The Singular Structure of Collider Observables at N3LO & Beyond

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Galileo Galilei Institute

“Theory Challenges in the Precision Era of the Large Hadron Collider”

Firenze, 5 September 2023

Based on:

“Collinear expansion for color singlet cross sections”
M.Ebert, B.Mistlberger, **GV**
[2006.03055]

“TMD PDFs at N3LO”
M.Ebert, B.Mistlberger, **GV**
[2006.05329]

“N-jettiness beam functions at N3LO”
M.Ebert, B.Mistlberger, **GV**
[2006.03056]

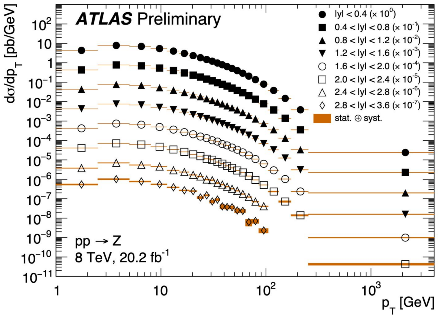
“Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond”
C.Duhr, B.Mistlberger, **GV**
[2205.04493]

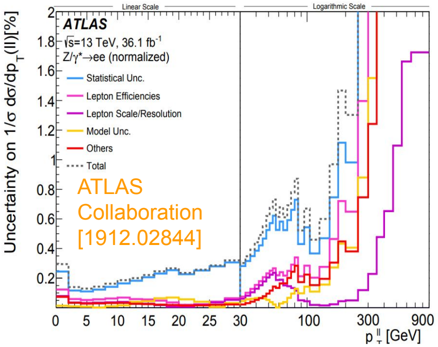
“The Four-Loop Rapidity Anomalous Dimension and Event Shapes to Fourth Logarithmic Order”
C.Duhr, B.Mistlberger, **GV**
[2205.02242]

Testing the SM at Percent Level Accuracy

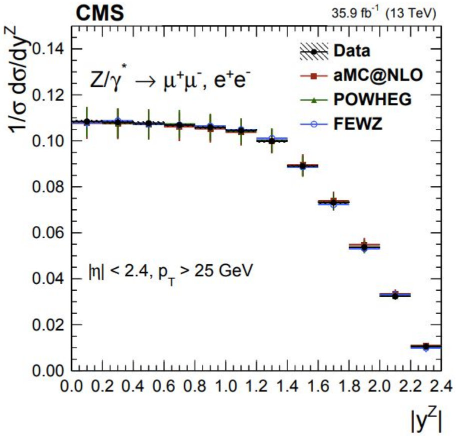
Astonishing level of precision in experimental measurements of key benchmark processes.

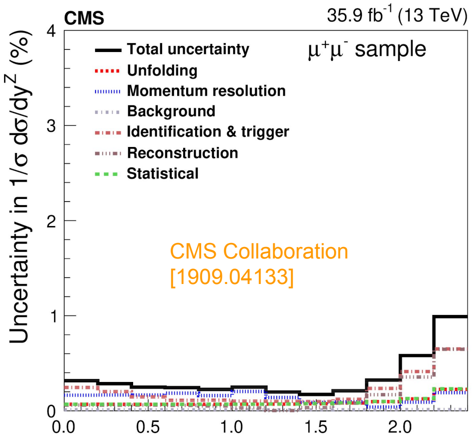
Example: normalized differential distributions in Drell-Yan measured with few **per-mille** level accuracy





ATLAS Collaboration [1912.02844]

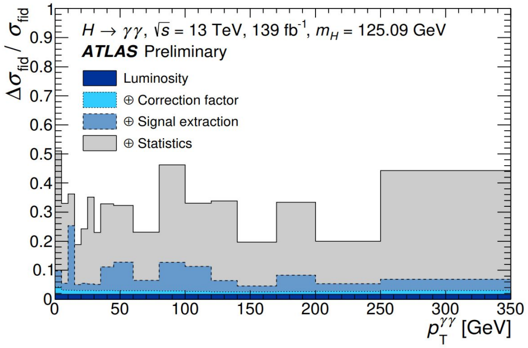




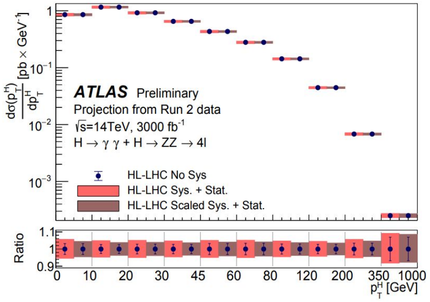
CMS Collaboration [1909.04133]

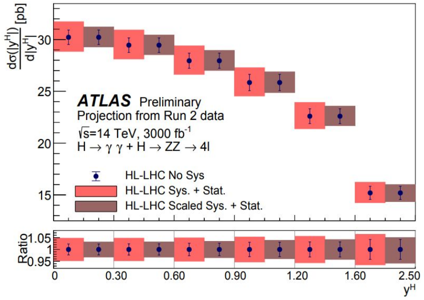
...and plethora of very precise differential distributions from LEP, future EIC measurements, possible future colliders, etc...

Higgs measurements at the moment are limited by statistics

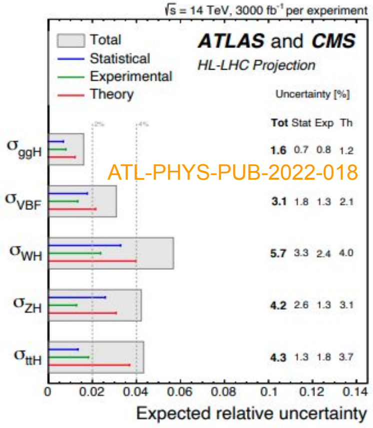


...but statistics **will improve dramatically** with HL LHC...





