Jets and differential operators in noncommutative geometry

Keegan Flood

Joint work with Mauro Mantegazza and Henrik Winther

UniDistance Suisse, Switzerland

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Setting and algebraic preliminaries

- To generalize the notion of jet and differential operator from classical differential geometry to the noncommutative setting all that is required is the following.
  1. A first order differential calculus
  2. An exterior algebra

Definition

A first order calculus (Ω^1_d, d) *for a unital associative* $\mathbb{k}$*-algebra* A *is*

1. *An* A*-bimodule* Ω^1_d*,*
2. *a map* $d\colon A \to \Omega^1_d$ *satisfying* $d(ab) = adb + (da)b$*,*
3. *such that* $AdA = \Omega^1_d$*.*

Definition

An exterior algebra $\Omega_d^\bullet$ over a $\Bbbk$-algebra A, is an associative graded algebra $(\Omega_d^\bullet = \bigoplus_{n\geq 0}\Omega_d^n, \wedge)$ equipped with a map d such that

1. $\Omega_d^0 = A$;
2. *d is a differential, that is a $\Bbbk$-linear map $d\colon \Omega_d^\bullet \to \Omega_d^\bullet$ such that $d(\Omega_d^n) \subseteq \Omega_d^{n+1}$ for all $n \geq 0$, which satisfies $d^2 = 0$ and*
$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^n\alpha\wedge d\beta, \qquad \forall \alpha\in\Omega_d^n, \beta\in\Omega_d^h.$$
3. *A generates $\Omega_d^\bullet$ via the d and the $\wedge$.*

Remark

Given an exterior algebra $\Omega_d^\bullet$, the first grade and $d\colon \Omega_d^0 = A \to \Omega_d^1$ form a first order calculus for A. Vice versa, given a first order differential calculus, Ω_d^1, a maximal exterior algebra, $\Omega_{d,\max}^\bullet$, is given by quotienting the tensor algebra by the minimal relations for $d^2 = 0$ to hold.

Quantum symmetric forms

Given an exterior algebra $\Omega_d^\bullet$ over A, we can define the tensor functors

$$\Omega_d^n \colon {}_A\mathrm{Mod} \longrightarrow {}_A\mathrm{Mod} \qquad E \longmapsto \Omega_d^n \otimes_A E.$$

We define the functors

$$S_d^0 = \Omega_d^0 = \mathrm{id}_{{}_A\mathrm{Mod}}, \qquad S_d^1 = \Omega_d^1 := \Omega_d^1 \otimes_A -.$$

For $n \geq 0$, the functor of quantum symmetric forms S_d^n is defined by induction as the kernel of the following composition

$$\Omega_d^1 \circ S_d^{n-1} \xrightarrow{\Omega_d^1(\iota_\wedge^{n-1})} \Omega_d^1 \circ \Omega_d^1 \circ S_d^{n-2} \xrightarrow{\wedge_{S_d^{n-2}}} \Omega_d^2 \circ S_d^{n-2}$$

and $\iota_\wedge^n \colon S_d^n \longrightarrow \Omega_d^1 \circ S_d^{n-1}$ is the inclusion.

Spencer cohomology

For all $k, h \geq 0$, consider the functor $\Omega_d^k \circ S_d^h$, and define $\delta^{h,k}$ as the following composition

$$\Omega_d^k \circ S_d^h \xrightarrow{\Omega_d^k(\iota_\wedge^n)} \Omega_d^k \circ \Omega_d^1 \circ S_d^{h-1} \xrightarrow{(-1)^k \wedge^{k,1}_{S_d^{h-1}}} \Omega_d^{k+1} \circ S_d^{h-1}, \qquad \delta^{h,k} \colon \Omega_d^k \circ S_d^h \to \Omega_d^{k+1} \circ S_d^{h-1}$$

We thus obtain a complex in the category of endofunctors on ${}_A\mathrm{Mod}$.

$$0 \longrightarrow S_d^n \xrightarrow{\delta^{n,0}} \Omega_d^1 \circ S_d^{n-1} \xrightarrow{\delta^{n-1,1}} \Omega_d^2 \circ S_d^{n-2} \xrightarrow{\delta^{n-2,2}} \Omega_d^3 \circ S_d^{n-3} \xrightarrow{\delta^{n-3,3}} \cdots$$

Definition (Spencer cohomology)

We call this the Spencer δ-complex, its cohomology the Spencer cohomology, and we denote the cohomology at $\Omega_d^k \circ S_d^h$ by $H^{h,k}$.

