Signal communication and modular theory

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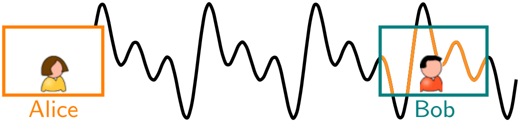
Signal communication

Suppose Alice sends a signal to Bob that is codified by a function of time f. Bob can measure the value f only within *a certain time interval*; moreover, the frequency of f is filtered by the signal device within a *certain interval in the spectrum amplitude*



Say both intervals are equal to $B = (-1, 1)$. As is well known, if a function f and its Fourier transform $\hat{f}$ are both supported in bounded intervals, then f is the zero function. So one is faced with the problem of simultaneously maximizing the portions of energy and amplitude spectrum within the intervals

$$||f||_B^2/||f||^2 , \quad ||\hat{f}||_B^2/||\hat{f}||^2 ,$$

the concentration problem.

The concentration problem

The problem of best approximating, with support concentration, a function and its Fourier transform is a classical problem; in particular, it lies behind Heisenberg uncertainty relations in Quantum Mechanics and is studied in Quantum Field Theory too (Jaffe, etc.)

In the '60ies, this problem was studied in seminal works by Slepian, Pollak and Landau. Denote by $\mathcal{F} : f \mapsto \hat{f}$ the Fourier transform and by $\mathcal{F}_B$ the *truncated Fourier transform*

$$\mathcal{F}_B = E_B \mathcal{F} E_B$$

$$(\mathcal{F}_B f)(p) = \frac{\chi_B(p)}{\sqrt{2\pi}} \int_B f(x) e^{-ixp} dx$$

as an operator on L^2.

The functions that best maximize the concentration problem are eigenfunctions of $\mathcal{F}_B$ with higher eigenvalues.

Since $||\mathcal{F}_B^*\mathcal{F}_B|| = ||\mathcal{F}_B||^2$, one can equivalently consider the angle operator

$$T_B \equiv \mathcal{F}_B^*\mathcal{F}_B = E_B\hat{E}_B E_B$$

where $\hat{E}_B = \mathcal{F}^* E_B \mathcal{F}$. This is a Hilbert-Schmidt integral

$$T_B = \int_B k_B(x-y)f(y)dy$$

$$k(x) = \frac{1}{(2\pi)^{1/2}}\frac{\sin x}{x}$$

and one has the eigenvalue problem

$$T_B f = \lambda f$$

This spectral analysis is not easily doable a priori.

The lucky accident

However, by the lucky accident figured out in by Slepian et al. , this integral operator commutes with a linear differential operator, the prolate operator

$$W = \frac{d}{dx}(1 - x^2)\frac{d}{dx} - x^2 \,,$$

that commutes with the angle operator, so these eigenfunctions were computed.

W is a classical operator, it arises by separating the 3-dimensional scalar wave equation in a prolate spheroidal coordinate system.

Connes raised new interest in this operator. Connes and Moscovici show an impressive relation of the prolate spectrum with the asymptotic distribution of the zeros of the Riemann ζ-function.

We want to understand the role of the prolate operator on a conceptual basis, in relation to the mentioned lucky accident: the prolate operator as an *entropy operator*.

We shall generalize the prolate operator in higher dimensions, guided by QFT

$$W = (1 - r^2)\nabla^2 - 2r\partial_r - r^2$$

on $S(\mathbb{R}^d)$ admits a natural extension W that commutes with the truncated Fourier transform F_B (In the one-dimensional case, W itself is selfadjoint (Connes-Moscovici)). The expectation values of W on $L^2(B)$ will indeed be entropy quantities.

Tomita-Takesaki modular theory

$\mathcal{M}$ a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\ {\scriptstyle X \to X\Omega} \big\downarrow & & \big\downarrow {\scriptstyle X \to X\Omega} \\ \mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H} \end{array}$$

$S = \bar{S}_0 = J\Delta^{1/2}$, Δ and J modular operator and conjugation

$$t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^\varphi(X) = \Delta^{it} X \Delta^{-it}$$

modular automorphisms **intrinsic dynamics associated with** φ!