With **percent level measurement** of Higgs distributions, theory errors are projected to be a major limiting factor for Higgs precision program



ATL-PHYS-PUB-2022-018

Standard Model Phenomenology at percent level

We should aim at comparable precision from the theory side!

$$\sigma_{pp\to X} \sim \int \overset{\text{Non Perturbative}}{f_a(x_1) f_b(x_2)} \otimes \overset{\text{Perturbative}}{\hat{\sigma}_{ab\to X}}$$

$$\hat{\sigma}_{ab\to X} = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s \sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2 \sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3 \sigma_3}_{\text{N}^3\text{LO}} + \ldots$$

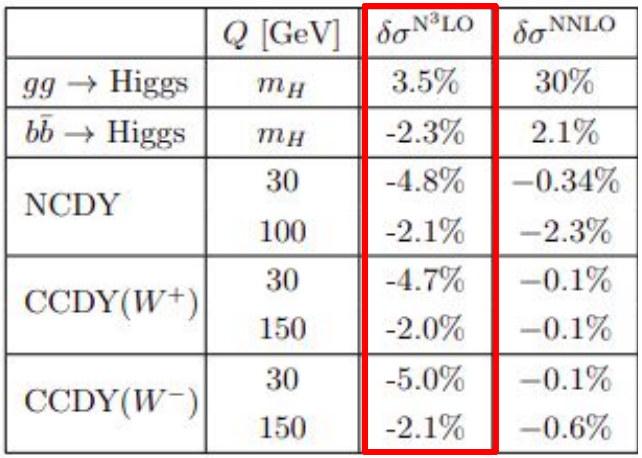
N3LO corrections (or at least *good* estimates of them) will be **necessary for percent level** phenomenology

"The Path Forward to N3LO"
Snowmass Whitepaper
[Caola, Chen, Duhr, Liu, Mistlberger, Petriello, GV, Weinzierl]

CAVEAT!

Often times convergence turns out to be slower than naive estimate

=> **N3LO gives few <u>percent</u> (not per-mille) shift**



	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	$\delta\sigma^{\text{NNLO}}$
$gg \to$ Higgs	m_H	3.5%	30%
$b\bar{b} \to$ Higgs	m_H	-2.3%	2.1%
NCDY	30	-4.8%	−0.34%
	100	-2.1%	−2.3%
CCDY(W^+)	30	-4.7%	−0.1%
	150	-2.0%	−0.1%
CCDY(W^-)	30	-5.0%	−0.1%
	150	-2.1%	−0.6%

n3loxs [Baglio, Duhr, Mistlberger, Szafron '22]

Predictions for Differential Cross Sections: IR singularities

$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2$$

- Cross sections require **integration over phase space**
- Complexity of infrared singularities grows with loop order
- Extremely challenging to systematize their treatment order by order
- Use **EFT methods** to systematize study of **collinear and soft radiation at the cross section level**
- Obvious applications: building universal counterterms (e.g. EFT-based subtractions) and improve resummation

Differential Distributions via Slicing

- **EFT-based subtractions** (AKA slicing methods)

q_T subtraction

[Catani, Grazzini '07]

***N*-Jettiness subtraction**

[Boughezal, Focke, Liu, Petriello '15]
[Gaunt, Stahlhofen, Tackmann, Walsh '15]

$$\sigma(X) = \boxed{\int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T}} + \boxed{\int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T}} + \boxed{\Delta\sigma(X, q_{T\,\mathrm{cut}})}$$

Below the cut region:

- Singular distribution
- Contains most complicated cancellation of IR divergences
- Control it analytically via factorization theorems

Above the cut region:

- Resolved extra radiation
- No events in Born configuration
- Lower number of loops
- Calculate numerically and/or with lower order subtraction schemes

Residual:

Non singular terms from below the cut (power correction). Minimized by going to very small $q_{T\,\mathrm{cut}}$

- Extremely successful program for many color singlet (and top) processes at **NNLO**

[Matrix collaboration]

- With ***N*-Jettiness** ability to tackle also processes with **jets in the final state**

[Boughezal, Focke, Liu, Petriello + Campbell, Ellis, Giele '15, '16] [Campbell, Ellis, Williams '16] [Campbell, Ellis, Seth '19][Mondini, Williams '21]

- Very CPU intensive. Efficiency can be improved by calculating residual power corr. analytically

[Ebert, Moult, Stewart, Tackmann, **GV**, Zhu, '18, '18, '19] [Boughezal, Isgro', Petriello '19] [Michel et al. '21] [Wiesemann et al. '21]

- Note also recent work on extending $\mathbf{q_T}$ to processes with jets [Grazzini et. al '22-23]

Differential Distributions via Slicing

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How do we extend this to N3LO?

Singular Region of LHC Observables

- **Singular region** (i.e. below the cut) can be understood at all orders via

Leading power factorization for ***Transverse-Momentum Distributions*** *in pp*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}Y\,\mathrm{d}^2\vec{q}_T} = \sigma_0 \sum_{i,j} \underbrace{\boxed{H_{ij}(Q^2,\mu)}}_{\text{Hard Function}} \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, \underbrace{\tilde{B}_i\Big(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\Big)\tilde{B}_j\Big(x_2^B, b_T, \mu, \frac{\nu}{\omega_b}\Big)}_{q_T\text{ Beam Functions}}\, \underbrace{\boxed{\tilde{S}(b_T,\mu,\nu)}}_{\text{Soft Function}}$$

- Perturbatively: H, B, and S take generic form in terms of **logs** and **boundaries**

$$\mathcal{F} = \sum_n \alpha_s^n(\mu)\left(\sum_{m=1}^{2n} c_{n,m}^{\mathcal{F}} \log^m(\mu/\mu_{\mathcal{F}}) \;+\; c_{n,0}^{\mathcal{F}}\right)$$

- For **N3LO slicing** we need Hard, Beam and Soft functions **boundaries** at N3LO

- For ***H*** and ***S***, **boundaries** are **constants:** known at N3LO since 2010 (H) and 2016 (S) [Li, Zhu '16] [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10]

- **Beam function boundaries** are **full functions** (of the collinear splitting variable)

Beam Functions

- **Beam Functions** can be understood as **generalization** of **Parton Distribution Functions** (PDFs)

Longitudinal momentum fraction

PDF: $$f_q(x) = \langle p_n | \bar{\chi}_n \frac{\bar{n}\!\!\!/}{2} \left[\delta(p^- - \bar{n} \cdot \mathcal{P}) \chi_n \right] | p_n \rangle$$

Longitudinal momentum fraction

Beam Function: $$B_q(x, q_T) = \langle p_n | \bar{\chi}_n \frac{\bar{n}\!\!\!/}{2} \left[\delta(p^- - \bar{n} \cdot \hat{\mathcal{P}}) \delta(q_T - \hat{k}_T) \chi_n \right] | p_n \rangle$$

Additional observable (q_T, beam thrust, etc...)

