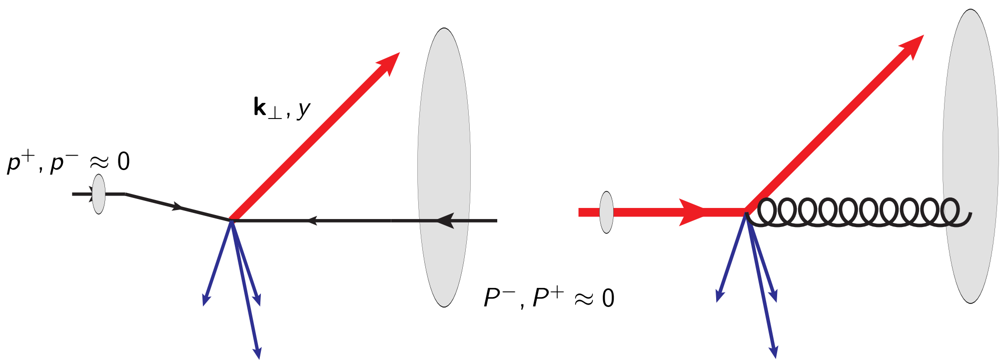


Figure 1: Nuclear modification ratio R_{pA} from ALICE [17] at midrapidity $y \approx 0$ and LHCb [18, 19] at forward rapidities.

very small $p^- = m_p^2/(2p^+)$. It collides with a heavy ion in the negative z direction, with a large P^- and a small $P^+ = m_A^2/(2P^-)$. Usually with nuclear collisions one calculates energies per nucleon, and thus the center of mass collision energy per nucleon is $s = (P+p)^2/A \approx 2p \cdot (P/A) = 2p^+P^-/A$. For simplicity let us assume that we are in the proton-nucleon center of mass frame. This means that the longitudinal momentum of the proton (p^+) is the same as the longitudinal momentum per nucleon in the nucleus (P^-/A). At the LHC, this is not exactly the frame where the detectors sit, but close enough for the purposes of this discussion; for a more careful calculation for LHC data one needs to take this small difference into account better.

We now produce and measure a particle with transverse momentum $\mathbf{k}_\perp$ at rapidity y, with y large. Four-momentum must be conserved. Since the proton has very little p^-, all the k^- of the produced particle must come from the nucleus. Similarly all the k^+ of the produced particle must come from the proton. If the produced particle is massless, we calculate its plus and minus momenta as follows.



$$\text{C.M.S} \quad \Longrightarrow \quad p^+ = \frac{P^-}{A} = \sqrt{\frac{s}{2}} \tag{44}$$

$$k^+ = \frac{1}{\sqrt{2}} k_T e^y = \frac{k_T e^y}{\sqrt{2} p^+} p^+ = \overbrace{\frac{k_T e^y}{\sqrt{s}}}^{x_p \lesssim 1} p^+ \tag{45}$$

$$k^- = \frac{1}{\sqrt{2}} k_T e^{-y} = \frac{k_T e^{-y}}{\sqrt{2}(P^-/A)} \frac{P^-}{A} = \underbrace{\frac{k_T e^{-y}}{\sqrt{s}}}_{x_A \ll 1} \frac{P^-}{A} \tag{46}$$

The fractions x_p and x_A are the fractions of the proton and nucleus longitudinal momenta carried by the produced particle. We see that when one is looking in the proton-going direction, where y is large, the process requires a large x_p and a small x_A. To a first approximation, the degrees of freedom at large x_p in the proton are valence quarks, and the ones at small x_A in the nucleus gluons. Thus, the process of particle production at forward rapidity in proton-nucleus collisions provides us a simple example of a dilute probe interacting with the color field of a dense nucleus. We shall next proceed to calculating the cross section for such a process.

6.2 Quark-nucleus scattering

To quantify the scattering of dilute probes on a dense gluon field target in a systematical way, we need to specify more precisely the theoretical framework that we use. In this context, the best way to do this is to use Light Cone Perturbation theory. A classic review is provided by the review of Brodsky, Pauli and Pinsky [13]. This review is not, however, very convenient for the kind of calculations we do here: the notations are different, and the discussion does not systematially use the concept of the Light Cone Wave Function (LCWF). A more useful reference for our calculation is the Kovchehov-Levin book [1], and the papers on DIS in the dipole picture [20, 21, 22, 23, 24, 25, 26]. We will, as in these papers, use a covariant normalization for the states (unlike the review [13]) and the Kogut-Soper metric for light cone variables introduced in Sec.4 (unlike the book [1]). A classic paper where the treatment of scattering off a classical background field is introduced is [16], which provides the more formal field theoretical justification for what we are doing here.

In our conventions, quark momentum eigenstates are normalized as

$$\left\langle q(\mathbf{p}_\perp, p^+, \lambda_p, i_p) \middle| q(\mathbf{q}_\perp, q^+, \lambda_q, i_q) \right\rangle = 2p^+(2\pi)^3 \delta(p^+ - q^+)\delta^{(2)}\left(\mathbf{p}_\perp - \mathbf{q}_\perp\right)\delta_{i_p,i_q}\delta_{\lambda_p,\lambda_q}, \tag{47}$$

where $\lambda_{p,q}$ is the *light front helicity* (spin, for the purposes of our discussion here), and $i_{p,q}$ the color, in the fundamental representation. For brevity of notation we will not write out the helicity explicitly for now. Because the eikonal interaction with the target preseves helicity, we will not write it out explicitly from now on.

Our incoming state for the quark is a single quark in a specific momentum state. The eikonal interaction, as we recall, preserves the transverse coordinate of the particle. We therefore need to Fourier transform to coordinate space. The Fourier-transforms and coordinate space states are defined as

$$\begin{aligned}
\left|q(\mathbf{p}_\perp, p^+, i)\right\rangle &= \int \mathrm{d}^2\mathbf{x}_\perp e^{i\mathbf{p}_\perp\cdot\mathbf{x}_\perp} \left|q(\mathbf{x}_\perp, p^+, i)\right\rangle && (48)\\
\left|q(\mathbf{x}_\perp, p^+, i)\right\rangle &= \int \frac{\mathrm{d}^2\mathbf{p}_\perp}{(2\pi)^2} e^{-i\mathbf{p}_\perp\cdot\mathbf{x}_\perp} \left|q(\mathbf{p}_\perp, p^+, i)\right\rangle && (49)\\
\left\langle q(\mathbf{x}_\perp, p^+, i_p) \middle| q(\mathbf{y}_\perp, q^+, i_q)\right\rangle &= 2p^+(2\pi)^3\delta(p^+ - q^+)\delta^{(2)}\left(\mathbf{x}_\perp - \mathbf{y}_\perp\right)\delta_{i_p,i_q} && (50)
\end{aligned}$$

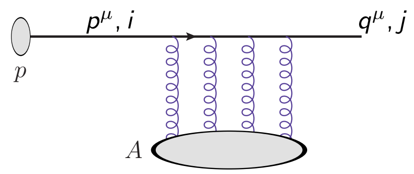
The state $|\text{in}\rangle = |q(\mathbf{p}_\perp, p^+, i)\rangle$ is the state of the incoming particle just before the shockwave of the target color field, i.e. at light cone time $x^+ \to 0^-$. As we have discussed, the effect of the color field shock wave is to color rotate the particle. This means that the state vector (bra) of the scattered particle in coordinate space is rotated by the Wilson line. We denote this rotation as the effect of the "scattering operator" $\hat{S}$.

Digression: Interaction picture

To be more specific, we are, as is common in scattering theory, using the Interaction or Dirac picture of time evolution. This means that we split the Hamiltonian of our system into two parts. The free part, corresponding to the kinetic energy k^- of the partons in the probe, causes a time evolution of the operators of the theory. The interaction part of the Hamiltonian, on the other hand, generates a time dependence in the state vector of the system. In this case, the interaction part consists of the interaction with the background field. Later, we will also include the interactions between different Fock states of the probe, most importantly gluon emission, in the interaction part of the Hamiltonian.

$$|\text{in}\rangle = \left|q(\mathbf{p}_\perp, p^+, i)\right\rangle = \int \mathrm{d}^2\mathbf{x}_\perp e^{i\mathbf{p}_\perp\cdot\mathbf{x}_\perp}\left|q(\mathbf{x}_\perp, p^+, i)\right\rangle \quad x^+ < 0 \tag{51}$$

$$\hat{S}\left|q(\mathbf{x}_\perp, q^+, i)\right\rangle = \sum_j [V(\mathbf{x}_\perp)]_{ji}\left|q(\mathbf{x}_\perp, q^+, j)\right\rangle \quad x^+ > 0 \tag{52}$$



Now we know what is the quantum state of the outgoing particle. To calculate the scattering amplitude, we need to project out the final state that we are measuring from the outgoing particle state. Now, because our background field does not (in the eikonal approximation) depend on x^-, it does not carry any longitudinal momentum p^+. Thus the scattering amplitude will be proportional to a delta function $\delta(p^+ + q^+)$. As is customary in all field theory calculations, this delta function is factorized out. This is done because to calculate the cross section we will need to square the amplitude. The square of the delta function becomes, in the calculation relating the square of the matrix element to the cross section, a single delta function the cross section level times a dimensional factor that is needed to match the correct interpretation of the cross section as a number of scatterings per unit time. It is also customary to factor out a factor $2q^+(2\pi)$ and an imaginary unit together with the delta function. This is just a convention (in field theory one writes $S = 1 + iT$), but helps to keep the terminology straight, since one often ends up discussing the meaning of the "real" and "imaginary" parts of the scattering amplitude separately. We also need to subtract the original incoming state, i.e. subtract identity from the interaction operator $\hat{S}$: when there is no background field and $\hat{S} = \hat{1}$, the scattering amplitude should be zero.

This leads us to define the invariant scattering amplitude $\mathcal{M}$ in the following way from the matrix element between the incoming and outgoing states. Given the explicit expression (52) it is straightforward to calculate the amplitude for our particular case of quark-color field scattering.

$$2q^+(2\pi)\delta(p^+ - q^+) i \overbrace{\mathcal{M}(i, \mathbf{p}_\perp \to j, \mathbf{q}_\perp)}^{\text{invariant amplitude}} \equiv \left\langle q(\mathbf{q}_\perp, q^+, j) \right| (\hat{S} - 1) \left| q(\mathbf{p}_\perp, p^+, i) \right\rangle \tag{53}$$

$$\begin{aligned}
&\left\langle q(\mathbf{q}_\perp, q^+, j) \right| (\hat{S} - 1) \left| q(\mathbf{p}_\perp, p^+, i) \right\rangle \\
&\qquad = \int \mathrm{d}^2\mathbf{x}_\perp \, \mathrm{d}^2\mathbf{y}_\perp e^{i\mathbf{p}_\perp \cdot \mathbf{x}_\perp - i\mathbf{q}_\perp \cdot \mathbf{y}_\perp} \left\langle q(\mathbf{y}_\perp, q^+, j) \right| (\hat{S} - 1) \left| q(\mathbf{x}_\perp, p^+, i) \right\rangle \\
&\qquad = \int \mathrm{d}^2\mathbf{x}_\perp \, \mathrm{d}^2\mathbf{y}_\perp e^{i\mathbf{p}_\perp \cdot \mathbf{x}_\perp - i\mathbf{q}_\perp \cdot \mathbf{y}_\perp} \left([V(\mathbf{x}_\perp)]_{ji} - \delta_{ij} \right) 2p^+(2\pi)\delta(p^+ - q^+)\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)
\end{aligned} \tag{54}$$

$$\mathcal{M}(i, \mathbf{p}_\perp \to j, \mathbf{q}_\perp) = \int \mathrm{d}^2\mathbf{x}_\perp e^{i(\mathbf{p}_\perp - \mathbf{q}_\perp)\cdot \mathbf{x}_\perp} \left([V(\mathbf{x}_\perp)]_{ji} - \delta_{ij} \right) \tag{55}$$

Note that we now, again, have the interpretation of the Wilson line V as the *eikonal scattering amplitude of a dilute probe off the target color field*.

Now, the square of the invariant amplitude, multiplied by the invariant phase space element and a momentum conserving delta function, is just the cross section. There are some factors of (2π) that, conventionally, go along with the momenta. In the case of light cone coordinates, there are additional factors of $2p^+$ that always go together with integrals or delta functions in p^+ to turn them into Lorentz-invariant phase space integrals (see Eq. (10)). To make a properly normalized cross section, we have to average over incoming colors i and sum over outgoing ones j. At this point let us also take the transverse momentum of the incoming quark to be zero, since we are interested in collinear high-x quarks coming from a high energy proton.