$$J\mathcal{M}J = \mathcal{M}' \quad \text{on } \mathcal{H}$$

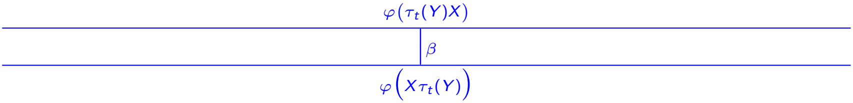
KMS equilibrium condition (Haag-Hugenoltz-Winnink)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi(X\tau_t(Y))$

(b) $F_{XY}(t + i\beta) = \varphi(\tau_t(Y)X)$

F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$, continuous on $\bar{S}_\beta$



KMS states generalise Gibbs states, equilibrium condition for infinite systems

The modular Hamiltonian

The generator of the modular operator unitary group Δ_φ^{it} is called the modular Hamiltonian $\log \Delta_\varphi$

One may consider the the relative modular operator $\Delta_{\xi,\eta}$, and the more general

$$\text{modular Hamiltonian } \log \Delta_{\xi,\eta}$$

The study of the modular Hamiltonian is presently a hot topic in Theoretical Physics

Entropy of finite systems

$X = \{x_1, \dots x_n\}$ a set of events. If x_i occurs with probability p_i, its information is $-\log p_i$

$$\text{Shannon entropy} : S(P) = -\sum p_i \log p_i \ .$$

If $Q = \{q_1, \dots q_n\}$ other probability distribution (state)

$$\text{Relative entropy} : S(P\|Q) = \sum p_i(\log p_i - \log q_i)$$

mean value in the state P of the difference between the information carried by the state P and the state Q.

Quantum entropy: $\varphi = -\mathrm{Tr}(\rho_\varphi \,\cdot)$ state on a matrix algebra

$$\text{von Neumann entropy} : S(\varphi) = \ -\mathrm{Tr}(\rho_\varphi \log \rho_\varphi)$$

Umegaki's relative entropy

$$S(\varphi\|\psi) =: \mathrm{Tr}\big(\rho_\varphi(\log \rho_\varphi - \log \rho_\psi)\big)$$

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exist; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace. *Symplectic complement*:

$$H' = \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* if $H \cap iH = \{0\}$.

A standard subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. H is standard iff H' is standard.

H standard subspace $\rightarrow$ anti-linear operator $S : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}$,

$$S : \xi + i\eta \rightarrow \xi - i\eta, \ \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. S is closed and densely defined, indeed

$$S_H^* = S_{H'}$$

Modular theory for standard subspaces

Set $S = J\Delta^{1/2}$, polar decomposition of $S = S_H$.

Then J is an anti-unitary involution, $\Delta > 0$ is non-singular called the modular conjugation and the modular operator; they satisfy $J\Delta J = \Delta^{-1}$ and

$$\Delta^{it} H = H, \quad JH = H'$$

(one particle Tomita-Takesaki theorem).

cf. Rieffel, van Daele; Leyland, Roberts, Testard

Entropy of a vector relative to a real linear subspace

Let $\mathcal{H}$ be a complex Hilbert space and $H \subset \mathcal{H}$ a standard subspace

The entropy of a vector $k \in \mathcal{H}$ with respect to $H \subset \mathcal{H}$ is defined by

$$S_k = S_k^H = \Im(k, P_H A_H\, k) = (k, P_H^* \log \Delta_H\, k)$$

(in a quadratic form sense), where P_H is the cutting projection

$$P_H : H + H' \to H\,, \quad h + h' \mapsto h$$

and $A_H = -i \log \Delta_H$, the semigroup generator $\frac{d}{ds}\Delta_H^{-is}|_{s=0}$ of the modular unitary group.

We have $P_H^* = -iP_H i$ and the formula

$$P_H = (1 - \Delta_H)^{-1} + J_H \Delta_H^{1/2}(1 - \Delta_H)^{-1}\,;$$

(P_H is the closure of the right-hand side).