Universal calculus and 1-jets

Definition

The universal calculus Ω^1_u *for* A *is given by the kernel of the multiplication map,*

$$\Omega^1_u = \ker(\cdot) \subset A \otimes A.$$

The corresponding universal differential is

$$d_u \colon a \mapsto 1 \otimes a - a \otimes 1$$

Proposition

We have

$$0 \to \Omega^1_u \to A \otimes A \to A \to 0$$

For the bimodule map $\pi^{1,0}_u : a \otimes b \mapsto ab$*. Moreover, the universal prolongation* $j^1_u \colon a \mapsto 1 \otimes a$ *splits the sequence in* Mod_A*.*

Universal property of the universal calculus

Recall the following result.

Proposition

For any first order calculus Ω^1_d, there is an epimorphism $p_d : \Omega^1_u \to \Omega^1_d$ which takes d_u to d, and we have

$$0 \to N_d \to \Omega^1_u \to \Omega^1_d \to 0$$

For a sub-bimodule $N_d \subset \Omega^1_u$.

We call N_d the space of first order differential relations. In the case of classical differential geometry, it contains things such as

$$d_u f - \sum_i f_{x^i} d_u x^i.$$

For any left A-module E, we define

$$N_d(E) = \ker(p_{d,E}) = \Big\{ \sum_i a_i \otimes e_i \Big| \sum_i a_i e_i = 0, \sum_i da_i \otimes_A e_i = 0 \Big\} \subset A \otimes E$$

Definition

The 1-jet module for a left A-module E is

$$J^1_d E := J^1_u E / N_d(E) = A \otimes E / N_d(E).$$

The prolongation operator is $j^1_{d,E} \colon E \to J^1_d E$, given by $j^1_{d,E}(e) = [1 \otimes e]$. The projection map is $\pi^{1,0}_{d,E} \colon [\sum_i a_i \otimes e_i] \mapsto \sum_i a_i e_i$.

A plethora of jet functors

We construct the following three families of functors:

- The nonholonomic jet functors $J_d^{(n)} \colon {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$
- The semiholonomic jet functors $J_d^{[n]} \colon {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$
- The holonomic jet functors $J_d^{n} \colon {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$

In particular we have $J_d^{(0)} = J_d^{[0]} = J_d^0 = \mathrm{id}_{{}_A\mathrm{Mod}}$, and $J_d^{(1)} = J_d^{[1]} = J_d^1$. These functors come equipped with natural transformations

$$j_d^{(n)} \colon \mathrm{id}_{{}_A\mathrm{Mod}} \longrightarrow J_d^{(n)} \qquad j_d^{[n]} \colon \mathrm{id}_{{}_A\mathrm{Mod}} \longrightarrow J_d^{[n]} \qquad j_d^{n} \colon \mathrm{id}_{{}_A\mathrm{Mod}} \longrightarrow J_d^{n},$$

which are respectively called the nonholonomic, semiholonomic, and holonomic jet prolongation maps. We also have the natural transformations,

$$\pi_d^{(n,n-1;m)} \colon J_d^{(n)} \longrightarrow J_d^{(n-1)}, \quad \pi_d^{[n,n-1]} \colon J_d^{[n]} \longrightarrow J_d^{[n-1]}, \quad \pi_d^{n,n-1} \colon J_d^{n} \longrightarrow J_d^{n-1},$$

respectively called the nonholonomic, semiholonomic, and holonomic jet projections.

Holonomic 2-jets

We can describe $J_d^2 E$ implicitly as the kernel of a left-linear (bilinear for $E = A$) map

$$\widetilde{Đ}_E \colon J_d^{(2)} E \longrightarrow (\Omega_d^1 \ltimes \Omega_d^2)(E),$$

where $(\Omega_d^1 \ltimes \Omega_d^2)(E) \cong (\Omega_d^1 \ltimes \Omega_d^2) \otimes_A E$.
As a right A-module, $\Omega_d^1 \ltimes \Omega_d^2 \cong \Omega_d^1 \oplus \Omega_d^2$, but as an A-bimodule, it comes equipped with a non-trivial left action

$$f \star (\alpha + \omega) = f\alpha + df \wedge \alpha + f\omega, \qquad \forall f \in A,\ \alpha \in \Omega_d^1,\ \omega \in \Omega_d^2.$$

Explicitly, we have

$$\begin{aligned} \widetilde{Đ}_E \colon J_d^{(2)} E &\longrightarrow (\Omega_d^1 \ltimes \Omega_d^2)(E) \\ [a \otimes b] \otimes_A [c \otimes e] &\longmapsto (ad(bc) \otimes_A e, da \wedge d(bc) \otimes_A e)\,. \end{aligned}$$

Definition (Holonomic n-jet functor)

Let A be a $\mathbb{k}$-algebra endowed with an exterior algebra $\Omega^\bullet_d$. We define J^n_d as the kernel of the natural transformation

$$\widetilde{\mathrm{Đ}}_{J^{n-2}_d} \circ J^1_d(l^{n-1}_d)\colon J^1_d \circ J^{n-1}_d \longrightarrow (\Omega^1_d \ltimes \Omega^2_d) \circ J^{n-2}_d,$$

where we denote the natural inclusion by $l^n_d\colon J^n_d \longrightarrow J^1_d \circ J^{n-1}_d$. We call J^n_d the (holonomic) n-jet functor.