- Beam functions are **non-perturbative** objects!

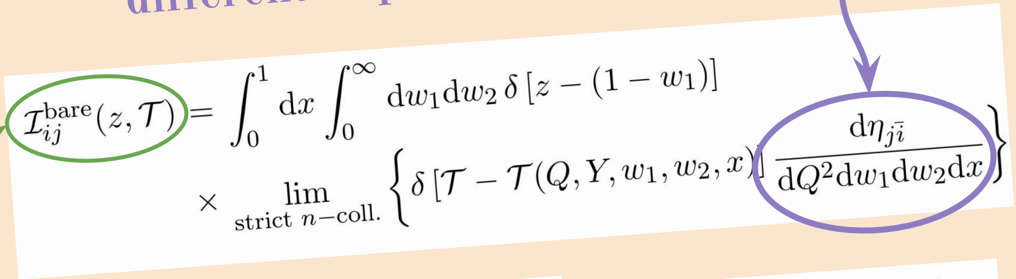
However, in **perturbative regime** of the observable $\mathcal{T} \gg \Lambda_{\rm QCD}$, they can be matched perturbatively onto PDF, via an **observable dependent matching kernel** $\mathcal{I}_{ij}(x, \mathcal{T}, \mu)$

$$B_i(x, \mathcal{T}, \mu) = \sum_j \mathcal{I}_{ij}(x, \mathcal{T}, \mu) \otimes_x f_j(x, \mu) \times \left[1 + \mathcal{O}(\Lambda_{\rm QCD}/\mathcal{T}) \right]$$

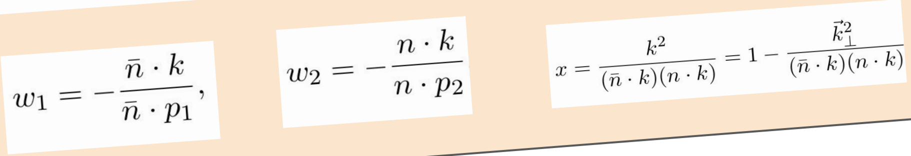
Beam Functions

- **Beam Functions** can be understood as **generalization** of **Parton Distribution Functions** (PDFs)

Bare **matching kernel** can be calculated using **collinear expansion of differential partonic cross sections for LHC processes!**



$$\mathcal{I}_{ij}^{\mathrm{bare}}(z,\mathcal{T}) = \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}w_1 \mathrm{d}w_2\, \delta\left[z-(1-w_1)\right] \times \lim_{\text{strict } n-\text{coll.}} \left\{ \delta\left[\mathcal{T}-\mathcal{T}(Q,Y,w_1,w_2,x)\right] \frac{\mathrm{d}\eta_{j\bar{i}}}{\mathrm{d}Q^2 \mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}x} \right\}$$



$$w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \qquad w_2 = -\frac{n\cdot k}{n\cdot p_2} \qquad x = \frac{k^2}{(\bar{n}\cdot k)(n\cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}$$

"Collinear expansion for color singlet cross sections"

[2006.03055]

Ebert, Mistlberger, **GV**

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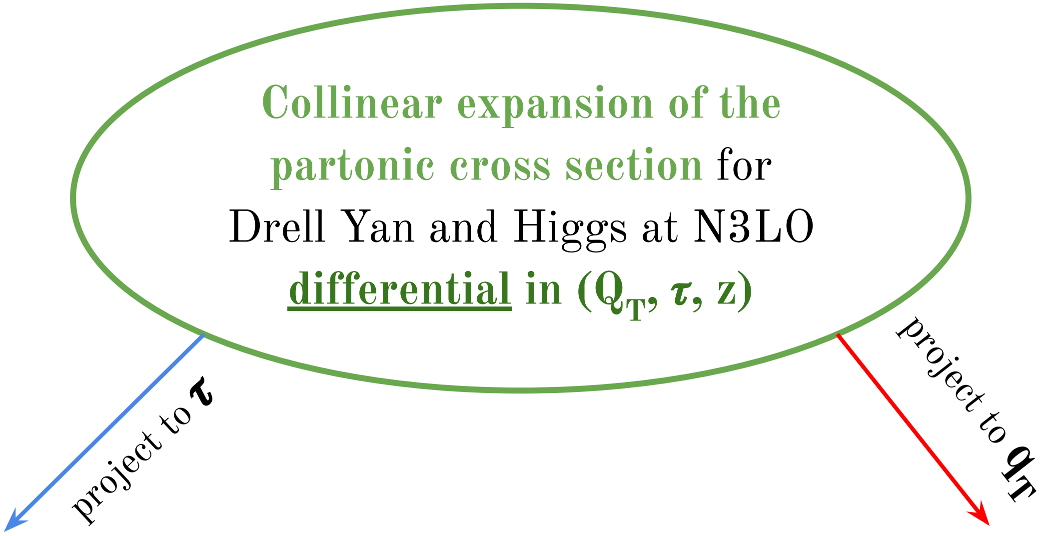
Beam Functions at N3LO



Collinear expansion of the partonic cross section for Drell Yan and Higgs at N3LO **differential in (Q_T, τ, z)**

$$B_a(t_a, x_1^B, \mu)$$

"N-Jettiness Beam Functions at N3LO"

M.Ebert, B.Mistlberger, **GV**
[2006.03056]

- ○ Quark **τ beam functions**
  (Quark N-Jettiness Beam Function)
- ○ Gluon **τ beam functions**
  (Gluon N-Jettiness Beam Function)

$$\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$$

"Transverse Momentum Dependent PDFs at N3LO"

M.Ebert, B.Mistlberger, **GV**
[2006.05329]

- ○ Quark **TMDPDF**
  (Quark q_T Beam Function)
- ○ Unpolarized **Gluon TMDPDF**
  (Gluon q_T Beam Function)

Beam Functions calculation at N3LO

(Ebert, Mistlberger, **GV**) [2006.05329], [2006.03056]

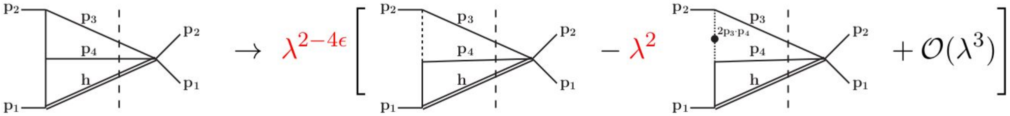
- Calculation of the collinear expansion of the partonic cross section for DY and Higgs @N3LO **differential in (Q_T, τ, z)**