Properties of the entropy of a vector

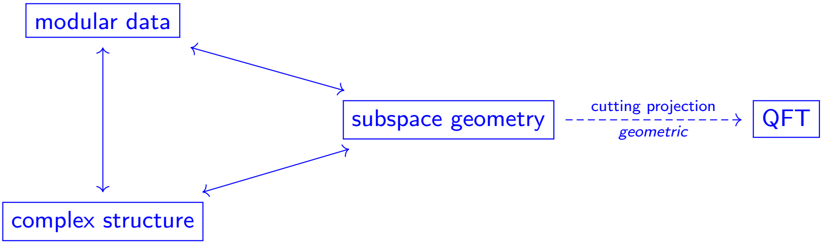
Some of the main properties of the entropy of a vector are:

- $S_k^H \geq 0$ or $S_k^H = +\infty$, positivity
- If $K \subset H$, then $S_k^K \leq S_k^H$ monotonicity
- If $k_n \to k$, then $S_k^H \leq \liminf_n S_{k_n}^H$ lower semicontinuity
- If $H_n \subset H$ is an increasing sequence with $\overline{\bigcup_n H_n} = H$, then $S_k^{H_n} \to S_k^H$ monotone continuity
- If $k \in D(\log \Delta_H)$ then $S_k^H < \infty$ finiteness on smooth vectors

Cutting Projection

In QFT, P_H *cuts the Cauchy data, so it is* *geometric*.

The following diagram illustrates the interplay among the three equivalent structures associated with standard subspaces and the geometric way out to QFT:



Weyl algebra and Gaussian states

If H is a real linear space with a non-degenerate symplectic form β, The Weyl C^*-algebra $C^*(H)$ linearly generated by the (unitaries) $V(h)$, $h \in H$, that satisfy the commutation relations (CCR)

$$V(h+k) = e^{i\beta(h,k)} V(h)V(k)\,, \quad h, k \in H\,,$$

$V(h)^* = V(-h)$. A state φ_α on $C^*(H)$ is called Gaussian, if

$$\varphi_\alpha\big(V(h)\big) = e^{-\frac{1}{2}\alpha(h,h)}\,,$$

with α a real bilinear form α on H, compatible with β.

H standard linear subspace of $\mathcal{H} \to$ von Neumann $\mathcal{A}(H) = C^*(H)''$ algebra on Fock space $e^{\mathcal{H}}$, i.e. GNS of $\omega = \varphi_\alpha$ with $\alpha = \Re(\cdot,\cdot)$

Entropy of coherent sectors

Given $\Phi \in \mathcal{H}$ consider the automorphism of the Weyl algebra $\mathcal{A}(H)$

$$\beta_\Phi = \mathrm{Ad}\, V(\Phi)^*|_{\mathcal{A}(H)} \ .$$

The vacuum relative entropy of the Gaussian state $\omega \cdot \beta_\Phi$ on $\mathcal{A}(H)$ is given by the entropy of the vector Φ w.r.t. H. Namely by

$$S(\omega \cdot \beta_\Phi \| \omega) = S_\Phi$$

Araki's relative entropy

Entropy of vector

The entropy operator $\mathcal{E}_H$ is defined by

$$\mathcal{E}_H = iP_H i \log \Delta_H$$

(closure of the right-hand side). We have

$$S_k = (k, \mathcal{E}_H k)\,, \quad k \in \mathcal{H}\,.$$

real quadratic form sense.

The entropy operator $\mathcal{E}_H$ is real linear, positive, and selfadjoint w.r.t. to the real part of the scalar product.

In my view, an entropy operator $\mathcal{E}$ is a real linear operator on a real or complex Hilbert space $\mathcal{H}$, such $\mathcal{E}$ is positive, selfadjoint and its expectation values $(f, \mathcal{E}f)$, $f \in \mathcal{H}$, correspond to entropy quantities (w.r.t. B).

Entropy operators. II

It is convenient to consider more entropy operators by performing natural operations.

Basic. If $\mathcal{E}$ is a real linear operator on a real Hilbert space H of the above form, we say that $\mathcal{E}$ is an entropy operator.

Restriction and direct sum. If $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ on a *real* Hilbert space direct sum $H = H_+ \oplus H_-$, then $\mathcal{E}$ is an entropy operator on H, iff both $\mathcal{E}_\pm$ are entropy operators.

Change of metric. Suppose that $\mathcal{S} \subset H$ is a core for the entropy $\mathcal{E}$ on H and $(\cdot,\cdot)'$ is a scalar product on $\mathcal{S}$; denote by H' the corresponding real Hilbert space completion. $\mathcal{E}'$ is an entropy operator on H' if

$$(f, \mathcal{E}' f)' = (f, \mathcal{E} f)\,, \quad f \in \mathcal{S}\,.$$

Entropy operators. III

Sum, difference. If $\mathcal{E}_1, \mathcal{E}_2$ are entropy operators and $\mathcal{E} = \mathcal{E}_1 \pm \mathcal{E}_2$ is densely defined and positive, the Friedrichs extension $\mathcal{E}$ is an entropy operator.