(Set $\mathbf{p}_\perp = 0$)

$$\mathrm{d}\sigma = \frac{1}{N_c} \sum_{i,j} 2p^+(2\pi)\delta(p^+ - q^+) \left| \mathcal{M}(i, \mathbf{p}_\perp \to j, \mathbf{q}_\perp) \right|^2 \frac{\mathrm{d}^2\mathbf{q}_\perp}{(2\pi)^2} \frac{\mathrm{d}q^+}{4\pi q^+} \tag{56}$$

$$\begin{aligned}
\frac{1}{N_c} \sum_{i,j} \left| \mathcal{M}(i, \mathbf{p}_\perp \to j, \mathbf{q}_\perp) \right|^2 &= \int \mathrm{d}^2\mathbf{x}_\perp \, \mathrm{d}^2\mathbf{y}_\perp e^{-i\mathbf{q}_\perp \cdot (\mathbf{x}_\perp - \mathbf{y}_\perp)} \\
&\qquad \times \frac{1}{N_c} \left[\mathrm{tr}\, V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp) \overbrace{- \mathrm{tr}\, V(\mathbf{x}_\perp) - \mathrm{tr}\, V^\dagger(\mathbf{y}_\perp) + 1}^{\sim \delta^{(2)}(\mathbf{q}_\perp)} \right]
\end{aligned} \tag{57}$$

In simplifying the Wilson lines it is useful to note that $[V(\mathbf{x}_\perp)]^*_{ji} = [V^\dagger(\mathbf{x}_\perp)]_{ij}$. Note that out of the four terms resulting from the square of the amplitude, the last three will give a delta function in the produced quark transverse momentum, after integration over $\mathbf{x}_\perp$ or $\mathbf{y}_\perp$. While they need to be there to preserve unitarity, we shall promptly disregard these terms as not physically interesting, since in the experimental situation we will be measuring quarks that actually scatter off the transverse plane

The crucial, general formulae here are Eq. (53) defining the invariant amplitude in terms of the incoming and outgoing states, and Eq. (56) which tells us how to calculate the cross section from this amplitude. These equations are valid for any process, the other equations just for the particular case of quark-nucleus scattering. These general formulae look quite similar to the corresponding ones in covariant field theory ("instant form," i.e. equal-t quantization), apart from the fact that only longitudinal momentum is conserved and some factors $2p^+$ are needed to make things Lorentz-covariant. Contrary to conventional Feynman perturbation theory, transverse momentum, and consequently k^-, are now not conserved for the "particles" that we are quantizing, because these particles are scattering off an external classical field that can also contribute momentum. To my knowledge, these general expressions were first written down in the very important paper of Bjorken, Kogut and Soper [16].

Let us now perform a final step and see how one goes from quark-nucleus scattering to proton-nucleus scattering. The quarks come from a collinear distribution in the proton $q(x, Q^2)$, and we must integrate over the momentum fraction x. The energy of the incoming quark is $p^+ = xP^+$, and we can insert an identity in terms of an integral over p^+ with a delta function. We change the variable for the longitudinal momentum from q^+ to the rapidity by $\mathrm{d}y = \mathrm{d}q^+/q^+$. This leads us to

$$\frac{\mathrm{d}\sigma^{p+A\to q+A}}{\mathrm{d}^2\mathbf{q}_\perp\,\mathrm{d}y} = \int \mathrm{d}x q(x,Q^2) \int \mathrm{d}p^+ \delta(p^+ - xP^+)\, q^+ \overbrace{\frac{\mathrm{d}\sigma^{q+A\to q+A}}{\mathrm{d}^2\mathbf{q}_\perp\,\mathrm{d}q^+}}^{\sim\delta(q^+-p^+)}$$
$$= \int \mathrm{d}x q(x,Q^2) \overbrace{\int \mathrm{d}p^+ \delta(p^+ - xP^+) q^+ \delta(p^+ - q^+)}^{q^+\delta(q^+-xP^+)=x\delta(x-q^+/P^+)}$$
$$\times \frac{1}{(2\pi)^2} \int \mathrm{d}^2\mathbf{x}_\perp\,\mathrm{d}^2\mathbf{y}_\perp e^{-i\mathbf{q}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)} \frac{1}{N_c} \operatorname{tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) \tag{58}$$

Hybrid formula for quark production

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\mathbf{q}_\perp\,\mathrm{d}y} = \frac{1}{(2\pi)^2} xq(x,\mu^2) \int \mathrm{d}^2\mathbf{x}_\perp\,\mathrm{d}^2\mathbf{y}_\perp e^{-i\mathbf{q}_\perp\cdot(\mathbf{x}_\perp-\mathbf{y}_\perp)} \boxed{\overbrace{\frac{1}{N_c}\operatorname{tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)}^{\text{dipole operator}}} \tag{59}$$

Now we have obtained a key result! This is our first example of how to measure the gluon field of a dense target. Of course, there are caveats. Firstly, one does of course not measure quarks in a collider experiment. There are typically two things one could do to actually connect (59) to experiment.

1. Measure jets. Experimentally, a jet is a collimated (close in angle) set of high-ish momentum particles. These are then "clustered" together into jets, and one is interested in the total momentum of the jet. One should think of a jet a direct product of a high momentum parton being produced in the primary collision. Jets are where perturbation theory and experiment meet, jets cross sections are well defined (infrared and collinear safe) observables. In a theory calculation a jet is a leading order a one-parton state, and at higher orders in perturbation theory also a 2-, 3- etc particle state. Thus a jet is in some sense the closest one can get experimentally to single quark production, but with the understanding that the relation between a jet and the quarks and gluons that create it must be systematically improved order by order in perturbation theory. The inconvenient thing about jets for our purpose is that experimentally identifying a jet as being separate from the background of the other particles produced in a hadronic collision, the jet must have a sufficiently high transverse momentum. This means that it will be typically at higher x and, more importantly, removed from the saturation region of $p_T \sim Q_s$.

2. The other option is to measure individual hadrons. In this case one needs a way to calculate the transformation of the produced quark to hadrons. Typically this is done by using "fragmentation functions," which are extracted from data (in particular $e^+e^- \to$hadrons) and satisfy their own version of the DGLAP evolution equation.

In addition to relating the quark to a measurable observable, one of course also needs to know where to get the dipole amplitude. One option for this is to use a model: later in these lectures we will discuss the McLerran-Venugopalan (MV) model [27, 28, 29] for the classical color fields, and calculate the dipole amplitude from this model. We can also derive renormalization group equations that allow one to predict, from a weak coupling QCD calculation, the dependence of the dipole amplitude on the collision energy. These go by the name of the Balitsky-Kovchegov (BK) [30, 31, 32] or JIMWLK [33, 34, 35, 36, 37, 38, 39, 40] equations. One an also, together with a model, parametrization of the BK/JIMWLK equations, extract the dipole cross amplitude from Deep Inelastic Scattering data.

To address the nuclear modification ratio R_{pA} we need to have a dipole operator separately for protons and for nuclei. We then calculate the cross sections using our hybrid formula, and take the ratio. This is the way the "CGC" curves in the LHCb plot have been obtained.

We will do an explicit calculation of the dipole operator in a specific model, the MV model, later. But even without such a calculation, it is possible to discuss the physics of gluon saturation in terms of the dipole operator.

Gluon saturation

- $\mathbf{x}_\perp = \mathbf{y}_\perp \implies \frac{1}{N_c}\left\langle \mathrm{tr}\, V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right\rangle = 1$
- $|\mathbf{x}_\perp - \mathbf{y}_\perp| \to \infty \implies \frac{1}{N_c}\left\langle \mathrm{tr}\, V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right\rangle \to 0$

$\implies \exists$ length scale $|\mathbf{x}_\perp - \mathbf{y}_\perp| \sim 1/Q_s$ s.t. $\frac{1}{N_c}\left\langle \mathrm{tr}\, V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right\rangle \sim \frac{1}{2}$ $\boxed{Q_s = \text{Saturation scale}}$

Perturbation theory in a theory with point-like particles (partons in the target) would always give you a power-law dependence. But a power law dependence on $|\mathbf{x}_\perp - \mathbf{y}_\perp|$ cannot be true at all values: it is not possible to satisfy the group theory constraints on possible values of the trace of an SU(N_c) matrix with a pure power law. Here the expectation values $\langle\rangle$ refer to some, quantum or classical, average over the states of the target.

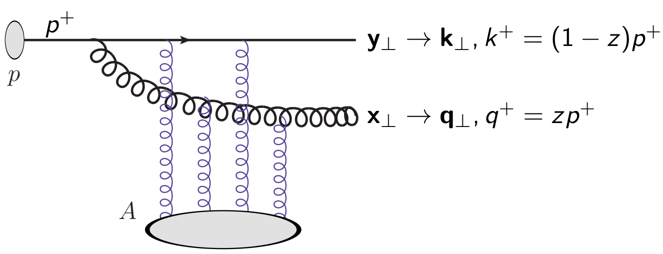
6.3 Going to higher orders

Without doing an actual calculation here, let us discuss what happend when we go to higher order in perturbation theory. Recall that we all already working to all orders in the classical field. Our probe, the quark, on the other hand is a quantum mechanical particle that we understand in perturbation theory. Starting from the single quark state, the next order in perturbation theory is when this quark emits a gluon. The emission can then happen either before or after the target. In the former case the quark-gluon system must pass thrugh the target color field. In stead of a single fundamental representation Wilson line this process will result in the product of an adjoint representation Wilson line for the gluon and a fundamental one for the quark. The adjoint representation Wilson line can be related to the fundamental one by

$$(V_{\mathrm{adj}})^{ab} = 2\,\mathrm{tr}\, t^a V t^b V^\dagger. \tag{60}$$

If one uses this relation to express everyting in terms of the fundamental representation, one ends up with three Wilson lines in the scattering amplitude. Squaring the amplitude, this means that we have operators made up of up to six Wilson lines in the cross section. We can express the result in terms of a $q + A \to q + g + X$ cross section, first derived in this formalism in Ref. [41]. Schematically, the expression looks like this:

Emit a gluon



$$\frac{\mathrm{d}\sigma^{qA\to qgX}}{\mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{k}} \propto \int_{\mathbf{x}_\perp,\bar{\mathbf{x}}_\perp,\mathbf{y}_\perp,\bar{\mathbf{y}}_\perp} e^{-i\mathbf{q}_\perp\cdot(\mathbf{x}_\perp-\bar{\mathbf{x}}_\perp)}\, e^{-i\mathbf{k}_\perp\cdot(\mathbf{y}_\perp-\bar{\mathbf{y}}_\perp)}\, \mathcal{F}(\bar{\mathbf{x}}_\perp-\bar{\mathbf{y}}_\perp,\mathbf{x}_\perp-\mathbf{y}_\perp)$$

$$\Bigg\langle \hat{Q}(\mathbf{y}_\perp,\bar{\mathbf{y}}_\perp,\bar{\mathbf{x}}_\perp,\mathbf{x}_\perp)\,\hat{D}(\mathbf{x}_\perp,\bar{\mathbf{x}}_\perp) - \hat{D}(\mathbf{y}_\perp,\mathbf{x}_\perp)\hat{D}(\mathbf{x}_\perp,\bar{\mathbf{z}}_\perp) - \hat{D}(\mathbf{z}_\perp,\bar{\mathbf{x}}_\perp)\hat{D}(\bar{\mathbf{x}}_\perp,\bar{\mathbf{y}}_\perp)$$

$$+\frac{C_F}{N_\mathrm{c}}\hat{D}(\mathbf{z}_\perp,\bar{\mathbf{z}}_\perp) + \frac{1}{N_\mathrm{c}{}^2}\left(\hat{D}(\mathbf{y}_\perp,\bar{\mathbf{z}}_\perp)+\hat{D}(\mathbf{z}_\perp,\bar{\mathbf{y}}_\perp)-\hat{D}(\mathbf{y}_\perp,\bar{\mathbf{y}}_\perp)\right)\Bigg\rangle_{\text{target}} \tag{61}$$

$(\mathbf{z}_\perp = z\mathbf{x}_\perp + (1-z)\mathbf{y}_\perp, \bar{\mathbf{z}}_\perp = z\bar{\mathbf{x}}_\perp + (1-z)\bar{\mathbf{y}}_\perp.)$

$$\hat{D}(\mathbf{x}_\perp-\mathbf{y}_\perp) \equiv \frac{1}{N_\mathrm{c}}\,\mathrm{tr}\,V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) \quad \hat{Q}(\mathbf{x}_\perp,\mathbf{y}_\perp,\mathbf{u}_\perp,\mathbf{v}_\perp) \equiv \frac{1}{N_\mathrm{c}}\,\mathrm{tr}\,V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)V(\mathbf{u}_\perp)V^\dagger(\mathbf{v}_\perp) \tag{62}$$

Some things can be pointed out about these equations. The function $\mathcal{F}(\bar{\mathbf{x}}_\perp - \bar{\mathbf{y}}_\perp, \mathbf{x}_\perp - \mathbf{y}_\perp)$ includes the dynamics of the vertex of gluon emission from the quark. We could calculate it from LCPT even in these lectures, if we had a bit more time. The transverse coordinates $\mathbf{y}_\perp$ and $\mathbf{x}_\perp$ are the coordinates of the quark and the gluon in the amplitude (remember that in the eikonal approach the coordinates stay fixed when propagating through the background field). Correspondingly, $\bar{\mathbf{y}}_\perp$ and $\bar{\mathbf{x}}_\perp$ are the coordinates of the quark and the gluon in the conjugate amplitude. The difference between the coordinates in the amplitude and the conjucate, e.g. $\mathbf{x}_\perp - \bar{\mathbf{x}}_\perp$ is Fourier-conjugate to the momentum f the produced particle. There are diagrams where the gluon is emitted either before or after the classical gluon field. This sum of two diagrams in the amplitude is then squared, yielding even more terms. The gluon emission vertex includes a fundamental representation generator t^a. When expressing the adjoint Wilson lines in terms of funamental ones, one gets more fundamental representation generators. One then sums over all colors of final state particles, and averages over colors of the incoming quark. These color summations turn into traces of Wilson lines. One then finally gets rid of the generators t^a by the Fierz identity:

$$(t^a)_{ij}(t^a)_{k\ell} = \frac{1}{2}\left(\delta_{i\ell}\delta_{jk} - \frac{1}{N_\mathrm{c}}\delta_{ij}\delta_{k\ell}\right), \tag{63}$$

which still increases the number of terms.

This is a wonderful result, that leads to further kinds of physics in two different ways.