Higher jet exact sequence

Theorem (Holonomic jet exact sequence)

Let A be a $\mathbb{k}$-algebra endowed with an exterior algebra $\Omega_d^\bullet$ such that Ω_d^1, Ω_d^2, and Ω_d^3 are flat in Mod_A. For $n \geq 1$, if the Spencer cohomology $H^{m,2}$ vanishes, for all $1 \leq m < n-2$, then the following sequence is exact,

$$0 \longrightarrow S_d^n \xhookrightarrow{\iota_d^n} J_d^n \xrightarrow{\pi_d^{n,n-1}} J_d^{n-1} \longrightarrow H^{n-2,2}.$$

Therefore, if $H^{n-2,2} = 0$ we obtain a short exact sequence

$$0 \longrightarrow S_d^n \xhookrightarrow{\iota_d^n} J_d^n \xrightarrow{\pi_d^{n,n-1}} J_d^{n-1} \longrightarrow 0.$$

Definition

Let $E, F \in {}_A\mathrm{Mod}$. *A* $\Bbbk$*-linear map* $\Delta\colon E \to F$ *is called a* ***linear differential operator*** *of order at most* n *with respect to the exterior algebra* $\Omega^\bullet_d$*, if it factors through the holonomic prolongation operator* j^n_d*, i.e. there exists an* ***A-module map*** $\widetilde{\Delta} \in {}_A\mathrm{Hom}(J^n_d E, F)$ *such that the following diagram commutes:*

$$\begin{array}{ccc} J^n_d E & & \\ {\scriptstyle j^n_d}\uparrow & \overset{\widetilde{\Delta}}{\searrow} & \\ E & \xrightarrow{\ \Delta\ } & F \end{array}$$

If n *is* ***minimal****, we say that* Δ *is a linear differential operator of* ***order*** n*.*

Examples of differential operators

- The differential d.
- Suppose Ω^1_d is free and finitely-generated as a left A-module, i.e. parallelizable. Given a basis $\theta_1, \ldots, \theta_n$, we can define the partial derivative operator $\partial_i \colon A \to A$, by $da = \sum_i \partial_i(a)\theta_i$.
- A (left) connection with respect to the first order differential calculus Ω^1_d on a left A-module E is a k-linear map

$$E \longrightarrow \Omega^1_d \otimes_A E,$$

satisfying the identity

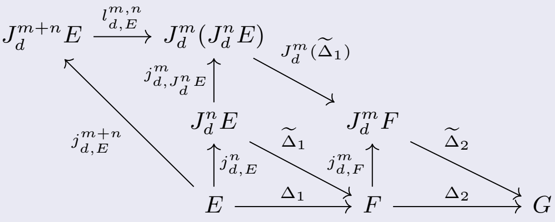
$$\nabla(fe) = df \otimes_A e + f\nabla e.$$

- The operator $Ð_E \colon J^1_d(E) \to (\Omega^1_d \ltimes \Omega^2_d)(E)$, whose lift $\widetilde{Ð}_E$ defines second order jets.
- Vector fields: $\mathfrak{X}_d := \mathrm{Diff}^1_d(A, A) \cap \mathrm{Ann}(\ker(d))$. Note that $\mathfrak{X}_d \simeq {}_A\mathrm{Hom}(\Omega^1_d, A)$.

Proposition

Let $\Delta_1 \colon E \to F$ and $\Delta_2 \colon F \to G$ be differential operators of order at most n and m, respectively. Then the composition $\Delta_2 \circ \Delta_1 \colon E \to G$ is a differential operator of order at most $n + m$.

Proof.



□

Corollary

There is a category $\mathrm{Diff}_d^{\mathrm{fin}}$ with the same objects as ${}_A\mathrm{Mod}$ and with morphisms given by the finite order differential operators.

Representability

Definition

We say that the functor $\mathrm{Diff}^n_d(E,-)$ *is* *representable* *if there is an object* Q *such that* $\mathrm{Diff}^n_d(E,-) \simeq {}_A\mathrm{Hom}(Q,-)$.