Born entropy. πE_B, with E_B the orthogonal projection onto $L^2(B)$, is an entropy operator on $L^2(\mathbb{R}^d)$.

In Quantum Mechanics, with the normalization $||f||^2 = 1$, $||f||_B^2$ is the probability of the particle to be localized in B, accordingly to Born's interpretation.

In Communication Theory, $||f||_B^2$ represents the part of energy of f contained in B. We set

$$\pi(f, E_B f) = \pi||f||_B^2 = \pi \int_B f^2 dx = \textit{Born entropy of } f \textit{ in } B\,,$$

$f \in L^2(\mathbb{R}^d)$ real.

Abstract field/momentum entropy

We are given *two real linear spaces* $\mathcal{S}_+$ *and* $\mathcal{S}_-$ *with duality*

$$f, g \in \mathcal{S}_+ \times \mathcal{S}_- \mapsto \langle f, g \rangle \in \mathbb{R}$$

and *a real linear, invertible operator*

$$\mu : \mathcal{S}_+ \to \mathcal{S}_-$$

μ *is symmetric and positive*, i.e.

$$\langle f_1, \mu f_2 \rangle = \langle f_2, \mu f_1 \rangle , \quad f_1, f_2 \in \mathcal{S}_+ ,$$

$$\langle f, \mu f \rangle \geq 0 , \quad f \in \mathcal{S}_+ ,$$

with $\langle f, \mu f \rangle = 0$ only if $f = 0$.

So $\mathcal{S}_\pm$ are real pre-Hilbert spaces with scalar products

$$(f_1, f_2)_+ = \langle f_1, \mu f_2 \rangle , \ (g_1, g_2)_- = \langle \mu^{-1} g_2, g_1 \rangle , \quad f_1, f_2 \in \mathcal{S}_+, \ g_1, g_2 \in \mathcal{S}_-$$

and μ *is a unitary operator*. $H_\pm$ be the real Hilbert space completion of $\mathcal{S}_\pm$

Set $\mathcal{H} = H_+ \oplus H_-$. The bilinear form β on $\mathcal{H}$

$$\beta(\Phi, \Psi) = \langle g_1, f_2 \rangle - \langle f_1, g_2 \rangle$$

$\Phi \equiv f_1 \oplus g_1$, $\Psi \equiv f_2 \oplus g_2$, is symplectic and non-degenerate (will be the imaginary part of the complex scalar product)

Now, the operator

$$\imath = \begin{bmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{bmatrix},$$

namely $\imath : f \oplus g \mapsto \mu^{-1} g \oplus -\mu f$, is a unitary on $\mathcal{H} = H_+ \oplus H_+$.

As $\imath^2 = -1$, the unitary $\imath$ defines a complex structure on $\mathcal{H}$ that becomes a complex Hilbert space

Suppose now $K_\pm \subset H_\pm$ are closed, real linear subspaces and $K \equiv K_+ \oplus K_-$ standard, factorial.

The cutting projection

$$P_K = K + K' \to K$$

is diagonal

$$P_K = \begin{bmatrix} P_+ & 0 \\ 0 & P_- \end{bmatrix},$$

with $P_\pm$ the projection $P_\pm : K_\pm + K^o_\mp \to K_\pm$.

$\log \Delta_K$ is diagonal, so $A_K = -\imath \log \Delta_K$ is off-diagonal

$$A_K = \pi \begin{bmatrix} 0 & \mathbf{M} \\ \mathbf{L} & 0 \end{bmatrix}$$

with $\mathbf{M}$ and $\mathbf{L}$ operators $H_\pm \to H_\mp$.