- One can use it to study correlations between produced hadrons or jets. Particularly interesting are correlations in the azimuthal angle, between particles that have relatively low momenta, not too much larger than the saturation scale. Here, the esential thing for the physics is that the produced particles receive, in addition to the back-to-back momentum that they have in recoiling against each other, intrinsic momentum from the classical field. In this case the larger saturation scale of the nucleus leads to a larger momentum being transfered from the target, which leads to a washing out of the back-to-back azimuthal correlation dictated by momentum conservation if there is no moemntum transfer from the target. Observing such an effect in two-particle correlations would be a direct probe of the natura of the saturation scale as a transverse momentum scale, and thus very valuable. There are hints of such a nuclear effect in the dihadron correlations observed at RHIC, but there is not yet a unanimity of the interpretation of these observations.

- One can also take the qg cross section result and integrate over the momentum of one of the particles. This procedure gives a part of the NLO correction to single particle production. Note that an integration over a momentum sets the coordinates of the corresponding particle in the amplitude and the conjugate amplitude to be equal. This significantly simplifies the Wilson line operator expression. A particularly important class of NLO corrections are the ones that are enhanced by some large logarithm. In this case, there are two relevant ones:
  - Collinear divergences: there can be an infrared divergence in the integral over the tranverse momentum of either of the two partons. This divergence can be regularized, and then absorbed into the DGLAP-evolution of the quark distribution in the leading order cross section. To my knowledge, this was first demonstrated in [42]. The fact that there is a collinear divergence at

NLO that can be absorbed into DGLAP evolution of the parton distribution function in the LO cross section expression is the real "proof" of the hybrid factorization formula, at least at this order.

– Rapidity divergences (or soft divergences). There is also a divergence in the limit of $z \to 0$. This divergence has to be absorbed into the BK/JIMWLK evolution of the Wilson lines of the target. That this happens was demonstrated in [43, 44] (see also [45, 46]).

Taking into account not only the logaritmically enhanced contributions but the full expression is needed for the full NLO calculation of single inclusive particle production in the hybrid formalism.

7 Gluon saturation & CGC

7.1 Spacetime structure in 2 gauges

Let us now reflect the z-axis and consider a nucleus (in stead of the probe) moving in the $+$-direction, just to confuse the reader... This means that the color field of the nucleus has a large A^+ component.

Until now we have been describing the high energy nucleus as a classical field with one Lorentz-component (which was A^- and is now A^+), which we then developed into a path-ordered exponential or Wilson line, which is the eikonal scattering amplitude for a high energy probe passing through the nucleus. What does this picture imply in terms of the partonic content of the nucleus, i.e. the gluon distribution? For this we need some kind of a microscopical picture of where the classical field comes from. We know that it should represent small-x gluonic degrees of freedom. We also remember from our discussion of the eikonal vertex that a gluon carrying only a small longitudinal momentum fraction (a soft gluon) is essentially *classical* radiation, that only cares about color charge. This leads to the basic idea of the CGC: a classical field approximation for small x gluons, radiated by classical color charges representing the large-x degrees of freedom.

The basic idea of the CGC is that we separate the microscopic degrees of freedom inside a high energy nucleus into two things:

CGC separation of scales:

- small x: classical field
- large x: color charge

For a first simplistic picture one can think of the color charge as valence quarks, which then radiate gluons, represented by the classical field. A classical field radiated by a classical current follows the classical equation of motion. For Yang-Mills theory this is given by

$$[D_\mu, F^{\mu\nu}] = J^\nu \tag{64}$$

Note that here we are using a matrix notation, with

$$J^\mu = J^\mu_a t^a \quad F_{\mu\nu} = F^a_{\mu\nu} t^a \quad D_\mu = \partial_\mu + igA_\mu \tag{65}$$

so that the commutator of the generators

$$[t^a, t^b] = if^{abc}t^c \tag{66}$$

provides just a short way of writing

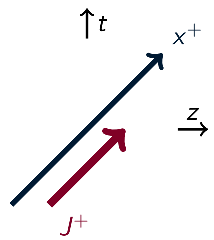
$$\partial_\mu F^{\mu\nu}_a - gf^{abc}A^b_\mu F^{\mu\nu}_c = J^\mu_a \tag{67}$$

Equation (64) involves some yet unspecified color current J^μ. The commutator in (64) refers to the fact that the gauge potential A_μ, field strength $F_{\mu\nu}$ and color current J^μ are matrices in color space. Incidentally, note that since the field strength is a commutator of covariant derivatives, $F_{\mu\nu} = -i/g[D_\mu, D_\nu]$, one can show using the Jacobi identity that the color current must be *covariantly conserved*,

$$[D_\nu, J^\nu] = 0 \tag{68}$$

in stead of the usual abelian charge conservation $\partial_\mu J^\mu = 0$. This means that the conservation of color charge actually requires J^μ to rotate in color space.

For a nucleus moving in the $+z$-direction, the classical current should be proportional to the momenta of the large-x degrees of freedom. We therefore expect that the most important (or, only nonzero) component of the current should be a J^+-component.



Now we have to somehow implement the statement that the charge is supposed to correspond to large x and the fields to small x in a spacetime picture. The consequence of this is that we can treat the color current J^+ as a very thin sheet in x^-. Sometimes one quotes a somewhat naive justification for this approximation in terms of a Lorenz-contraction of the nucleus from its rest frame thickness $2R_A$ to $\Delta z \sim 2R_A m_A/\sqrt{s}$ by the boost from the rest frame energy m_A to the large energy $\sqrt{s}$. However, the more proper argument for this assumption is the combination of the uncertainty principle with the assumption about a scale separation. The current represents large x degrees of freedom, which for a fast nucleus have a large p^+ momentum. The classical field, on the other hand, corresponds to degrees of freedom with smaller p^+. As a consequence of this the charges are better localized x^- than the field, so the field (which is what we are interested in) sees the charges as localized in x^-.

Charges are thin sheet in x^-

1. Naive explanation: Lorentz-contraction
2. Real explanation: Heisenberg $\Delta x^- \sim 1/p^+$, charges have large p^+

A similar argument must be made for the dependence of the current on the light cone time coordinate x^+. A simplistic argument states that for a fast-moving nucleus time is Lorentz-dilated so that all the internal dynamics happens very slowly in x^+. A more detailed statement of the same effect is that any probe of the small x (small p^+) degrees of freedom will have a larger $p^- \sim {p_T}^2/p^+$ than the light cone energies p^- of the color charges. Since the color charges have a small p^-, they evolve more slowly in x^+ than the fields, or an external probe of the system operating at the timescale of the small-x fields. Therefore from the point of view of the field, or of this external probe, the charges appear as slowly dependent of x^+. The extreme limit of this argument is that we can consider the charge density to be independent of the light cone time x^+, i.e. static.

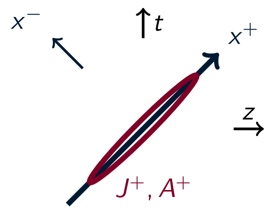
Digression: Glass

A system that is dime dependent, but whose time dependence is unnaturally slow compared to the timescales that one is probing it with, is known as a glass in statistical physics. An actual glass is a "glass" in the statistical physics sense, because glasses are non-crystalline like liquids, but nevertheless do not flow. In a loose (and not technically correct) sense one can describe a glass as a liquid that flows so slowly that on the timescales of interest it behaves like a solid.

Taking these energy- and momentum scale arguments to the extreme now arrive at the desired form for the color current as a δ-function in x^-, independent of x^+, which leaves us with a purely 2-dimensional transverse color charge density. The notation $\delta(x^-)$ should be thought of as a somewhet formal limit, we will still need to maintain the concept of path ordering (now in x^- because we switched the direction of motion of the nucleus): so we should still think of the nucleus as slightly elongated in x^-, but very thin compared to any other x^--scale in the scattering problem.

Current *static*, independent of x^+ $\Longrightarrow$ *glass*

1. Time is dilated for the nucleus
2. Charges: small p^-: slow dependence on x^+



We can now easily find a solution to the equation of motion (64), in which there is only is only one component in the gauge field, A^+, with

$$J^+(x^-, \mathbf{x}_\perp) \approx \delta(x^-)\rho(\mathbf{x}_\perp) \tag{69}$$

$$A^+(x^-, \mathbf{x}_\perp) \approx -\delta(x^-)\frac{1}{\boldsymbol{\nabla}_\perp^2}\rho(\mathbf{x}_\perp) \tag{70}$$

$$F^{+i} = \partial_i A^+ \tag{71}$$

This looks very nice, the big $+$-component of the field corresponds to a color current in the $+$-direction. This is precisely what we would expect in the physical situation of a color charged object (which the nucleus is, if you only look at it locally within a region of size $\ll 1/\Lambda_{\text{QCD}}$ in the transverse plane) moving in the $+z$-direction, and generating a field A^+ that a leftmoving high energy probe would like to couple to. It is easy to see that the current is covariantly conserved, because it is independent of x^+, and simultaneously $A^- = 0$.

Note that the only nonzero component of the field strength tensor $F^{\mu\nu}$ is F^{+i}, which corresponds to F^{0i} and F^{zi}, i.e. chromoelectric and - magnetic fields perpendicular to the z-axis. This is the same as the Weizsäcker-Williams field of an electric charge boosted to the speed of light, which can be thought of as an "equivalent photon" cloud accompanying the photon.

The physical picture of "gluons as partons" requires two things

- Infinite momentum frame: we have to boost to a frame where the nucleus is moving fast, so that the partons are collinear (at least approximately) to the nucleus.
- Light cone gauge: in order to have a partonic interpretation we also have to gauge transform to *light cone gauge* $A^+ = 0$.

The choice of the light cone gauge is often just stated without any particular justification or deep reason. We can give two reasons for why we need to work in light cone gauge.

1. In order to talk about gluons as partons we need to quantize the gluon field. In high energy scattering one is probing the hadron or nucleus at an instant in light cone time, with an approximately light-like probe. This means that we want to light cone quantize the gluon field. Canonical quantization proceeds by imposing canonical commutation relations between fields $\phi(x)$ and their canonical conjugate variables. The canonical conjugate of a generalized coordinate is the derivative of the action with respect to the time derivative of the variable, so in light cone quantization

$$\pi = \frac{\delta}{\delta(\partial_+\phi)} \int \mathrm{d}^4 x \mathcal{L}. \tag{72}$$

But since the QCD Lagrangian is built from the antisymmetric field tensor $F_{\mu\nu}$, the light cone time x^+ derivative of the component A^- is not present in the Lagrangian. In stead, $A^- = \dots$ will be a constraint. We want to solve this constraint equation explicitly so that we can completely eliminate A^- from the theory and only work with physical degrees of freedom. This can be achieved using the $\nu = +$ component of the gauge field equation of motion

$$[D_\mu, F^{\mu\nu}] = J^\nu, \tag{73}$$

where J^ν is the color current from the matter fields. Writing the $\nu = +$ component out we get

$$[D_-, F^{-+}] + [D_i, F^{i+}] = J^+. \tag{74}$$

Now by choosing the gauge $A^+ = 0$ we see that $F^{-+} = -\partial_- A^-$ is a simple function of A^- and does not even depend on its time derivative. Likewise the covariant derivative D_- is just a normal (spatial, so does not interfere with the dynamics) derivative, and altogether we have

$$(\partial_-)^2 A^- = [D_i, F^{i+}] - J^+, \tag{75}$$

from which we can solve, at any light cone time x^+ the constrained field A^- in terms of the physical degrees of freedom: A_i and the matter particle fields. Thus the light cone gauge $A^+ = 0$ conveniently enables a complete reduction of the 4 components of the gauge field to 2 dynamical degrees of freedom. This is similar to the way the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ enables one to solve A^0 as an explicit constraint in terms of the matter fields in the canonical quantization of QED.

2. In order to talk about partons, we need to be able to look at their longitudinal momentum p^+, which is some fraction of the longitudinal momentum of the proton or nucleus, $p^+ = xP^+$. Because p^+ is the conjugate variable to x^-, a parton distribution will be defined as a Fourier-transform from x^- to p^+ of a two-point function of fields. Schematically: the parton distribution for partons ϕ, characterized by some field operator $\hat{\phi}$ in a hadron h with momentum P^+ will be

$$f_\phi(x) \overset{?}{\sim} \int \mathrm{d}x^- e^{ixP^+x^-} \langle h| \hat{\phi}^\dagger(x^-)\hat{\phi}(0) |h\rangle \tag{76}$$

For the parton ϕ =gluon, the relevant operator would be $\hat{\phi} \sim F^{i+}$, where $i = 1, 2$ is connected to the polarization of the gluon. Now if the particle ϕ carries color, such a two-point function is not gauge invariant, because the gauge transformations of the field operators $\hat{\phi}^\dagger(x^-)\hat{\phi}(0)$ at different values of x^- do not cancel. Thus the proper, physical, measurable definition must include a Wilson line between the coordinates of the operators:

$$f_\phi(x) \sim \int \mathrm{d}x^- e^{ixP^+x^-} \langle h| \hat{\phi}^\dagger(x^-) \left[\mathbb{P}\exp\left\{ ig \int_0^{x^-} \mathrm{d}y^- A^+(y^-) \right\}\right] \hat{\phi}(0) |h\rangle \, . \tag{77}$$

A consequence of this is that in an arbitrary gauge, the parton distribution for a parton ϕ does not have a clear interpretation as measuring just partons ϕ, but depends in a complicated way also on the gluon fields. But *if*, and only if, we choose the gauge $A^+ = 0$, the parton distribution of parton ϕ is just a two-point function of the field operator $\hat{\phi}$, and thus has (at least roughly) a physical interpretation as measuring the distribution of partons ϕ in the hadron.