○ ~ 100k Feynman diagrams

○ Reverse unitarity for phase space integrals

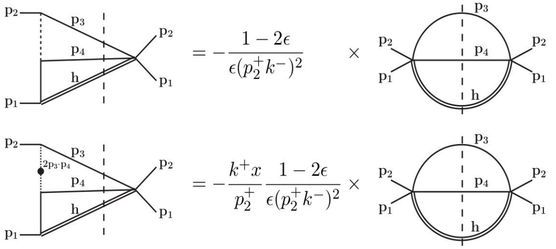
○ Collinear Expansion at the XS level

"Collinear expansion for color singlet cross sections" [Ebert, Mistlberger, **GV**]



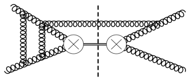
○ Reduction to basis of **Master Integrals** via Integration By Parts (IBPs) using Water

Expanded diagrams admit (simplified) **IBPs** identities

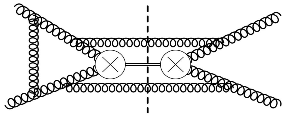


○ RVV: known in full kinematics

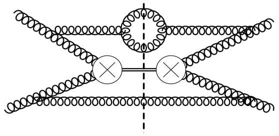
[Duhr, Gehrmann] [Duhr, Gehrmann, Jaquier] [Dulat, Mistlberger]



○ **RRV**: 170 Collinear Master Integrals



○ **RRR**: 320 Collinear Master Integrals



○ Derived system of Differential Equations for the Master Integrals

○ System has 2 non trivial scales with algebraic dependence on the variables (not something solvable algorithmically)

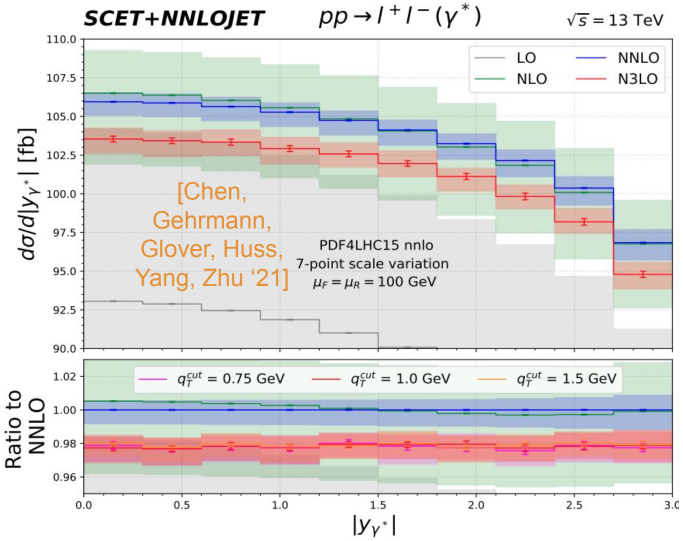
○ Algebraic sectors: constructed dlog integrand basis via calculation of **leading singularities** of candidate integrals on maximal cut surface

○ Boundaries from soft integrals [Anastasiou, Duhr, Dulat, Mistlberger] and constraints on singular behavior

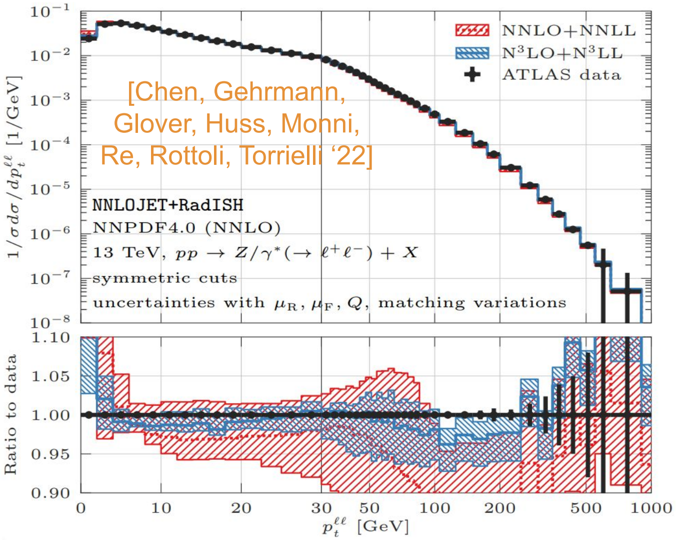
Slicing at N3LO:
Enabling N3LO differential predictions for the LHC

- **q_T beam functions** at N3LO were **last missing ingredient** for:
  - q_T subtraction for differential and fiducial Drell-Yan and Higgs production at N3LO
  - q_T resummation at N3LL`

- Many new exciting phenomenological results at N3LO employing them!

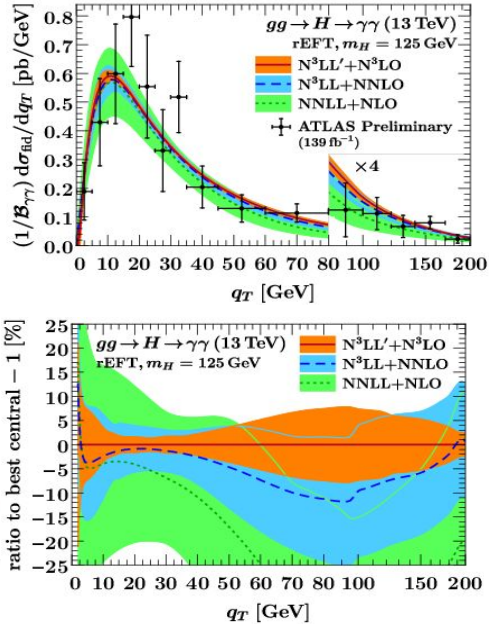


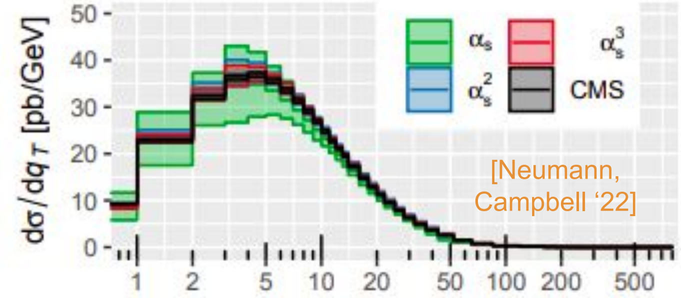
[Chen, Gehrmann, Glover, Huss, Yang, Zhu '21]