The entropy of $\Phi \equiv f \oplus g \in \mathcal{H}$ with respect to K is given by

$$S_\Phi = -\pi\langle f, P_-\mathbf{L}f\rangle + \pi\ \langle g, P_+\mathbf{M}g\rangle$$

The entropy operator is given by

$$\mathcal{E}_K = \pi \begin{bmatrix} -\mu^{-1} P_- \mathbf{L} & 0 \\ 0 & \mu P_+ \mathbf{M} \end{bmatrix}$$

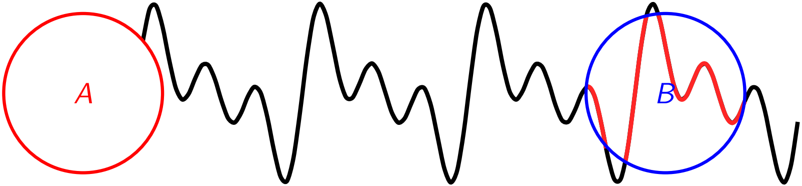
We then define:

$$\begin{aligned} -\pi\langle f, P_- \mathbf{L} f\rangle & \quad \textit{field entropy of } f \in \mathcal{S}_+ \ \textit{w.r.t.}\ K_+ \,, \\ \pi\langle g, P_+ \mathbf{M} g\rangle & \quad \textit{momentum entropy of } g \in \mathcal{S}_- \ \textit{w.r.t.}\ K_- \,. \end{aligned}$$

(quadratic form sense). Note that only the duality, not the Hilbert space structure, enters directly into the definitions of the above entropies.

Wave transmission

Suppose that Alice encodes and sends information by an undulatory signal, what information can Bob get by the wave packet in a given region at later time?



Bob has access only the portion of the wave that is in his lab at a given time. We are interested in the local information or information density of the wave packet.

Wave packets

By a wave (or wave packet), we mean a real solution of the wave equation

$$\Box\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes Φ by the stress-energy tensor $T_{\mu\nu}$, which provides the energy-momentum density of Φ at any time.

But, how to define the information, or entropy, carried by Φ in a given region at a given time?

We give a classical answer to such a classical question by Operator Algebras and Quantum Field Theory

Joint work with Fabio Ciolli and Giuseppe Ruzzi

Steps

• Symplectic form. Define a natural symplectic form on the real linear space $\mathcal{T}$ of waves

• Complex Hilbert structure. Define a *complex structure* the *real* linear space $\mathcal{T}$ and compatible complex scalar product on $\mathcal{T}$ giving *complex* Hilbert space $\mathcal{H}$

• Local subspaces. Waves with Cauchy data supported in a region O will form a *real* linear subspace $H(O)$ of $\mathcal{H}$

• Local entropy. Define the entropy of a wave Φ in the region O as $S_\Phi^{H(O)}$, the entropy of the vector Φ w.r.t. $H(O)$.

• Computation. Give the explicit formula for the local entropy of a wave Φ in the region O

Entropy of a wave

Waves are given by Cauchy data $\Phi \leftrightarrow \langle f, g\rangle \in S(\mathbb{R}^d) \times S(\mathbb{R}^d)$.

The symplectic form is the time-independent form

$$\beta(\Phi, \Psi) = \frac{1}{2} \int_{x^0=t} (\Psi \partial_0 \Phi - \Phi \partial_0 \Psi) dx \ ,$$

The complex structure is then

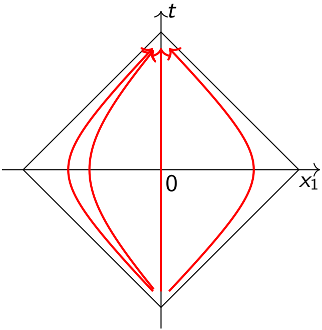
$$\imath = \begin{bmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{bmatrix}, \quad \mu = \sqrt{-\nabla^2}$$

Waves with Cauchy data supported in region O (causal envelop of a space region B) form a real linear subspace $H(O) \equiv H(B)$.

The information carried by the wave Φ in the region O is the entropy S_Φ of the vector Φ w.r.t. $H(O)$

Double cone, conformal case

For a bounded region O (double cone, causal envelop of a space ball B), in the conformal case the modular group is given by the geometric transformation (Hislop, L. '81)



local modular trajectories

$$(u, v) \mapsto \big((Z(u, s), Z(v, s)\big)$$

$$Z(z, s) = \frac{(1+z)+e^{-s}(1-z)}{(1+z)-e^{-s}(1-z)}$$

$$u = x_0 + r, \quad v = x_0 - r, \quad r = |\mathbf{x}| \equiv \sqrt{x_1^2 + \cdots + x_d^2}$$

Entropy density and modular Hamiltonian, massless case

What is the entropy density of a wave? We have to compute the modular Hamiltonian of space ball B.