So, let us now transform our field to the light cone gauge $A^+ = 0$. In addition to satisfying the requirement of a formal partonic interpretation in terms of a light cone quantization of the degrees of freedom in the nucleus, this gauge transformation will be required to calculate what happens in the collision of our nucleus with another one. We will come back to this in Sec. 8. But already at this stage it is useful to note that the other colliding nucleus will have a color current J^-, and if our color field has a A^+-component it will cause the color current of the other nucleus to precess in color space. But if we transform our field to $A^+ = 0$-gauge, the current of the other nucleus can stay unaffected by it, leaving only the fields to interact. This is easier to deal with, in fact a major reason for using the classical field approximation in the first place is that classical fields are easier to deal with.

So let us now make this gauge transformation to $A^+ = 0$. We will keep our static approximation, nothing depends on x^+, but maintain a general x^--dependence remembering that the support of our color current in x^- is very narrow. Note that since there is no dependence on x^+, the gauge transform will not generate an A^--component. This should not be seen as a gauge choice, but as a result of the high energy kinematics.

Gauge transform:

$$A^+ \Rightarrow V^\dagger(\mathbf{x}_\perp, x^-)A^+V(\mathbf{x}_\perp, x^-) - \frac{i}{g}V^\dagger(\mathbf{x}_\perp, x^-)\partial_- V(\mathbf{x}_\perp, x^-) = 0 \tag{78}$$

$$A^- \Rightarrow -\frac{i}{g}V^\dagger(\mathbf{x}_\perp, x^-)\partial_+ V(\mathbf{x}_\perp, x^-) = 0, \text{ still} \tag{79}$$

$$A^i \Rightarrow \frac{i}{g}V^\dagger(\mathbf{x}_\perp, x^-)\partial_i V(\mathbf{x}_\perp, x^-) \qquad \textit{transverse pure gauge} \tag{80}$$

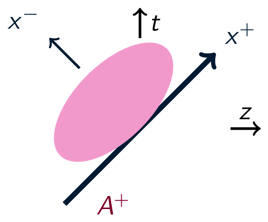
(78) solved by

$$V(\mathbf{x}_\perp, x^-) = \mathbb{P}\exp\left[-ig \int_{-\infty}^{x^-} \mathrm{d}y^- A^+ \right]$$

■ $\Longrightarrow$ Wilson line!

Now since for $A^+(y^-)$ is very localized around $y^- = 0$, the Wilson line V jumps very rapidly from unity (independently of the transverse coordinate) to a nontrivial $\mathbf{x}_\perp$-dependent value around $y^- = 0$. Thus its derivative, which gives A^i, has a $\sim \theta(x^-)$-function like discontinuity around $x^- = 0$. Now our small-x gluons look like we would expect them to look like based on the Heisenberg uncertainty principle: they are small p^+ gluons and thus should be delocalized in x^-. This was not the case in the covariant gauge.

$A^i \sim \theta(x^-)$



The current also is gauge transformed as

$$J^\mu \to V^\dagger J^\mu V. \tag{81}$$

This, however, does not change the region of support in x^-: the charges are still localized as $\sim \delta(x^-)$ as we want. Usually practical calculations are in fact (implicitly) done in terms of the covariant gauge charge density ρ that determines the Wilson lines, even if for the gauge fields one uses light cone gauge.

7.2 McLerran-Venugopalan model

In order to be able to actually calculate things we will now need a concrete model for what the charge density ρ in Eqs. (69), (70) could look like. Such a model is provided by the Mclerran-Venugopalan (MV) model [28, 27, 29]. The MV model is not the same thing as the CGC, but it is a model that makes within the CGC classical field picture.

Let us briefly review the physics argument behind this model. The situation one is thinking about is that of small enough x so that the CGC spacetime picture discussed above is valid; but not necessarily asymptotically small. Instead, we are working in the limit where the nuclear mass number A is very large. This implies that at a fixed coordinate in the transverse plane there is an parametrically large number $\sim A^{1/3}$ of valence quark-like colored degrees of freedom. Since these color charges come from separate nucleons, they should be uncorrelated with each other in the longitudinal direction. Also in the transverse plane, at two different coordinates, the total color charge comes from a sum of at only partially overlapping color charges in the different nucleons, and is therefore different. Thus one can also assume the color charges at different transverse coordinates to be uncorrelated. Finally, since the color charge at one coordinate is a sum of many uncorrelated color charges, the central limit theorem states that its distribution should be Gaussian.

MV model: charge density $\rho(\mathbf{x}_\perp)$ is

- stochastic, Gaussian random
- local in x^- and $\mathbf{x}_\perp$

$$\left\langle \rho^a(\mathbf{x}_\perp, x^-)\rho^b(\mathbf{y}_\perp, y^-)\right\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp) \tag{82}$$

The importance of this model is that it is based on a very general physical argument, but is nevertheless is very explicit, enabling one to calculate a large variety of things in various approximations. It is also simple in terms of parameters. So far $\mu^2(x^-)$ is just some unknown function, but should be thought of as being rather narrowly peaked in x^-: $\mu^2(x^-) \sim \delta(x^-)$. It will turn out that the physics in the high energy limit does not care about the functional form, but everything will just depend on the integral of $\mu^2(x^-)$ over x^-, which is essentially the saturation scale

$$Q_s^2 \sim g^4 \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-). \tag{83}$$

Digression: Powers of g

There are different conventions for where to put powers of g. Here, μ^2 is a number density. From this we construct ρ as a charge density, thus $\rho \sim g\mu$. Then, from the gluon field $A^+ \sim \rho/\nabla_\perp^2 \sim g\mu$, one calculates the Wilson line which involves $gA^+ \sim g^2\mu$. Thus in fact physics only depends on the combination $g^2\mu$ in the end.

For further developments it is important to note that one is making two independent approximations that need to be considered separately.

- We assumed that the charge density correlator is proportional to $\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$. This, as we will see, leads to a very specific dependence of the unintegrated gluon distribution $\varphi^{\mathrm{WW}}(\mathbf{k}_\perp)$ on k_T at high momentum. The resulting distribution is roughly consistent with a DGLAP-like behavior of the gluon distribution, but not for example with the solution of the BK equation at very small-x. This assumption is therefore usually thought of as appropriate for moderately small x and large A, but not parametrically small x.

- The other assumptions; namely that the charges at different x^- are uncorrelated, and that the distribution is Gaussian (which implies a specific relation between the higher 4-, 6- etc. correlators and the 2-point function (82)) are much more general. These assumptions can be used to to calculate relations between different Wilson line correlators, and in the cases where this has been checked (see e.g. [47]) this assumption has been found to be consistent with JIMWLK evolution. What this implies in practice is that one can use the assumption of locality in x^- and Gaussian correlators to calculate multiparticle correlations, and combine this with a transverse coordinate structure that is not given by $\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$, but taken e.g. from the solution of the BK equation. This procedure or approximation is often used in the literature (see e.g. [48, 49, 50, 51, 52]). Depending on the authors it is referred to in the literature as the "nonlinear Gaussian" [47], "nonlocal Gaussian" [50] or "Gaussian truncation" [10] approach.

7.3 Dipole cross section

In order to get a more physical picture of what the MV model means, let us now use it to calculate two kinds of unintegrated gluon distributions, the dipole distribution or dipole amplitude (appearing in the hybrid formalism in Sec. 6), and the so-called Weizsäcker-Williams (WW) distribution.

Dipole

$$S(r = |\mathbf{x}_\perp - \mathbf{y}_\perp|) \equiv \frac{1}{N_c}\left\langle \operatorname{tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right\rangle \qquad V(\mathbf{x}_\perp, x^-) = \mathbb{P}\exp\left[-ig\int_{-\infty}^{x^-} \mathrm{d}y^- A^+\right] \tag{84}$$

$$\left\langle A_a^+(\mathbf{x}_\perp, x^-)A_b^+(\mathbf{y}_\perp, y^-)\right\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)L(\mathbf{x}_\perp - \mathbf{y}_\perp) \tag{85}$$

Here we have introduced a notation $L(\mathbf{x}_\perp - \mathbf{y}_\perp)$ for the two-point function of the covariant gauge field A^+, which is related to the color charge density by Eq. (70). Since our MV model color charge density has a two-point function $\sim \delta(x^- - y^-)$, also the two point function of the fields will be proportional to the same delta function. The transverse coordinate dependence is more complicated because of the 2-dimensional Laplacian, therefore we just denote it by L and come back to it later.

In order to make sense of the path ordered exponential, it is not in practice very convenient to use the path-ordered product definition (33). In stead, we will discretize the integral over x^- into many infinitesimally thin pieces of width $\Delta^- \to 0$. Now a path ordered exponential of an integral turns into a path ordered exponential of a sum, which is nicely interpreted as a path ordered product of exponentials of individual terms in the sum.

Discretize

$$x^- = n\Delta^- \quad \Longrightarrow \quad \left\langle A_{ma}^+(\mathbf{x}_\perp)A_{nb}^+(\mathbf{y}_\perp)\right\rangle = g^2\delta^{ab}\mu_n^2\,\overbrace{\frac{1}{\Delta^-}\delta_{mn}}^{\delta(x^- - y^-)}\,L(\mathbf{x}_\perp - \mathbf{y}_\perp) \tag{86}$$

$$V(\mathbf{x}_\perp) = \lim_{N\to\infty} e^{-ig\Delta^- A_N^+(\mathbf{x}_\perp)}\cdots e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)}\cdots e^{-ig\Delta^- A_{-N}^+(\mathbf{x}_\perp)} \tag{87}$$

Plugging the infinite produce representation of the Wilson lines into the dipole operator (84) we obtain

$$S(r) = \lim_{N\to\infty} \frac{1}{N_c} \operatorname{tr} \Big\langle e^{-ig\Delta^- A^+_N(\mathbf{x}_\perp)} \dots e^{-ig\Delta^- A^+_n(\mathbf{x}_\perp)} \dots e^{-ig\Delta^- A^+_{-N}(\mathbf{x}_\perp)} \\ \times e^{ig\Delta^- A^+_{-N}(\mathbf{y}_\perp)} \dots e^{ig\Delta^- A^+_n(\mathbf{y}_\perp)} \dots e^{ig\Delta^- A^+_N(\mathbf{y}_\perp)} . \Big\rangle \tag{88}$$

Notice that if the path ordering in the Wilson line is from right to left, in the hermitian conjugate the corresponding ordering is from left to right. Thus in the trace (88) the elements at the lower limit in x^- are next to each other.

Now according to the MV model assumption the color fields at different steps in x^- are indpendent of each other. This means that the expectation value of the product is the product of expectation values, and we can start calculating the expectation values one rapidity step at the time, starting with the innermost $n = -N$. Now we also use the fact that Δ^- is small and expand

$$\Big\langle e^{-ig\Delta^- A^+_n(\mathbf{x}_\perp)} e^{ig\Delta^- A^+_n(\mathbf{y}_\perp)} \Big\rangle \approx \Big\langle 1 - ig\Delta^- A^+_n(\mathbf{x}_\perp) + ig\Delta^- A^+_n(\mathbf{y}_\perp) + g^2(\Delta^-)^2 A^+_n(\mathbf{x}_\perp) A^+_n(\mathbf{y}_\perp) \\ - \frac{1}{2} g^2 (\Delta^-)^2 \left(\left(A^+_n(\mathbf{x}_\perp)\right)^2 + \left(A^+_n(\mathbf{y}_\perp)\right)^2 \right) \Big\rangle + \mathcal{O}(\Delta^-)^{3/2}) \tag{89}$$

Here the power counting combines the explicit Δ^- in the exponent with

$$A^+ \sim \frac{1}{\sqrt{\Delta^-}}, \tag{90}$$

which is to be understood as following in a r.m.s. sense from the discretized two-point function (86).

The expectation value of a single A^+ vanishes and we use (86) for the two point function:

$$\begin{aligned} &\Big\langle e^{-ig\Delta^- A^+_n(\mathbf{x}_\perp)} e^{ig\Delta^- A^+_n(\mathbf{y}_\perp)} \Big\rangle \\ &\approx 1 + g^4 (\Delta^-)^2 \frac{\mu_n^2}{\Delta^-} \left[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - \frac{1}{2} L(\mathbf{x}_\perp - \mathbf{x}_\perp) - \frac{1}{2} L(\mathbf{y}_\perp - \mathbf{y}_\perp) \right] \overbrace{t^a t^a}^{= C_F \mathbb{I}_{N_c \times N_c}} \\ &\approx \exp\left\{ g^4 \Delta^- C_F \mu_n^2 \underbrace{\left[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - \frac{1}{2} L(\mathbf{x}_\perp - \mathbf{x}_\perp) - \frac{1}{2} L(\mathbf{y}_\perp - \mathbf{y}_\perp) \right]}_{-\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)} \right\} \mathbb{I}_{N_c \times N_c} \end{aligned} \tag{91}$$

Here we have defined a new notation Γ for the field-field correlator L minus its value at the origin. As we will see, the value of L at zero has a power law (=bad) IR divergence. This subtraction removes this leading divergence.

Now that this expectation value is just a number times the identity matrix, it commutes with all the other infinitesimal Wilson line factors in the infinite product (88). We can therefore pull it out and repeat the same procedure for the next factor (next n). Doing this one by one we finally get just a product of terms like (91), each of them proportional to the identity matrix. The overall trace now just gives a factor N_c that cancels with the normalization. We can now return from the discrete to the continuous notation as

$$S(r) = \exp\left\{ -g^4 C_F \int_{-\infty}^{\infty} dx^- \mu^2(x^-) \Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) \right\} \tag{92}$$

This is now practically our final result. Note that, as advertized, the Wilson line correlator only depends on the integral of $\mu^2(x^-)$ over x^-; at this stage the functional form (which we never specified) of the dependence on x^- does not matter any more.