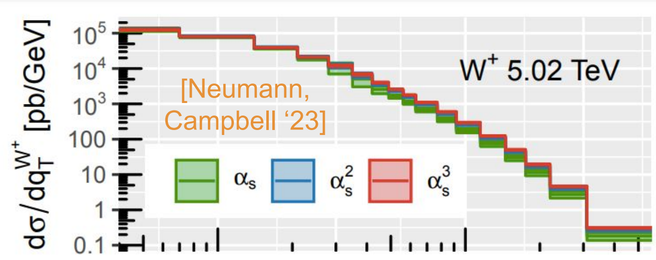
[Chen, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli '22]

[Billis, Dehnadi, Ebert, Michel, Tackmann '21]





[Neumann, Campbell '22]

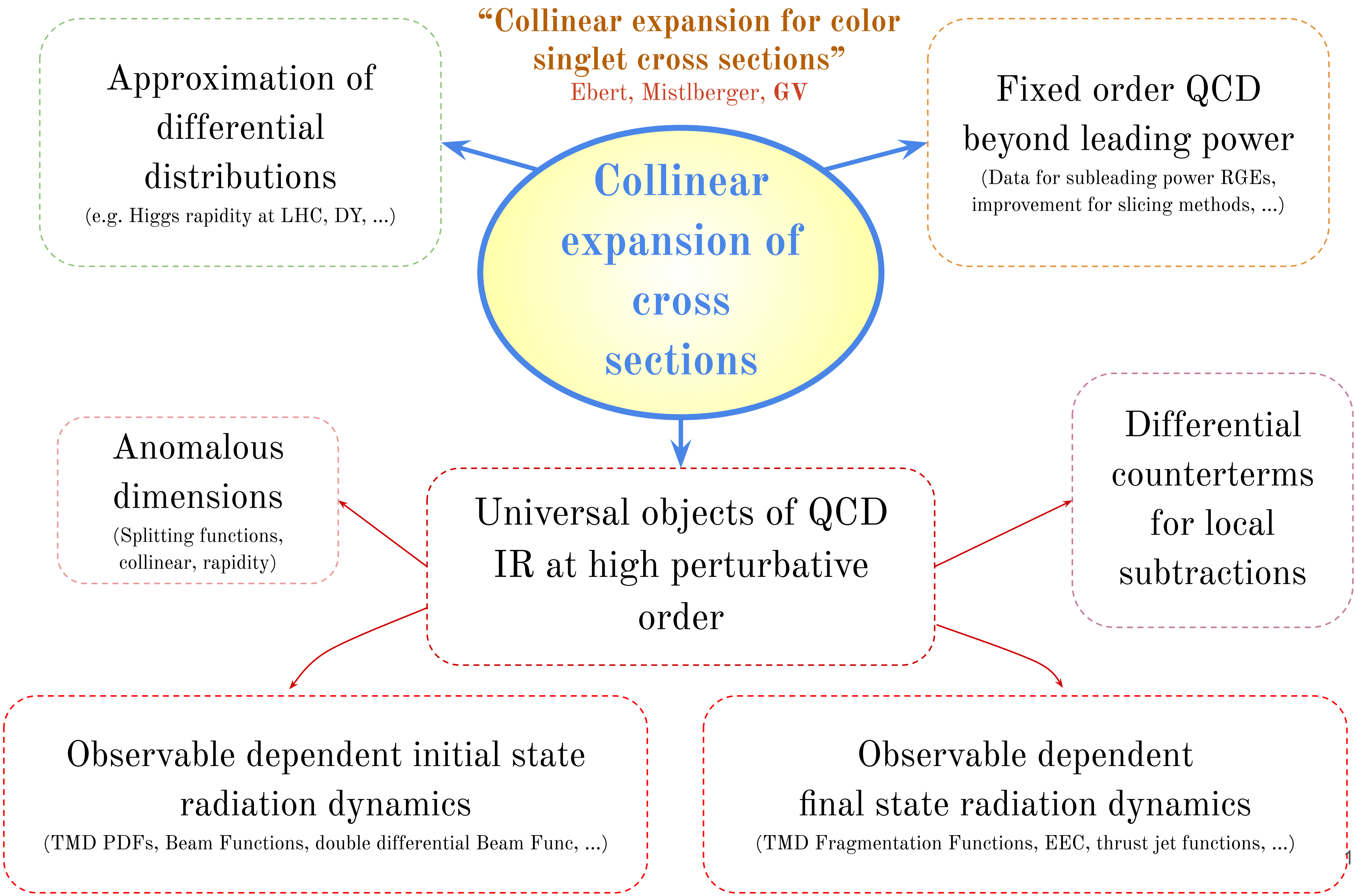


[Neumann, Campbell '23]

And many more:

[Ju, Schönherr '21]
[Camarda, Cieri, Ferrera '21]
[Re, Rottoli, Torrielli '21]
[ATLAS alpha_S extraction '23]

Collinear expansion of cross sections: Applications



“Collinear expansion for color singlet cross sections”
Ebert, Mistlberger, **GV**

Collinear expansion of cross sections

Approximation of differential distributions
(e.g. Higgs rapidity at LHC, DY, ...)

Fixed order QCD beyond leading power
(Data for subleading power RGEs, improvement for slicing methods, ...)

Universal objects of QCD IR at high perturbative order

Anomalous dimensions
(Splitting functions, collinear, rapidity)

Differential counterterms for local subtractions

Observable dependent initial state radiation dynamics
(TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics
(TMD Fragmentation Functions, EEC, thrust jet functions, ...)