In terms of the wave Cauchy data, the local massless modular Hamiltonian associated with the unit space ball B is given by

$$\log \Delta_B = -2\pi\imath_0 \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D & 0 \end{bmatrix}$$

$D = (d-1)/2$ the scaling dimension of the free scalar field. Namely

$$\log \Delta_B = -2\pi\imath_0 \begin{bmatrix} 0 & M \\ L - D & 0 \end{bmatrix}$$

with

$$M = \text{ Multiplication operator by } \frac{1}{2}(1-r^2)\,,$$

$$L = \text{ Legendre operator } \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r$$

In terms of the classical stress-energy tensor

$$\langle T^{(0)}_{00} \rangle_\Phi = \frac{1}{2}\big((\partial_0\Phi)^2 + |\nabla_{\mathbf{x}}\Phi|^2\big) \ .$$

$$-(\Phi, \log \Delta_B \Phi) = 2\pi \int_{x_0=0} \frac{1-r^2}{2} \langle T^{(0)}_{00} \rangle_\Phi(x) dx + \pi D \int_{x_0=0} \Phi^2 dx$$

The entropy of a wave Φ in the ball B is (massless case)

$$S^B_\Phi = 2\pi \int_B \frac{1-r^2}{2} \langle T^{(0)}_{00} \rangle_\Phi(x) dx + \pi D \int_B \Phi^2 dx$$

(Work with G. Morsella)

Massive case: numerical results by H. Bostelmann, D. Cadamuro, C. Minz

Higher-dimensional Legendre operator

The Legendre operator is the one-dimensional Sturm-Liouville linear differential operator $\frac{d}{dx}(1-x^2)\frac{d}{dx}$. We consider a natural higher-dimensional generalization.

We denote by L the d-dimensional Legendre operator, on $L^2(\mathbb{R}^d)$, initialliy defined on $S(\mathbb{R}^d)$

$$L = \nabla(1-r^2)\nabla = (1-r^2)\nabla^2 - 2r\partial_r\,;$$

The quadratic form associated with L is

$$(f, Lg) = -\int_{\mathbb{R}^d}(1-r^2)\nabla\bar{f}\cdot\nabla g\,dx\,, \quad f, g \in S(\mathbb{R}^d)\,,$$

L is a Hermitian operator.

Higher-dimensional prolate operator

Let W be the operator on $L^2(\mathbb{R}^d)$ given by

$$W = \nabla(1 - r^2)\nabla - r^2 = L - r^2$$

with $D(W) = S(\mathbb{R}^d)$. W is a higher-dimensional generalisation of the prolate operator.

W is a Hermitian, being a Hermitian perturbation of L on $S(\mathbb{R}^d)$; moreover,

$$-W \geq -L \geq 0$$

on $D(W) \cap L^2(B)$

Note the equality

$$W = L + M - 1$$

with M multiplication by $(1 - r^2)$. This makes a conenction with the modular Hamiltonian

- W commutes with the Fourier transformation $\mathcal{F}$:

$$\widehat{W} = W\,.$$

- Any linear combination of L and M commuting with $\mathcal{F}$ is proportional to W

- W has a natural Hermitian extension that commutes with $\mathcal{F}$ and E_B, thus with $\hat{E}_B$ and $\mathcal{F}_B$ too

• Connes: in one dimension, W is Hermitian on $\mathcal{S}(\mathbb{R})$ with defect indices 4 - 4 and it has a selfadjoint extension that commutes with $\mathcal{F}$ and E_B, thus with $\hat{E}_B$ and $\mathcal{F}_B$ too

• in higher dimension d, W is Hermitian on $S(\mathbb{R}^d)$; and it has a natural Hermitian extension that commutes with $\mathcal{F}$ and E_B, thus with $\hat{E}_B$ and $\mathcal{F}_B$ too. The domain of the extension is

$$\mathcal{D} \equiv S(\mathbb{R}^d) + \chi_B S(\mathbb{R}^d) + \chi_B \widehat{S(\mathbb{R}^d)}$$

and is given by

$$W(f + \chi_B g + \hat{\chi}_B * h) = Wf + \chi_B Wg + \hat{\chi}_B * Wh\,, \quad f, g, h \in S(\mathbb{R}^d)\,;$$