The only thing we have used so far from the MV model is the more general of the two approximations involved: the independence of different x^- color charges and the Gaussian expectation value [2]. We have not yet actually used the assumption that the correlator (82) is proportional to $\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$ at all. Therefore the result (92) can be used independently of the MV model assumption. In particular, we can take a solution of the BK equation for $S(r)$, and consider (92) as the definition of the correlator $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) \equiv -L(\mathbf{x}_\perp - \mathbf{y}_\perp) + L(0)$. But in order to see what the more restrictive local charge correlation assumption gives us, let us now calculate $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)$ in the actual model, recalling that physically we expect this to be a good approximation in the limit of smallish x and very large A. In order to get the A^+ correlator from the MV model assumption for the correlator of ρ's we need to solve the Poisson equation (70). This is done by Fourier-transforming; we also need to introduce an additional IR cutoff to deal with the very divergent integrals in intermediate stages of our calculation. For now we will suppress the argument x^- for brevity, the following steps happen locally in x^-.

The covariant gauge field A^+ is related to the color charge by the Poisson equation (70). This equation is best solved by Fourier-transforming:

At each x^-

$$A^+(k_T) = \frac{1}{\mathbf{k}_\perp^2}\rho(\mathbf{k}_\perp) \tag{93}$$

To work in momentum space we also need the Fourier-transform of the color charge 2-point function (82)

$$\begin{aligned}\left\langle \rho^a(\mathbf{k}_\perp)\rho^b(\mathbf{p}_\perp)\right\rangle &= g^2\delta^{ab}\mu^2 \int \mathrm{d}^2\mathbf{x}_\perp\, \mathrm{d}^2\mathbf{y}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp + i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\\ &= g^2\delta^{ab}\mu^2(2\pi)^2\delta^{(2)}(\mathbf{k}_\perp + \mathbf{p}_\perp)\end{aligned} \tag{94}$$

Combining the relation between A^+ and ρ, and the two-point function of ρ's, we get

$$\begin{aligned}\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) &= \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{p}_\perp}{(2\pi)^2}(2\pi)^2\frac{\delta^{(2)}(\mathbf{k}_\perp + \mathbf{p}_\perp)}{{p_T}^2{k_T}^2}\left[1 - e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp - i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\right]\\ &= \frac{1}{2\pi}\int \mathrm{d}k\frac{1 - J_0(kr)}{k^3} \approx \frac{1}{8\pi}r^2\ln\overbrace{\frac{1}{r\Lambda}}^{\gg 1}\end{aligned} \tag{95}$$

This integral is still IR divergent at $k = 0$, but only logarithmically so. The leading power law divergence cancels [3] in the combination $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) \equiv -L(\mathbf{x}_\perp - \mathbf{y}_\perp) + L(0)$. This is true generally: expectation values of nonsinglet operators are very very sick because they depend on $L(\mathbf{x}_\perp - \mathbf{y}_\perp)$ alone. They diverge (or vanish) as an exponential function of a power of the IR cutoff; they do not describe meaningful physical quantities. Expectation values of singlet operators, such as the dipole that we are calculating here, depend always on the combination $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)$ which, in the MV model, is only logarithmically divergent. One can always argue away the logarithm for coordinate space quantities like $S(r)$, but it does affect the analytical properties and therefore the momentum space behavior, so it is better to keep it for now. Note that since Λ in Eq. (95) is an IR cutoff in the k-integral, we should assume that it is always small compared to any coordinate value that we are interested in: $r\Lambda \ll 1$. The full $\Gamma(r)$ is, however finite, because of the power of r multiplying the logarithm.

Different ways of regulating the IR divergence lead to different values for the remaining finite term. The leading log, however, is universal. Our result is for the dipole cross section is now:

$$\boxed{S_{MV}(r) \equiv \frac{1}{N_c}\left\langle \mathrm{tr}\, V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right\rangle = \exp\left\{-\frac{g^4 C_F}{8\pi}\left[\int_{-\infty}^{\infty}\mathrm{d}x^-\mu^2(x^-)\right]r^2\ln\frac{1}{r\Lambda}\right\}} \tag{96}$$

$$Q_s^2 \sim \frac{g^4 C_F}{4\pi}\left[\int_{-\infty}^{\infty}\mathrm{d}x^-\mu^2(x^-)\right] \tag{97}$$

[2] In fact we did not even use the Gaussian property, because we expanded in powers of Δ^- and only used the two-point function. At least for a Gaussian correlator some anomalously large expectation values of higher orders in the series in powers of Δ^- do not matter, but for this it is enough that the higher point correlators are not too large, they don't need to be exactly Gaussian. In fact it is probably better to think of the Gaussianity as arising via the central limit theorem; a color charge that is built up of layers in x^- that are independent of each other will behave as a Gaussian.

[3] It is curious that a small-momentum, i.e. IR, divergence manifests itself in the short-distance behavior of the two-point function.

This provides a concrete link between the MV model variable μ^2 (as already promised, this only enters as the integral over x^-) and the dipole cross section, e.g. the quar-nucleus scattering or DIS cross sections. We can now make a fit to DIS data, use that to extract a value for μ^2, and apply this to calculating heavy ion collisions: this is indeed what is done in the very successful IPglasma model for the initial stage of a heavy ion collision [53].

The dipole cross section interpolates between 1 at $r = 0$ and 0 at $r \to \infty$ as it should, satisfying the expectations for a SU(3) matrix correlator. We can identify the saturation scale Q_s from the characteristic scale in this r-dependence. Here we calculated the dipole amplitude in the fundamental representation, which is why our result has the Casimir operator C_F, overall the saturation scale should have *Casimir scaling*, i.e. be proportional to the relevant Casimir. The glasma fields in the initial stage of a heavy ion collision are gluons, there the relevant scale is the adjoint Casimir $C_A = N_c$ which, in the MV model, is larger than the saturation scale measured e.g. in DIS. In addition to the color factor, the value of Q_s depends on the precise convention chosen to determine it from the dipole correlator (there are many ways to extract something so vaguely defined as "the" characteristic scale from a function). In the MV model specifically, it also depends on the details of how one regularized the IR divergence in the momentum integral, this adds another uncertainty factor into the interpretation of MV model calculations [54].

In momentum space, the Fourier-transform of the dipole expectation value, multiplied by ${k_T}^2$, is known as the dipole gluon distribution

$$\varphi_{\text{dip.}}(k_T) \sim {k_T}^2 \int \mathrm{d}^2\mathbf{r}_\perp e^{i\mathbf{r}_\perp\cdot\mathbf{k}_\perp} S(r) \tag{98}$$

Thanks to the logarithm of r in the MV model expression, it behaves [Exercise!] like $1/{k_T}^2$ at large k_T. At small k_T, the dipole distribution behaves as $\sim {k_T}^2$ (this is easy to see, since the integral of $S(r)$ over $\mathbf{r}_\perp$ is finite, therefore $S(k_T = 0)$ is a constant and $\varphi_{\text{dip.}}(k_T) \sim {k_T}^2 S(k)$).

7.4 Weizsäcker-Williams distribution

To characterize the number distribution of gluons in the nucleus one needs, as discussed above, to look at a 2-point function of the gluon field in the light cone gauge. Note that now (contrary to the dipole picture of DIS or the hybrid formulation of proton-nucleus scattering) we trying to analyze gluons, quanta, in the target nucleus, not the probe. It would be difficult to light cone quantize both at the same time because we cannot simultaneously fix the gauge to be both $A^+ = 0$ and $A^- = 0$.

This gluon number density corresponds to what is know as the Weizsäcker-Williams gluon distribution. The full calculation in the MV model was first done in [55], following a procedure very similar to the above. Repeating this would take up too much time in this lecture, so we will just quote the result below. But before that it is useful to look at the *dilute limit* of the calculation. This can be derived by assuming that $\rho \sim g^2\mu$ is small, in which case one can expand the Wilson lines to lowest order. Since in the MV model everything is expressed in terms of $A^+ \sim \rho/\boldsymbol{\nabla}_\perp^2$, the dilute limit of small $g^2\mu$ is at the same time the limit of large transverse momenta.

High $\mathbf{k}_\perp$ — small μ:

$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}_\perp\,\mathrm{d}y} = \varphi^{\text{WW}}(\mathbf{k}_\perp) \sim \left\langle A^i_a(\mathbf{k}_\perp)A^i_a(-\mathbf{k}_\perp)\right\rangle \tag{99}$$

$$A^i_a(\mathbf{k}_\perp) = \frac{k^i}{{k_T}^2}\rho^a(\mathbf{k}_\perp) \implies \left\langle A^i_a(\mathbf{k}_\perp)A^i_a(-\mathbf{k}_\perp)\right\rangle = \frac{1}{g^2}\overbrace{(2\pi)^2\delta^{(2)}(\mathbf{k}_\perp=0)}^{\pi R_A^2}({N_c}^2-1)\frac{\overbrace{g^4\mu^2}^{\sim Q_s^2}}{\mathbf{k}_\perp^2} \tag{100}$$

Here we used the momentum space correlator (94) and the fact that in the dilute limit the light cone gauge transverse gauge field in (80) reduces to just a derivative of covariant gauge A^+ field. This gluon distribution reduced in a very natural way to the DGLAP-evolved integrated gluon distribution, for which the definition in terms of the quantized light cone gauge gluonic field operators leads to

$$xG(x,Q^2) = \int^{Q^2} \mathrm{d}^2\mathbf{k}_\perp \frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}_\perp\,\mathrm{d}y} \sim \pi R_A^2({N_c}^2-1)\frac{Q_s^2}{g^2}\ln Q^2, \tag{101}$$

i.e. a function that grows logarithmically with Q^2. This growth can be identified with the growth in the number of gluons in DGLAP evolution when increasing the resolution scale Q^2.

The result from [55] (see e.g. [56] for a mathematically clearer but more formal derivation) for the Weizsäcker-Williams gluon distribution is

WW distribution

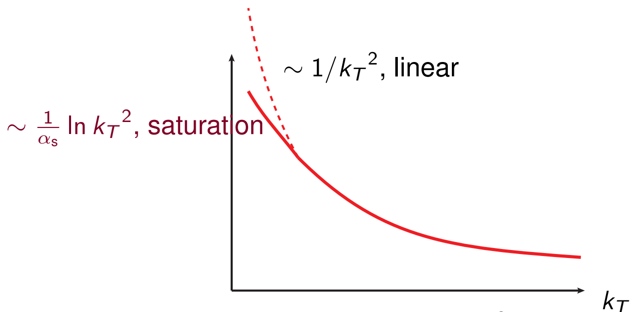
$$A^i = \frac{i}{g} V(\mathbf{x}_\perp) \partial_i V(\mathbf{x}_\perp) \tag{102}$$

$\Longrightarrow$

$$\left\langle A^i_a(\mathbf{x}_\perp) A^i_b(\mathbf{y}_\perp) \right\rangle = \delta^{ab} \frac{N_c}{g^2} \frac{\nabla^2_\perp \Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)}{\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)} \left(\exp\left\{ -g^4 N_c \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-) \Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) \right\} - 1 \right) \tag{103}$$

$\Longrightarrow$

$$\varphi_{WW}(k_T) \sim \left(\begin{array}{ll} \frac{1}{\alpha_s} \ln k_T, & k_T \ll Q_s^2 \\ \frac{1}{\alpha_s} \frac{Q_s^2}{k_T{}^2}, & k_T \gg Q_s^2. \end{array} \right. \tag{104}$$



Here we have again identified the saturation scale as $Q_s \sim g^2\mu$. Some points to note about the expression (104):

- The factor in the exponent is an adjoint representation Casimir C_A, compared to the fundamental Casimir $C_F = (N_c{}^2 - 1)/(2N_c)$ in the fundamental representation dipole (96). This is what one would expect for gluons as opposed to a fundamental representation probe.

- Both the dipole gluon distribution (98) and the WW distribution (104) have (when properly normalized) the same large k_T limit. However, they look quite different at small k_T. Both exhibit saturation: the behavior changes at $k_T \lesssim Q_s$; the dipole distribution changes to $\sim k_T{}^2$ and the WW distribution to $\sim \ln k_T$. You can deduce the latter property by noting that due to the 1 inside the bracket in (102), the integral over r of the coordinate space distribution diverges logarithmically; this would be the $k_T = 0$ value of the momentum space distribution.

- In a strict operatorial sense the dipole and WW distributions are independent quantities, some cross sections depend on one, some on the other. Without evoking a model such as MV one has to measure both of them separately experimentally. The MV model (or in general the nonlinear Gaussian assumption) provides an additional physical model that allows one to relate them to each other. For a discussion explaining how these two distributions result in the small x limit from transverse momentum distributons (TMD's) that have explicitly gauge invariant operatorial definitions, see [52].

- The WW distribution is, when one writes it in terms of the saturation scale Q_s in stead of the model parameter μ, proportional to $1/g^2$ or $1/\alpha_s$. The WW distribution is the best we can do if we try to make a bona fide effort to count gluons. In the CGC field the result is that, at weak coupling $\alpha_s \to 0$, there is a nonperturbatively large number of them. This is the justification for the word *condensate* in the CGC.