Going Beyond N3LO:
Rapidity Anomalous Dimension to Four Loops and Resummation at N4LL

C.Duhr, B.Mistlberger, **G.Vita**

[2205.02242]

$$
\begin{aligned}
\gamma^{i}_{r,4} = {} & C_A^3 C_R \bigg(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
& - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6} \bigg) + C_R n_f^3 \left(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187} \right) \\
& + C_R C_A^2 n_f \bigg(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
& - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3} \bigg) + C_R C_F n_f^2 \left(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243} \right) \\
& + \frac{C^4_{AR}}{d_R} \bigg(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 \\
& - 384 + 4b_{4,d4AF} \bigg) + C_A C_R n_f^2 \left(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} - \frac{898033}{2916} \right) \\
& + \frac{C^4_{FR}}{d_R} n_f \left(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}} \right) \\
& + C_A C_F C_R n_f \left(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486} \right) \\
& + C_F^2 C_R n_f \left(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54} \right)
\end{aligned}
$$

The Rapidity Anomalous dimension

- Key ingredients for the resummation of large logarithms for transverse observables is the **rapidity anomalous dimension**. It appears in many contexts under different names: *Collins Soper Kernel*, *Anomaly Exponent*, piece of *B coefficient* in Sudakov Exponent, *TMD anomalous dimension*, etc...

In short: if you want to do anything involving transverse momentum logs beyond NLL, you need this ingredient.

The Rapidity Anomalous dimension

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In short: if you want to do anything involving transverse momentum logs beyond NLL, you need this ingredient.

- Non cusp term vanishes at LO and NLO.
- NNLO: known for a long time. [Davies, Webber, Stirling '85] [de Florian, Grazzini '00]
- N3LO: determined in 2016 via bootstrap methods [Li, Zhu '16]
- **N4LO:** C.Duhr, B.Mistlberger, **GV** [2205.02242] (see also [Moult, Zhu, Zhu '22])

Rapidity Anomalous Dimension to Four Loops

- The calculation of the **Rapidity anomalous dimension** to 4 loops by brute force would require calculation of some **differential object** (e.g. p_T soft function) **to 4 loops**

- This is beyond the current technology for fixed order calculations (more difficult than 4 loop splitting functions)

- Anomalous dimensions known at 4 loops:
  - **Hard/Collinear** Anomalous Dimension to 4 loops [von Manteuffel, Panzer, Schabinger - 2002.04617]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} H_{ij}^B(\mu^2) = \gamma_H^r(\alpha_S(\mu^2), \mu^2) H_{ij}^B(\mu^2),$$

$$\gamma_H^r(\alpha_S(\mu^2), \mu^2) = \Gamma_{\text{cusp}}^r(\alpha_S(\mu)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2} \boxed{\gamma_H(\alpha_S(\mu^2))}$$

Hard anomalous dimension
(2 x collinear anomalous dimension of form factors)

  - **Virtual** Anomalous Dimension to 4 loops [Das, Moch, Vogt - 1912.12920]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} f_i^{\text{th}}(z, \mu^2) = \gamma_f^r(z, \alpha_S(\mu^2)) \otimes_z f_i^{\text{th}}(z, \mu^2),$$

$$\gamma_f^r(z, \alpha_S(\mu^2)) = \Gamma_{\text{cusp}}^r(\alpha_S(\mu^2)) \left[\frac{1}{1-z}\right]_+ + \frac{1}{2} \boxed{\gamma_f^r(\alpha_S(\mu^2))} \delta(1-z)$$

DGLAP at threshold

Rapidity Anomalous Dimension to Four Loops

- There is a **Rapidity**/**Threshold** **correspondence** for conformal theories, which holds at the critical dimension of QCD [Vladimirov - 1610.05791]

$$\gamma_r^i[\alpha_s,\epsilon^*]+\gamma_{\rm th}^i[\alpha_s,\epsilon^*]=0$$

$$\beta[\alpha_s,\epsilon]=-2\alpha_s\left[\epsilon+\frac{\alpha_s}{4\pi}\beta_0+\left(\frac{\alpha_s}{4\pi}\right)^2\beta_1+\ldots\right]$$

$$\beta[\alpha_s,\epsilon^*]=0$$

$$\epsilon^*=-\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0+\left(\frac{\alpha_s}{4\pi}\right)^2\beta_1+\ldots\right]$$

Critical dimension of QCD

- **Threshold anomalous dimension** is part of RGE of soft function

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln S_i(\vec{b}_T,\mu,\nu)=4\Gamma^i_{\rm cusp}[\alpha_s(\mu)]\ln\mu/\nu+\boxed{\gamma^i_{\rm th}[\alpha_s]}$$

$$\nu\frac{\mathrm{d}}{\mathrm{d}\nu}\ln S_i(\vec{b}_T,\mu,\nu)=-4\int_{b_0/b_T}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\Gamma^i_{\rm cusp}[\alpha_s(\mu')]+\boxed{\gamma^i_r[\alpha_s]}$$

- Via SCET I consistency relations, relate **Threshold** to **Virtual** and **Collinear** anomalous dimensions

$$\boxed{\gamma^r_{\rm thr.}(\alpha_S(\mu^2))}=\boxed{-2\gamma^r_f(\alpha_S(\mu^2))}-\boxed{\gamma^r_H(\alpha_S(\mu^2))}$$

Rapidity Anomalous Dimension to Four Loops

- Difference between threshold and rapidity anomalous dimension comes from **higher orders in dimensional regularization evaluated at critical point!**

$$\underline{\gamma_r^{\text{N4LO}}} \sim \underline{\gamma_{\text{th}}^{\text{N4LO}}} + \underline{\gamma_r^{\text{N3LO}}[\epsilon = \epsilon^*]} \qquad \epsilon^* = -\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \dots\right]$$

- To obtain these terms it is necessary to calculate the **TMD Soft Function at N3LO** to **higher orders in dimensional regularization**