The restriction W_B of the extension to $L^2(B)$ is selfadjoint

The angle operator $E_B \hat{E}_B E_B$ is of trace class, indeed $E_B \hat{E}_B|_{L^2(B)}$ is the positive Hilbert-Schmidt T_B on $L^2(B)$ with kernel $k_B(x-y)$

$$k_B(z) = \frac{1}{(2\pi)^{d/2}} \int_B e^{-ix\cdot z} dx \, \chi_B(z)$$

The eigenvalues of T_B are strictly positive, with finite multiplicity

$$\lambda_1 > \lambda_2 > \cdots \lambda_k > \cdots > 0$$

The eigenfunctions are concentrated at level λ_k in appropriate sense

$-E_B W$ is positive. Both W and L commute with E_B, and we consider their restrictions W_B and L_B to $L^2(B)$

Legendre and Parabolic entropies (concrete abstract/field entropies)

The entropy operator $\mathcal{E}'_B$ on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ corresponding to $\mathcal{E}_B$ is given by

$$\mathcal{E}'_B = \begin{bmatrix} -\pi E_B L_D & 0 \\ 0 & \pi E_B M \end{bmatrix}$$

With $f \in S(\mathbb{R}^d)$ real, we set

$$\pi(f, Mf)_B = \pi \int_B (1 - r^2) f^2 dx = \text{parabolic entropy of } f \text{ in } B\,.$$

$$-\pi(f, Lf)_B = \pi \int_B (1 - r^2) |\nabla f|^2 dx = \text{Legendre entropy of } f \text{ in } B\,.$$

Prolate entropy

The Parabolic/Legendre entropies are the field/momentum entropies associated with a wave

Now, $-LE_B = -WE_B + ME_B - E_B$, so πWE_B is an entropy operator too; we thus define:

$$-\pi(f, Wf)_B = \pi \int_B \left((1-r^2)|\nabla f|^2 + r^2\right) dx = \textit{prolate entropy} \text{ of } f \text{ in } B\,,$$

$f \in S(\mathbb{R}^d)$ real.

Conclusion

We summarize our discussion in the following.

$-\pi WE_B$ is an entropy operator on $L^2(\mathbb{R}^d)$. The sum of the prolate entropy and the parabolic entropy is equal to the sum of the Legendre entropy and the Born entropy, all with respect to B

WE_B commutes with the truncated Fourier transform $\mathcal{F}_B$.

Let V be a real linear combination of LE_B, ME_B and E_B commuting with $\mathcal{F}_B$; then $V = aWE_B + bE_B$ for some $a, b \in \mathbb{R}$. If V is also positive, and the spectral lower bound of $V|_{L^2(B)}$ is zero, then $V = aWE_B$, $a \geq 0$

One-dimensional case: As T_B is strictly positive and Hilbert-Schmidt, its eigenvalues can be ordered as $\lambda_1 > \lambda_2 > \cdots > 0$; they are simple.

the eigenvalues of $-W_B$ can be ordered as

$$\alpha_1 < \alpha_2 < \cdots < \infty$$

correspond to the λ_k's in inverse order. Then

$$(f_k, T_B f_k)_B = \lambda_k\,, \quad -(f_k, W_B f_k)_B = \alpha_k\,,$$

and $\pi\alpha_k$ is the prolate entropy of f_k.

lower prolate entropy $\longleftrightarrow$ higher concentration

where the concentration is both on space and in Fourier modes as above. This is intuitive since information is the opposite of entropy. In other words, in order to maximize simultaneously both quantities $||f||^2_{2,B}/||f||^2_2$ and $||\hat{f}||^2_{2,B}/||/f||^2_2$ we have to minimize the prolate entropy.

The lucky accident is not an accident

As $-W + M = -L + 1$,

$$-\pi(f, Wf)_B + \pi(f, Mf)_B = -\pi(f, Lf)_B + \pi(f, f)_B\,;$$

$-\pi(f, Wf)_B$ is the sum of the Legendre entropy of f and $\pi||f||_B^2$ (Born entropy), minus the parabolic entropy of f, i.e.

$$-\pi(f, Wf)_B + \pi\int_B (1-r^2)f^2 dx = \pi\int_B (1-r^2)|\nabla f|^2 dx + \pi\int_B f^2 dx\,.$$

We conclude that $-\pi(f, Wf)_B$ is an entropy quantity, i.e. a measure of information, the *prolate entropy* of f w.r.t. B. In other words, $-\pi E_B W$ is an entropy operator.

The *lucky accident*, that W commutes with the truncated Fourier transform, finds a conceptual clarification in this fact.