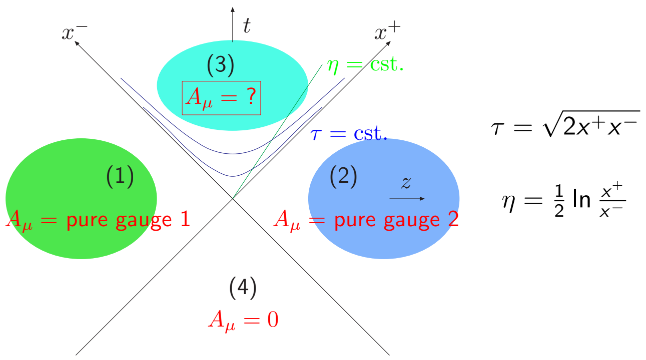
8 Glasma

8.1 Heavy ion collision: initial condition

Now let us finally move to the collision of two high energy nuclei. We follow here a calculation that was originally done in Ref. [57]. We will represent both nuclei in a symmetrical way, by their own classical color currents and radiated fields, with nucleus (1) moving in the positive z-direction and nucleus (2) in the negative one.

$$J^\mu = \underbrace{\delta^{\mu +}\rho_{(1)}(\mathbf{x}_\perp)\delta(x^-)}_{A^i\sim\theta(x^-)} + \underbrace{\delta^{\mu -}\rho_{(2)}(\mathbf{x}_\perp)\delta(x^+)}_{A^i\sim\theta(x^+)} \tag{105}$$

For each of these currents separately, we already know the solution of the field in the light cone gauge of each of the currents $A^\pm = 0$, and these solutions have the nice property that also the other longitudinal component vanishes: $A^\mp = 0$. I.e. the light cone field of nucleus (1) also "accidentally" satisfies the ligh cone gauge condition of the other one. Thus in some region of spacetime, which is causally connected to only one of the nuclei (we chose the boundary conditions for the Yang-Mills equation to be causal, so that one gets a color field only in the region after the passage of the nucleus).



Let us recall the expressions for these "transverse pure gauge" fields without the θ-functions, which we will deal with separately.

$$A^i_{(1,2)} = \frac{i}{g} V_{(1,2)}(\mathbf{x}_\perp)\partial_i V^\dagger_{(1,2)}(\mathbf{x}_\perp) \tag{106}$$

$$V_{(1,2)}(\mathbf{x}_\perp) = P e^{ig\int \mathrm{d}x^- \frac{\rho(\mathbf{x}_\perp, x^-)}{\nabla^2_\perp}} \tag{107}$$

Digression: Conventions for $\pm ig$
In my conventions the covariant derivative is

$$D_\mu = \partial_\mu + igA_\mu \tag{108}$$

the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \tag{109}$$

A gauge transformation is

$$A_\mu \to VA_\mu V^\dagger - \frac{i}{g}V\partial_\mu V^\dagger \tag{110}$$

and a pure gauge

$$-\frac{i}{g}V\partial_\mu V^\dagger. \tag{111}$$

All the signs of the ig's in these expressions are correlated with each other, and it becomes hard to keep one's conventions consistent. For the transverse pure gauge in Eq. (106) one has to note that $A^i = -A_i$, and thus the sign is

$$A^i = -A_i = \frac{i}{g}V\partial_i V^\dagger. \tag{112}$$

Equations (106) and (107) provide valid solutions to the equations of motion in the regions (1) and (2) in the figure above, which are causally connected to only one of the two nuclei. Let us now derive an initial condition for the field inside the future light cone, which is causally connected to both. *Inside* the region (3) there is no current (since the color currents are proportional to delta functions)

Region (3):

- Inside: $[D_\mu, F^{\mu\nu}] = 0$; no source
- Need initial conditions at $\tau = 0$
- $\sqrt{s} \to \infty$: should be boost invariant: $A_\mu(\tau, \mathbf{x}_\perp)$

Gauge choice:

- Avoid color precession: $A^- = 0$ at $x^- = 0$ where J^+ lives, + vice versa
- Boost invariant: τ, η-components

$\Longrightarrow$ Fock-Schwinger $A_\tau = (x^+ A^- + x^- A^-)/\tau = 0$

The Fock-Schwinger or temporal gauge condition has the very nice property that on the x^+-axis, i.e. $x^- = 0$, it reduces to $x^+ A^- = 0$, i.e. $A^- = 0$. Thus the current that lives on this light cone is not only covariantly conserved $[D_\mu, J^\mu] = \partial_+ + ig[A^-, J^+] = 0$, but in fact does not color precess and is constant $\partial_+ J^+ = 0$. This significantly simplifies our following calculation. The other advantage of the $A_\tau = 0$-condition is that in order to solve the equations of motion numerically in a Hamiltonian formalism, which is the standard thing to do, one needs to choose a temporal gauge condition. There are a few calculations of gluon production inside the forward light cone in another gauge, the covariant gauge, but this is done only in the dilute limit, not including any final state interactions [58, 59]. I am not aware of a numerical implementation of the full equations in another gauge.

Now, as discussed above, we need an initial condition for the field inside the future light cone. For the case of one nucleus we could have two components equal to zero, one as a gauge choice and the other one as a result of our current being independent of the light cone time x^+. In the case of two nuclei there is a dependence on both x^- and x^+ in the problem, and we cannot have this "accidental" vanishing of one component of the gauge field any more. So we can only make one component of the gauge field vanish with a gauge choice, but our solution inside the future light cone will have nonzero values for the other three. With $A_\tau = 0$, this third component is the longitudinal field orthogonal to it, namely A_η or $A^\eta = -A_\eta/\tau^2$. In the following we will use the latter, because A^η is finite at $\tau = 0$, and consequently $A_\eta(\tau = 0) = 0$.

Since the currents exist only on the light cones, the field can have discontinuities across the light cone. We can write an ansatz for the solution in terms of θ functions. Inserting this into the equations of motion will yield δ-functions arising from derivatives of the θ-functions. We can determine the initial conditions for the field at $\tau = 0^+$ by requiring that the coefficients of these δ-functions vanish. This is enough to determine the initial condition, we do not need the terms that do not have a $\delta(x^\pm)$.

$$\text{Ansatz: } A_i = \overbrace{A_i^{(1)}(\mathbf{x}_\perp)\theta(-x^+)\theta(x^-) + A_i^{(2)}(\mathbf{x}_\perp)\theta(x^+)\theta(-x^-)}^{\text{known}} + A_i^{(3)}(\mathbf{x}_\perp, \tau)\theta(x^+)\theta(x^-) \tag{113}$$

$$A^\pm = \pm\theta(x^+)\theta(x^-)x^\pm A^\eta(\mathbf{x}_\perp, \tau) \tag{114}$$

$$[D_\mu, F^{\mu\nu}] = J^\nu \quad \text{match} \quad \delta(x^\pm) \tag{115}$$

▼

$\delta(x^-)\delta(x^+)$ in:

$$J^i = \partial_- \partial_+ A^i + \cdots = \delta(x^-)\delta(x^+)(-A_i^{(1)}(\mathbf{x}_\perp) - A_i^{(2)}(\mathbf{x}_\perp) + A_i^{(3)}(\mathbf{x}_\perp)) = 0 \tag{116}$$

To get a contribution like $\delta(x^-)\delta(x^+)$ we need two derivatives $\partial_-\partial_+$ acting on the theta functions $\theta(\pm x^-)\theta(\pm x^+)$. When acting on the transverse component $A^- i$ such derivatives appear in the $J^i = 0$ equation. In principle we have derivatives like $\partial_-\partial_+$ acting on $\theta(\pm x^-)\theta(\pm x^+)$ in $A^\pm$ in the equations with $J^\pm$. However, in this case the contribution is killed by the additional $x^\pm$ in the relation between $A^\pm$ and A^η, which gives terms like $\delta(x^-)\delta(x^+)x^\pm = 0$. We now have our first equation.

Next, we have to start looking at terms like $\delta(x^\pm)\theta(x^\mp)$, where one derivative is acting on a theta function and other derivatives on something else. There are such terms in the $J^i = 0$ equation too, but it turns out they do not give us new constraints on the fields in region (3). In stead, we have to look at the equations for $J^{\pm}$. Since $+$ and $-$ are symmetric, we can look at either J^+ or J^-; let us look at the first one. Now $J^+ \sim \delta(x^-)$, so we want to be looking for terms $\sim \delta(x^-)$ in the equation for J^+. There are several:

$\delta(x^-)\theta(\pm x^+)$ in J^+ equation

$$\partial_- F^{-+} = \big[\partial_-\partial_+ \overbrace{(x^+\theta(x^+)\theta(x^-))}^{A^+} - \partial_-\partial_- \overbrace{(-x^-\theta(x^+)\theta(x^-))}^{A^-}\big] A^\eta(\mathbf{x}_\perp, \tau) \tag{117}$$

$$= 2\delta(x^-)\theta(x^+)A^\eta(\mathbf{x}_\perp,\tau) + \dots \tag{118}$$

$$\partial_i(\overbrace{\partial_- A_i}^{F^{i+}}) = \delta(x^-)\theta(x^+)\partial_i(\overbrace{-A_i^{(2)}(\mathbf{x}_\perp) + A_i^{(3)}(\mathbf{x}_\perp)}^{A_i^{(1)}}) = J^+|_{x^+>0} \tag{119}$$

$$ig[A_i, \overbrace{\partial_- A_i}^{F^{i+}}] = ig[\overbrace{\theta(-x^-)A_i^{(2)} + \theta(x^-)A_i^{(3)}}^{A_i^{(2)}+\theta(x^-)A_i^{(1)}}, \overbrace{-A_i^{(2)}(\mathbf{x}_\perp) + A_i^{(3)}(\mathbf{x}_\perp)}^{A_i^{(1)}(\mathbf{x}_\perp)}]\delta(x^-)\theta(x^+) \tag{120}$$

$$= ig[A_i^{(2)}, A_i^{(1)}(\mathbf{x}_\perp)]\delta(x^-)\theta(x^+) \tag{121}$$

In relating F^{i+} to A_i note both the antisymmetry of $F^{\mu\nu}$ and the Lorentz signature $A_i = -A^i$.

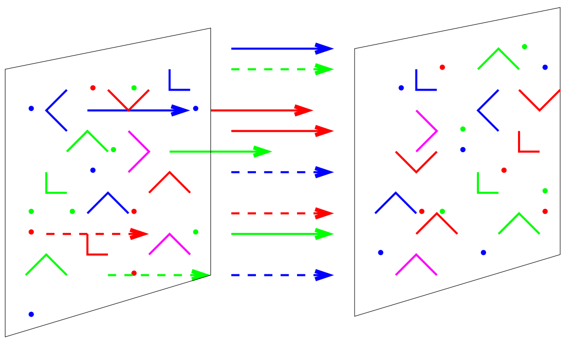
Requiring that all the equations are satisfied leads us to the initial conditions for the glasma fields first derived in Ref. [57]:

$$A_i^{(3)}|_{\tau=0} = A_i^{(1)} + A_i^{(2)} \tag{122}$$

$$A^\eta|_{\tau=0} = \frac{ig}{2}\left[A_i^{(1)}, A_i^{(2)}\right] \tag{123}$$

8.2 Properties of glasma fields

Now we have the gauge potentials at $\tau = 0$. From these we can calculate the field strength tensor, i.e. the chromomagnetic and chromoelectric fields. It is quite easy ton convince yourself that these are both in the z-directon. These leads to the following often seen picture:



- $\tau = 0$ longitudinal $E^z \sim \left[A_i^{(1)}, A_i^{(2)}\right]$ and $B^z \sim \varepsilon^{ij}\left[A_i^{(1)}, A_j^{(2)}\right]$
- $\perp$ correlation length $1/Q_s$.

One way to think of the longitudinal initial electric and magnetic fields is to note that in the absence of color charges the fields satisfy the nonabelian Gauss law and Bianchi identities, stating that the covariant divergences of the chromoelectric and -magnetic fields vanish. It is interesting to separate the commutator terms from the linear ones. Then one can interpret the nonlinear terms as kind of effective electric and magnetic field densities generated by the interaction of the electric and magnetic fields of one nucleus with the pure gauge potential of the other one.

- Gauss

$$[D_i, E^i] = 0 \implies \partial_i E^i = ig[A^i, E^i] \tag{124}$$

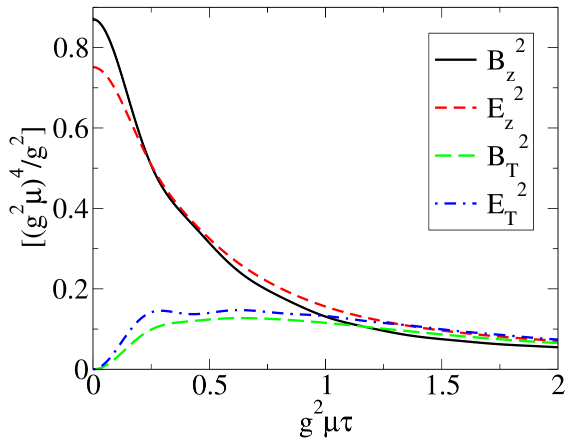
- Bianchi
$$[D_i, B^i] = 0 \implies \partial_i B^i = ig[A^i, B^i] \tag{125}$$

Loosely quating L. McLerran, the pure gauge field of one nucleus is a "Color Glass Fairy" that sprinkles magic dust on the Weizsäcker-Williams gluons (transverse fields that only live on the light cone, following the color charges) of the other nucleus. This magic fairy dust generates color charges (and chromomagnetic monopoles) on the gluons of the second nucleus, which serve as the end points of color flux tubes.