Question

When is $\mathrm{Diff}^n_d(E,-)$ *represented by* $J^n_d E$*?*

Higher order differential relations

There is a natural transformation

$$\hat{p}^n_d : J^1_u = A \otimes - \to J^n_d$$

Explicitly given by

$$a \otimes e \to a j^n_d(e)$$

We term its kernel the differential relations of order n, written $\ker \hat{p}^n_d(E) = N^n_d(E)$.

Proposition

Let the n-jet sequence be left exact. We have $\hat{p}^n_d|_{N^{n-1}} \to S^n_d$. The following are equivalent:

- $\hat{p}^n_d|_{N^{n-1}}(E) \to S^n_d E$ *is surjective.*
- $S^n_d(E) \subset A j^n_d(E)$.

When the latter condition is satisfied, we say symmetric forms of degree n are generated by differential relations.

Theorem

Suppose the n-jet sequence is left exact, and that $J_d^{n-1}E$ represents differential operators. Then $J_d^n E$ represents differential operators if and only if symmetric forms of degree n are generated by differential relations. In that case, we have

$$S_d^n E = (N^{n-1}/N^n)(E)$$

Skeletal jet functors

Let us introduce one more “jet functor”.

Definition

The skeletal n-jet functor is defined as the image subfunctor

$$\mathbb{J}^n_d = \operatorname{Im} \hat{p}^n_d$$

Theorem

We have the following:

- $\mathbb{J}^0_d = J^0_d$ *and* $\mathbb{J}^1_d = J^1_d$.
- *Let* $\operatorname{Tor}(\Omega^1_d, E) = 0$. *Then* $\mathbb{J}^2_d = J^2_d$ *if and only if* $\Omega^2_d = \Omega^2_{d,max}$, *the maximal prolongation*
- $\operatorname{Diff}^n_d(E, -)$ *is representable if and only if* $\mathbb{J}^n_d = J^n_d$, *and then the representing object is* J^n_d.

∞-jet functor

Consider the following diagram in the abelian category of endofunctors on ${}_A\mathrm{Mod}$, constructed using the jet projections.

$$\cdots J^n_d \xrightarrow{\pi^{n,n-1}_d} J^{n-1}_d \cdots \xrightarrow{\pi^{3,2}_d} J^2_d \xrightarrow{\pi^{2,1}_d} J^1_d \xrightarrow{\pi^{1,0}_d} \mathrm{id}_{{}_A\mathrm{Mod}}.$$

We call the above diagram the *(holonomic) jet tower*, and its limit in the category of endofunctors on ${}_A\mathrm{Mod}$ the (holonomic) ∞-jet functor, denoted

$$J^\infty_d := \lim_{n\in\mathbb{N}} J^n_d.$$

Jet comonad

Given an object E in ${}_A\mathrm{Mod}$, denote an element of $J^\infty_d E$ by

$$\mathbf{e}_0 := (\dots, \hat{e}_n, \dots, \hat{e}_1, \hat{e}_0)$$

Similarly, we will denote a truncated element as follows.

$$\mathbf{e}_k := (\dots, \hat{e}_{k+n}, \dots, \hat{e}_{k+1}, \hat{e}_k)$$

Proposition

J^∞_d is a comonad with comultiplication $\iota \colon J^\infty_d \longrightarrow J^\infty_d J^\infty_d$ and counit $\epsilon \colon J^\infty_d \longrightarrow \mathrm{id}_{{}_A\mathrm{Mod}}$ given, respectively, by

$$\mathbf{e}_0 \mapsto (\dots, \mathbf{e}_n, \dots, \mathbf{e}_1, \mathbf{e}_0) \quad \text{and} \quad \mathbf{e}_0 \mapsto \hat{e}_0$$

Proof.

Show that $(\dots, \mathbf{e}_n, \dots, \mathbf{e}_1, \mathbf{e}_0) \in J^\infty_d J^\infty_d E$, then check counitality and coassociativity. □

Noncommutative differential equations

This gives us several ways to proceed in developing a noncommutative theory of (linear) differential equations:

- Via $\mathcal{D}$-modules and $\mathcal{D}$-algebras give notions of linear and (polynomial) nonlinear differential equations.
- Via a noncommutative analogue of the geometric theory of differential equations developed by Spencer, Quillen, Goldschmidt, etc., where one considers monomorphisms $\mathcal{E} \hookrightarrow J^k_d E$.
- Via the comandic analogue of Vinogradov's category of differential equations. The Eilenberg-Moore category for the jet comonad (cf. Marvan, Schreiber, Khavkine) where the objects are diffieties (i.e. infinitely prolonged differential equations) and the morphisms are Cartan distribution preserving maps.