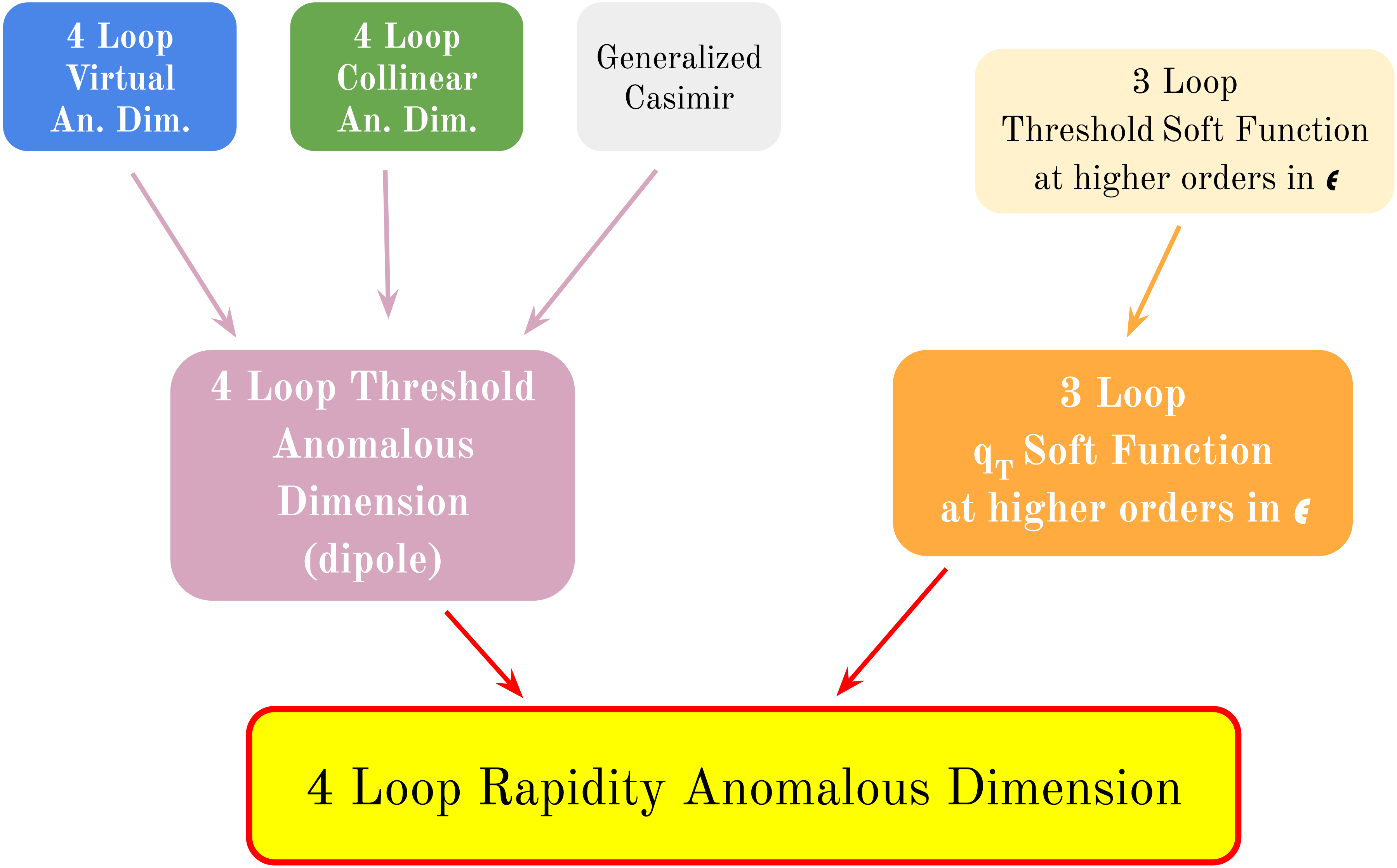
- We obtained this in

"Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond"

C.Duhr, B.Mistlberger, **GV** [2205.04493]

- Key point: Use method of differential equations and fix boundaries by relations between differential and inclusive threshold integrals

Rapidity Anomalous Dimension to Four Loops



4 Loop Virtual An. Dim.

4 Loop Collinear An. Dim.

Generalized Casimir

3 Loop Threshold Soft Function at higher orders in ϵ

4 Loop Threshold Anomalous Dimension (dipole)

3 Loop q_T Soft Function at higher orders in ϵ

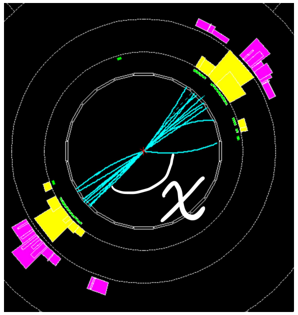
4 Loop Rapidity Anomalous Dimension

Resummation at N4LL

Energy-Energy Correlation

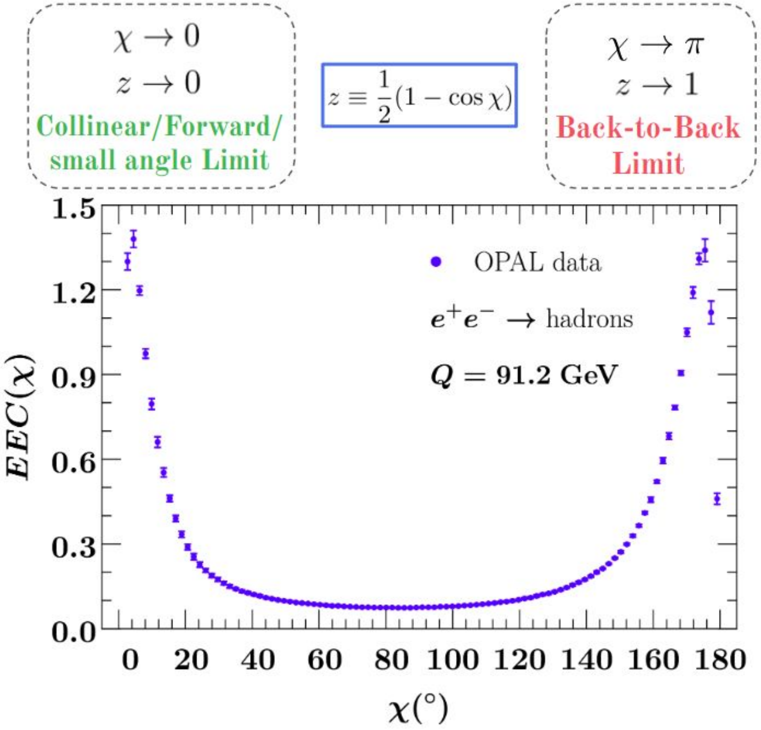
[Basham, Brown, Ellis, Love, PRL **41**, 1585 **(1978)**]

- Interesting TMD observable is the Energy-Energy Correlation (EEC)



$$\mathrm{EEC}(\chi) = \frac{\mathrm{d}\sigma}{\mathrm{d}\chi} = \sum_{i,j} \int \mathrm{d}\sigma_{e^+e^- \to ij+X} \frac{E_i E_j}{Q^2} \, \delta(\cos\theta_{ij} - \cos\chi)$$

- Measures **angle** χ between pairs of colored particles, weighted by energy
- Ton of interest in this observable: α_s extraction, precision QCD, related to correlators in CFT, playground for $\mathcal{N}=4$ and QCD connections, ...