Digression: Glasma strings and Lund strings

One might think that this picture is similar to the Lund string model. In some ways it is, but in some important ways it is not. The first seemingly very different feature is that there is also a magnetic field, parametrically (and in fact, if one averages over a rotationally invariant target, exactly) equally large. The second, perhaps more consequential, difference is the length scale. The gluon fields depend on the transverse coordinate with a typical length scale $1/Q_s$*, which we are assuming to be a weak coupling momentum scale. This is in contrast to Lund strings, whose width and string tension are understood as confinement scale quantities. This means that for Lund strings, the energy is stored in the potential energy of the string. The beam particles are constantly stretching the string and transferring energy to the central rapidity region. This energy is then eventually released in a nonperturbative mechanism of string breaking via the Schwinger mechanism. For the glasma, on the other hand, the field modes can be interpreted as gluons with typical transverse momenta* $k_T \sim Q_s$*. They can scatter, turn into a quark-gluon plasma, and then eventully hadronize by some mechanism which a weak coupling calculation cannot capture. The energy is not stored in potential energy, but in the kinetic energy (momentum) of these gluonic modes. After the collision, the beams (the currents) are gone and are no longer transfering energy to the midrapidity region. This means that the energy at an early stage in the collision (say* $\tau \sim 1\text{fm}$*) is larger in the glasma picture: all the energy is already there, while in the Lund string picture the midrapidity region is still gaining energy by pulling back on the receding beams (and other hard particles) that are stretching the strings.*

Inside the future light cone the field equations have to be solved numerically. This can be done with relatively standard real time lattice gauge field methods.



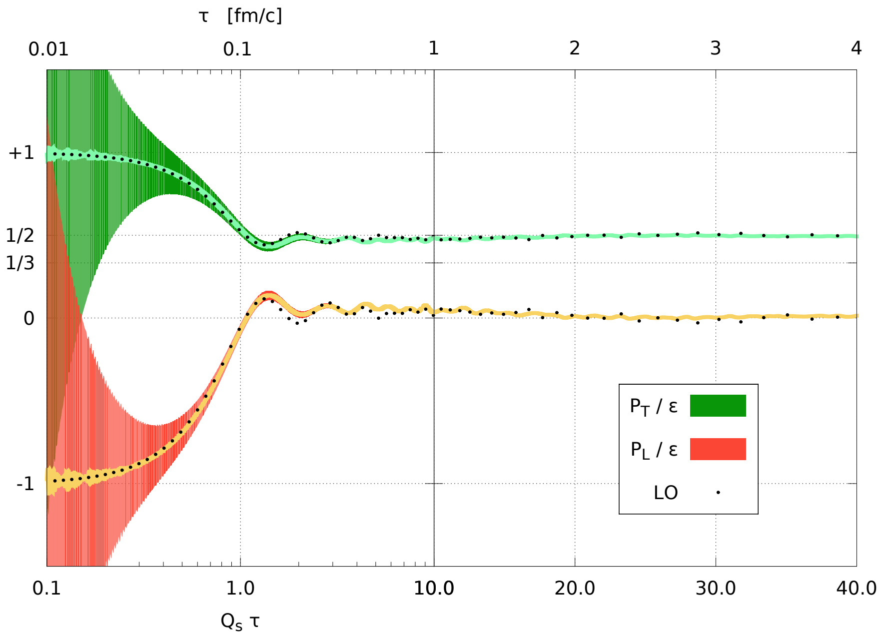
In this plot the transverse field notation means

$$E_T^2 = E_x^2 + E_y^2 \tag{126}$$
$$B_T^2 = B_x^2 + B_y^2 \tag{127}$$

It is interesting to see what this means in terms of the energy momentum tensor. For pure gauge Yang-Mills theory the diagonal components are given by

$$\varepsilon = = \frac{1}{2}\left[E_x^2 + E_y^2 + E_z^2 + B_x^2 + B_y^2 + B_z^2\right] \tag{128}$$
$$p_x = = \frac{1}{2}\left[-E_x^2 + E_y^2 + E_z^2 - B_x^2 + B_y^2 + B_z^2\right] \tag{129}$$
$$p_y = = \frac{1}{2}\left[E_x^2 - E_y^2 + E_z^2 + B_x^2 - B_y^2 + B_z^2\right] \tag{130}$$
$$p_z = = \frac{1}{2}\left[E_x^2 + E_y^2 - E_z^2 + B_x^2 + B_y^2 - B_z^2\right] \tag{131}$$

At the initial condition one starts with negative longitudinal pressure $p_z = -p_x = -p_z = -\varepsilon$. This is similar to the longitudinal chromoelectric field in a QCD string, which has a string tension, i.e. negative longitudinal pressure. here is a plot from [60]:



After a time $\tau \sim 1/Q_s$ the negative pressure approaches zero, and the transverse pressure becomes half of the energy density. This is the starting point for the isotropization of the system, which requires going beyond the classical approximation.

8.3 Gluon spectrum

In order to interpret these fields in terms of a number density of gluons (which is possible only after a time $\tau \gtrsim 1/Q_s$), one needs to decompose the solution in Fourier $\mathbf{k}_\perp$-modes. Knowing that this is a one scale problem, with the only dimensionful scale, one knows already beforehand that the gluon number distribution has to have the parametric form

$$\frac{dN_g}{dy\, d^2\mathbf{x}_\perp\, d^2\mathbf{p}_\perp} = \frac{1}{\alpha_s} f\left(\frac{p_T}{Q_s}\right) \tag{132}$$

In the dilute limit one can recover a perturbative k_T-factorization result for the gluon density. One first has to transform the fields to transverse Coulomb gauge $\partial_i A_i = 0$; this is done using an additional gauge freedom in the Fock-Schwinger gauge to perform τ-independent gauge transformations. Assuming that the fields are small, one neglects the nonlinear terms in the equations of motion for $\tau > 0$. The linearized equations are then wave equations in an expanding geometry, separately for each transverse momentum mode.

$$\left(\tau^2\partial_\tau^2 + \tau\partial_\tau + \tau^2\mathbf{k}_\perp^2\right) A_i(\tau,\mathbf{k}_\perp) = 0 \tag{133}$$

$$\left(\tau^2\partial_\tau^2 - \tau\partial_\tau + \tau^2\mathbf{k}_\perp^2\right) A_\eta(\tau,\mathbf{k}_\perp) = 0. \tag{134}$$

$$\Longrightarrow \quad A_i(\tau,\mathbf{k}_\perp) = A_i(\tau=0,\mathbf{k}_\perp) J_0(|\mathbf{k}_\perp|\tau) \qquad A^\eta(\tau,\mathbf{k}_\perp) = -\frac{1}{\tau|\mathbf{k}_\perp|} A^\eta(\tau=0,\mathbf{k}_\perp) J_1(|\mathbf{k}_\perp|\tau). \tag{135}$$

- These modes are (boost invariant) plane waves. Thus they can be interpreted as particles, gluons.
- In this approximation (interactions only at $\tau = 0$, free propagation after that), there is no contradiction between a classical field and particle description of the fields. The negative longitudinal pressure at $\tau = 0$ is due to the boost invariance of the situation. This imposes a specific restriction on the two

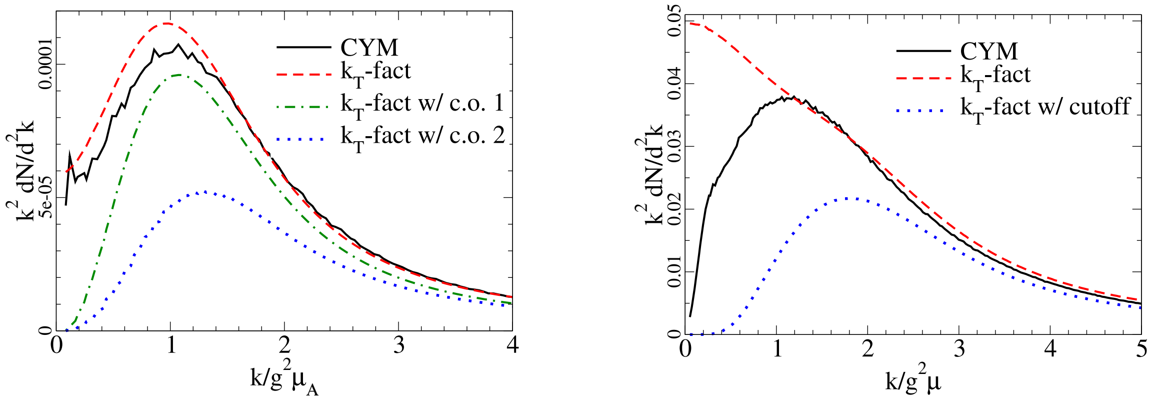


Figure 2: Full CYM calculation and different k_T-factorized approximations (with a cutoff on the momentum from one or the other nucleus). Left: dilute-dense case with very small $g^2\mu$ for one nucleus. Right: dense-dense case with similar $g^2\mu$ for both nuclei. These plots are from [62].

possible linear polarization states for gluons with only transverse momentum. The polarization state with a electric field in the z-direction and a magnetic field in a transverse direction starts in a phase where there is only an electric field, whereas the other polarization state with a magnetic field in the z-direction and an elecric field in a transverse direction has to start in a phase where there is only a magnetic field. The different modes present in these two polarization states oscillate in time with slightly different frequencies, and later get out of sync. Thus after a time $\tau \sim 1/Q_\mathrm{s}$ the phases of these oscillations are more randomly distributed, and the energy momentum tensor is just dictated by the boost invariance, i.e. absence of longitudinal momentum, i.e. zero longitudinal pressure.

- The initial fields related to Wilson lines, and via that to the dipole amplitude measured in DIS.

In this dilute limit, when we have turned off all the interactions between the gluons after $\tau = 0$ *and* expanded the color fields of both nuclei to the lowest nontrivial order in their color charge densities, the number distribution is given by a k_T-factorization formula

$$\frac{\mathrm{d}N}{\mathrm{d}y\,\mathrm{d}^2\mathbf{k}_\perp} = \frac{\alpha_\mathrm{s}}{S_\perp}\frac{2}{C_\mathrm{F}}\frac{1}{{k_T}^2}\int \mathrm{d}^2\mathbf{q}_\perp \varphi^{\mathrm{dip}}(\mathbf{q}_\perp)\varphi^{\mathrm{dip}}(|\mathbf{k}_\perp - \mathbf{q}_\perp|), \tag{136}$$

with $S_\perp$ the transverse area of the nuclei, and $\varphi^{\mathrm{dip}}(\mathbf{q}_\perp)$ the dipole distribution obtained from the Fourier-transform of the dipole operator (see Eq. (98), exact coefficients convention-dependent, [TBD: need to fix]). This calculation can also be repeated by assuming that *one* of the two colliding objects is dilute (a theorist's "pA" collision) [61].

However, no such analytic calculation exists for a dense-dense system , an "AA" collision. Also in the "AA" case, one can numerically verify that the k_T-factorization formula works for high k_T gluons. However, beacuse of the explicit $1/{k_T}^2$ in front, it cannot be integrated to give a finite total gluon multiplicity. The full result of the numerical calculation, on the other hand, can. Sometimes this is fixed by an ad hoc cutoff

$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{p}_\perp\,\mathrm{d}y} = \frac{1}{\alpha_\mathrm{s}}\frac{1}{{\mathbf{p}_\perp}^2}\int_{\mathbf{k}_\perp}\Big[\theta(p_T - k_T)\Big]\phi_y(\mathbf{k}_\perp)\phi_y(\mathbf{p}_\perp - \mathbf{k}_\perp) \tag{137}$$

Figure 2 shows a comparison of the k_T-factorized formula (136) to the result of the full CYM calculation, and also to the version with the cutoffs (137). The calculation for the dilute-dense case is done in the MV model with two very different values for $g^2\mu$ in the two colliding nuclei: in this case the result for ad hoc cutoff depends very much on whether one cuts off the spectrum at $k_T < p_T$ or $|\mathbf{p}_\perp - \mathbf{k}_\perp| < p_T$. The plot on the right in the same figure shows the same comparison in the case of a dense-dense collision, where the k_T-factorization formula does not work for $k_T \lesssim Q_\mathrm{s}$.

Figure 3 shows a comparison with experimental data from pA collisions; the theory predictions here are calculated with the k_T-factorization formula (136), combined with fragmentation functions to turn the produced gluons into hadrons.

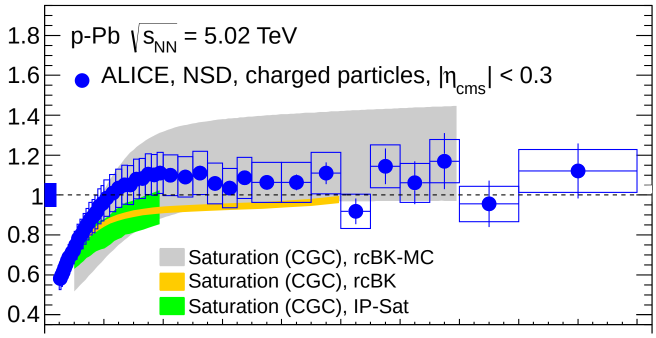


Figure 3: Comparison of CGC calculations to ALICE data on midrapidity nuclear modifications to single particle production. The plot is from ALICE [17], the "rcBK" and "IPsat" CGC calculations from [63] and the "rcBK-MC"-calculation from [64]; the difference between these is mostly in how one goes from a unintegrated gluon distribution in a proton to that in a nucleus, less in the formalism of how the particle spectrum is computed.

References

[1] Y. V. Kovchegov and E. Levin, *Quantum chromodynamics at high energy*, vol. 33. Cambridge University Press, 2012.

[2] V. Barone and E. Predazzi, *High-Energy Particle Diffraction*, vol. v.565 of *Texts and Monographs in Physics*. Springer-Verlag, Berlin Heidelberg, 2002.

[3] F. Gelis, *Initial state and thermalization in the Color Glass Condensate framework*, *Int. J. Mod. Phys.* **E24** (2015) 1530008 [arXiv:1508.07974 [hep-ph]].

[4] J. L. Albacete and C. Marquet, *Gluon saturation and initial conditions for relativistic heavy ion collisions*, *Prog. Part. Nucl. Phys.* **76** (2014) 1 [arXiv:1401.4866 [hep-ph]].

[5] S. Schlichting and D. Teaney, *The First fm/c of Heavy-Ion Collisions*, *Ann. Rev. Nucl. Part. Sci.* **69** (2019) 447 [arXiv:1908.02113 [nucl-th]].

[6] J. Berges, M. P. Heller, A. Mazeliauskas and R. Venugopalan, *QCD thermalization: Ab initio approaches and interdisciplinary connections*, *Rev. Mod. Phys.* **93** (2021) 035003 [arXiv:2005.12299 [hep-th]].

[7] E. Iancu and R. Venugopalan, *The Color glass condensate and high-energy scattering in QCD*, in *Quark-gluon plasma 4* (R. C. Hwa and X.-N. Wang, eds.), pp. 249–3363. 2003. arXiv:hep-ph/0303204 [hep-ph].

[8] F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, *The Color Glass Condensate*, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 463 [arXiv:1002.0333 [hep-ph]].

[9] T. Lappi, *Small x physics and RHIC data*, *Int. J. Mod. Phys.* **E20** (2011) 1 [arXiv:1003.1852 [hep-ph]].

[10] H. Weigert, *Evolution at small x(bj): The Color glass condensate*, *Prog. Part. Nucl. Phys.* **55** (2005) 461 [arXiv:hep-ph/0501087 [hep-ph]].

[11] J. Jalilian-Marian and Y. V. Kovchegov, *Saturation physics and deuteron-Gold collisions at RHIC*, *Prog. Part. Nucl. Phys.* **56** (2006) 104 [arXiv:hep-ph/0505052 [hep-ph]].

[12] F. Gelis, T. Lappi and R. Venugopalan, *High energy scattering in Quantum Chromodynamics*, *Int. J. Mod. Phys.* **E16** (2007) 2595 [arXiv:0708.0047 [hep-ph]].

[13] S. J. Brodsky, H.-C. Pauli and S. S. Pinsky, *Quantum chromodynamics and other field theories on the light cone*, *Phys. Rept.* **301** (1998) 299 [arXiv:hep-ph/9705477 [hep-ph]].

[14] C. Marquet, *Chromodynamique quantique à haute énergie, théorie et phénoménologie appliquée aux collisions de hadrons*, 2006. PhD thesis, Paris, LPTHE. https://tel.archives-ouvertes.fr/tel-00096416/fr/.

[15] H. Hänninen, *Deep inelastic scattering in the dipole picture at next-to-leading order*. PhD thesis, Jyvaskyla U, Jyvaskyla U., 2021. arXiv:2112.08818 [hep-ph].

[16] J. D. Bjorken, J. B. Kogut and D. E. Soper, *Quantum Electrodynamics at Infinite Momentum: Scattering from an External Field*, Phys. Rev. **D3** (1971) 1382.

[17] **ALICE** collaboration, B. Abelev *et. al.*, *Transverse momentum distribution and nuclear modification factor of charged particles in p-Pb collisions at* $\sqrt{s_{NN}} = 5.02$ *TeV*, Phys. Rev. Lett. **110** (2013) 082302 [arXiv:1210.4520 [nucl-ex]].

[18] **LHCb** collaboration, R. Aaij *et. al.*, *Measurement of the Nuclear Modification Factor and Prompt Charged Particle Production in* $p - Pb$ *and pp Collisions at* $\sqrt{s_{NN}}$*=5 TeV*, Phys. Rev. Lett. **128** (2022) 142004 [arXiv:2108.13115 [hep-ex]].

[19] **LHCb** collaboration, *Nuclear modification factor of neutral pions in the forward and backward regions in pPb collisions*, arXiv:2204.10608 [nucl-ex].

[20] G. Beuf, *Dipole factorization for DIS at NLO: Loop correction to the* $\gamma^*_{T,L} \to q\bar{q}$ *light-front wave functions*, Phys. Rev. D **94** (2016) 054016 [arXiv:1606.00777 [hep-ph]].

[21] G. Beuf, *Dipole factorization for DIS at NLO: Combining the* $q\bar{q}$ *and* $q\bar{q}g$ *contributions*, Phys. Rev. D **96** (2017) 074033 [arXiv:1708.06557 [hep-ph]].

[22] T. Lappi and R. Paatelainen, *The one loop gluon emission light cone wave function*, Annals Phys. **379** (2017) 34 [arXiv:1611.00497 [hep-ph]].

[23] H. Hänninen, T. Lappi and R. Paatelainen, *One-loop corrections to light cone wave functions: the dipole picture DIS cross section*, Annals Phys. **393** (2018) 358 [arXiv:1711.08207 [hep-ph]].

[24] G. Beuf, T. Lappi and R. Paatelainen, *Massive quarks in NLO dipole factorization for DIS: Longitudinal photon*, Phys. Rev. D **104** (2021) 056032 [arXiv:2103.14549 [hep-ph]].

[25] G. Beuf, T. Lappi and R. Paatelainen, *Massive Quarks at One Loop in the Dipole Picture of Deep Inelastic Scattering*, Phys. Rev. Lett. **129** (2022) 072001 [arXiv:2112.03158 [hep-ph]].

[26] G. Beuf, T. Lappi and R. Paatelainen, *Massive quarks in NLO dipole factorization for DIS: Transverse photon*, Phys. Rev. D **106** (2022) 034013 [arXiv:2204.02486 [hep-ph]].

[27] L. D. McLerran and R. Venugopalan, *Gluon distribution functions for very large nuclei at small transverse momentum*, Phys. Rev. **D49** (1994) 3352 [arXiv:hep-ph/9311205 [hep-ph]].

[28] L. D. McLerran and R. Venugopalan, *Computing quark and gluon distribution functions for very large nuclei*, Phys. Rev. **D49** (1994) 2233 [arXiv:hep-ph/9309289 [hep-ph]].

[29] L. D. McLerran and R. Venugopalan, *Green's functions in the color field of a large nucleus*, Phys. Rev. **D50** (1994) 2225 [arXiv:hep-ph/9402335 [hep-ph]].

[30] I. Balitsky, *Operator expansion for high-energy scattering*, Nucl. Phys. **B463** (1996) 99 [arXiv:hep-ph/9509348].

[31] Y. V. Kovchegov, *Unitarization of the BFKL pomeron on a nucleus*, Phys. Rev. **D61** (2000) 074018 [arXiv:hep-ph/9905214].

[32] Y. V. Kovchegov, *Small-x F2 structure function of a nucleus including multiple pomeron exchanges*, Phys. Rev. **D60** (1999) 034008 [arXiv:hep-ph/9901281].

[33] J. Jalilian-Marian, A. Kovner, L. D. McLerran and H. Weigert, *The intrinsic glue distribution at very small x*, Phys. Rev. **D55** (1997) 5414 [arXiv:hep-ph/9606337 [hep-ph]].

[34] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, *The BFKL equation from the Wilson renormalization group*, Nucl. Phys. **B504** (1997) 415 [arXiv:hep-ph/9701284].

[35] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, *The Wilson renormalization group for low x physics: Towards the high density regime*, Phys. Rev. **D59** (1998) 014014 [arXiv:hep-ph/9706377].

[36] J. Jalilian-Marian, A. Kovner and H. Weigert, *The Wilson renormalization group for low x physics: Gluon evolution at finite parton density*, Phys. Rev. **D59** (1998) 014015 [arXiv:hep-ph/9709432].

[37] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, *Unitarization of gluon distribution in the doubly logarithmic regime at high density*, Phys. Rev. **D59** (1999) 034007 [arXiv:hep-ph/9807462].

[38] E. Iancu, A. Leonidov and L. D. McLerran, *Nonlinear gluon evolution in the color glass condensate. I*, Nucl. Phys. **A692** (2001) 583 [arXiv:hep-ph/0011241].

[39] E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, *Nonlinear gluon evolution in the color glass condensate. II*, Nucl. Phys. **A703** (2002) 489 [arXiv:hep-ph/0109115].

[40] A. H. Mueller, *A simple derivation of the JIMWLK equation*, Phys. Lett. **B523** (2001) 243 [arXiv:hep-ph/0110169].

[41] C. Marquet, *Forward inclusive dijet production and azimuthal correlations in p(A) collisions*, Nucl. Phys. A **796** (2007) 41 [arXiv:0708.0231 [hep-ph]].

[42] A. Dumitru, A. Hayashigaki and J. Jalilian-Marian, *The Color glass condensate and hadron production in the forward region*, Nucl. Phys. A **765** (2006) 464 [arXiv:hep-ph/0506308].

[43] G. A. Chirilli, B.-W. Xiao and F. Yuan, *One-loop Factorization for Inclusive Hadron Production in pA Collisions in the Saturation Formalism*, Phys. Rev. Lett. **108** (2012) 122301 [arXiv:1112.1061 [hep-ph]].

[44] G. A. Chirilli, B.-W. Xiao and F. Yuan, *Inclusive Hadron Productions in pA Collisions*, Phys. Rev. D **86** (2012) 054005 [arXiv:1203.6139 [hep-ph]].

[45] B. Ducloué, T. Lappi and Y. Zhu, *Implementation of NLO high energy factorization in single inclusive forward hadron production*, Phys. Rev. D **95** (2017) 114007 [arXiv:1703.04962 [hep-ph]].

[46] E. Iancu, A. H. Mueller and D. N. Triantafyllopoulos, *CGC factorization for forward particle production in proton-nucleus collisions at next-to-leading order*, JHEP **12** (2016) 041 [arXiv:1608.05293 [hep-ph]].

[47] A. Dumitru, J. Jalilian-Marian, T. Lappi, B. Schenke and R. Venugopalan, *Renormalization group evolution of multi-gluon correlators in high energy QCD*, Phys. Lett. **B706** (2011) 219 [arXiv:1108.4764 [hep-ph]].

[48] E. Iancu, K. Itakura and L. McLerran, *A Gaussian effective theory for gluon saturation*, Nucl. Phys. A **724** (2003) 181 [arXiv:hep-ph/0212123].

[49] J. P. Blaizot, F. Gelis and R. Venugopalan, *High-energy pA collisions in the color glass condensate approach. 2. Quark production*, Nucl. Phys. A **743** (2004) 57 [arXiv:hep-ph/0402257].

[50] F. Gelis, A. M. Stasto and R. Venugopalan, *Limiting fragmentation in hadron-hadron collisions at high energies*, Eur. Phys. J. C **48** (2006) 489 [arXiv:hep-ph/0605087].

[51] F. Dominguez, C. Marquet and B. Wu, *On multiple scatterings of mesons in hot and cold QCD matter*, Nucl. Phys. A **823** (2009) 99 [arXiv:0812.3878 [nucl-th]].

[52] F. Dominguez, C. Marquet, B.-W. Xiao and F. Yuan, *Universality of Unintegrated Gluon Distributions at small x*, Phys. Rev. D **83** (2011) 105005 [arXiv:1101.0715 [hep-ph]].

[53] B. Schenke, P. Tribedy and R. Venugopalan, *Fluctuating Glasma initial conditions and flow in heavy ion collisions*, *Phys. Rev. Lett.* **108** (2012) 252301 [arXiv:1202.6646 [nucl-th]].

[54] T. Lappi, *Wilson line correlator in the MV model: Relating the glasma to deep inelastic scattering*, *Eur. Phys. J. C* **55** (2008) 285 [arXiv:0711.3039 [hep-ph]].

[55] J. Jalilian-Marian, A. Kovner, L. D. McLerran and H. Weigert, *The intrinsic glue distribution at very small x*, in *Phys. Rev.* [33], pp. 5414–5428, [arXiv:hep-ph/9606337 [hep-ph]].

[56] T. Lappi and S. Schlichting, *Linearly polarized gluons and axial charge fluctuations in the Glasma*, *Phys. Rev.* **D97** (2018) 034034 [arXiv:1708.08625 [hep-ph]].

[57] A. Kovner, L. D. McLerran and H. Weigert, *Gluon production from nonAbelian Weizsacker-Williams fields in nucleus-nucleus collisions*, *Phys. Rev.* **D52** (1995) 6231 [arXiv:hep-ph/9502289 [hep-ph]].

[58] Y. V. Kovchegov and D. H. Rischke, *Classical gluon radiation in ultrarelativistic nucleus-nucleus collisions*, *Phys. Rev.* **C56** (1997) 1084 [arXiv:hep-ph/9704201 [hep-ph]].

[59] J. P. Blaizot, F. Gelis and R. Venugopalan, *High-energy pA collisions in the color glass condensate approach. 1. Gluon production and the Cronin effect*, *Nucl. Phys.* **A743** (2004) 13 [arXiv:hep-ph/0402256 [hep-ph]].

[60] T. Epelbaum and F. Gelis, *Pressure isotropization in high energy heavy ion collisions*, *Phys. Rev. Lett.* **111** (2013) 232301 [arXiv:1307.2214 [hep-ph]].

[61] A. Dumitru and L. D. McLerran, *How protons shatter colored glass*, *Nucl. Phys. A* **700** (2002) 492 [arXiv:hep-ph/0105268].

[62] J. P. Blaizot, T. Lappi and Y. Mehtar-Tani, *On the gluon spectrum in the glasma*, *Nucl. Phys.* **A846** (2010) 63 [arXiv:1005.0955 [hep-ph]].

[63] P. Tribedy and R. Venugopalan, *QCD saturation at the LHC: Comparisons of models to p + p and A + A data and predictions for p + Pb collisions*, *Phys. Lett.* **B710** (2012) 125 [arXiv:1112.2445 [hep-ph]]. [Erratum: Phys. Lett.B718,1154(2013)].

[64] J. L. Albacete, A. Dumitru, H. Fujii and Y. Nara, *CGC predictions for p + Pb collisions at the LHC*, *Nucl. Phys. A* **897** (2013) 1 [arXiv:1209.2001 [hep-ph]].