- EEC has **two singular limits** with very different structure (no symmetry between them)
- **Single logarithmic** series in small angle limit

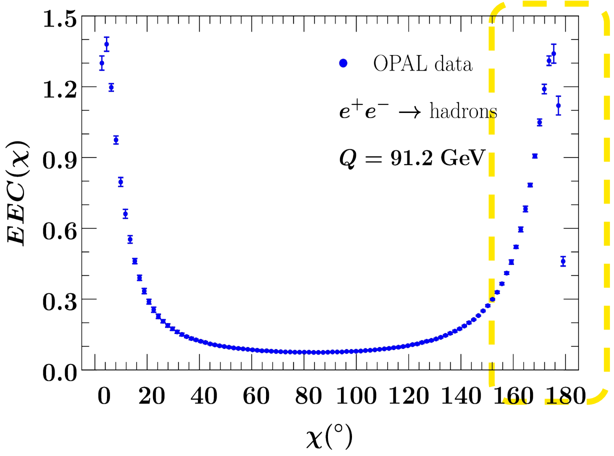
$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \overset{z\to 0}{\sim} \sum_{L=1}^{\infty} \sum_{m=0}^{L-1} \left(\frac{\alpha_s}{4\pi}\right)^L c_{L,m} \frac{\log^m z}{z}$$

- **Double logarithmic** series at $z \to 1$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \overset{z\to 1}{\sim} \sum_{L=1}^{\infty} \sum_{m=0}^{2L-1} \left(\frac{\alpha_s}{4\pi}\right)^L d_{L,m} \frac{\log^m(1-z)}{(1-z)}$$

- We have factorization theorems at both ends in SCET for resummation [Moult, Zhu] [Moult, Dixon, Zhu] [Ebert, Mistlberger, **GV**]

EEC in the back-to-back limit



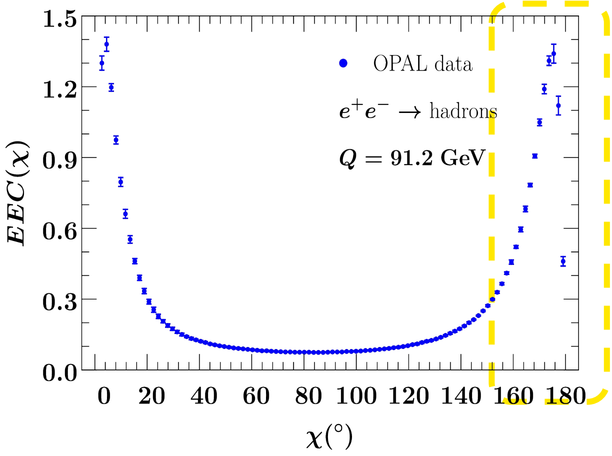
$$\chi \to \pi$$

$$z \to 1$$

Back-to-Back Limit

Back-to-back region of EEC has Sudakov peak and obeys **TMD-like** factorization theorem and resummation (“crossed version of q_T”)

EEC in the back-to-back limit



$\chi \to \pi$

$z \to 1$

Back-to-Back Limit

Back-to-back region of EEC has Sudakov peak and obeys **TMD-like** factorization theorem and resummation ("crossed version of q_T")

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\,\mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, \delta\Big(\overbrace{1 - z - \frac{q_T^2}{Q^2}}^{1-z\equiv(\cos\frac{\chi}{2})^2\approx\frac{q_T^2}{Q^2}}\Big) \underbrace{\mathcal{J}_q\Big(b_T,\mu,\frac{Qb_T}{\upsilon}\Big)\mathcal{J}_{\bar{q}}\big(b_T,\mu,Qb_T\upsilon\big)}_{\text{Pure Rapidity EEC Jet Functions}}$$

[C.Duhr, B.Mistlberger, **GV** '22]

Standard RGE

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln H_{q\bar{q}}(Q,\mu) = \gamma_H^q(Q,\mu)\,,$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\mathcal{J}_q\Big(b_T,\mu,\frac{Qb_T}{\upsilon}\Big) = \gamma_{\mathcal{J}_q}(\mu,\upsilon\mu/Q)$$

Rapidity RGE

$$\upsilon\frac{\mathrm{d}}{\mathrm{d}\upsilon}\ln\mathcal{J}_q\Big(b_T,\mu,\frac{Qb_T}{\upsilon}\Big) = \boxed{-\frac{1}{2}\gamma_r^q(b_T,\mu)}$$

Resummed cross section to all orders (at LP)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{8}\int_0^\infty \mathrm{d}(b_TQ)^2\, J_0\big(b_TQ\sqrt{1-z}\big)\, H_{q\bar{q}}(Q,\mu_H)$$
$$\times\,\mathcal{J}_q\Big(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\Big)\mathcal{J}_{\bar{q}}\big(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\big)\Big(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\Big)^{\boxed{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}}$$
$$\times\,\exp\left[4\int_{\mu_J}^{\mu_H}\frac{\mathrm{d}\mu'}{\mu'}\Gamma_{\mathrm{cusp}}^q[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right]$$

Logarithmic Accuracy for Resummed Predictions

- **Resummation accuracy** is determined by the perturbative accuracy of ingredients entering resummed cross section

- For **N4LL resummation:**
  - 3 Loop Hard Function
    [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10]
  - 3 Loop EEC Jet Function
    [Ebert, Mistlberger, **GV** 2012.07859]
  - 4 Loop Collinear Anom. Dim.
    [von Manteuffel, Panzer, Schabinger '20]
  - 4 Loop Rapidity Anomalous Dimension NEW!
  - 5 Loop Beta function
    [Baikov, Chetyrkin, Kuhn '16]
  - 5 Loop Cusp (approx)
    [Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt '18]

Resummed cross section to all orders (at LP)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{8}\int_0^\infty \mathrm{d}(b_T Q)^2\, J_0\big(b_T Q\sqrt{1-z}\big)\, H_{q\bar{q}}(Q,\mu_H)$$

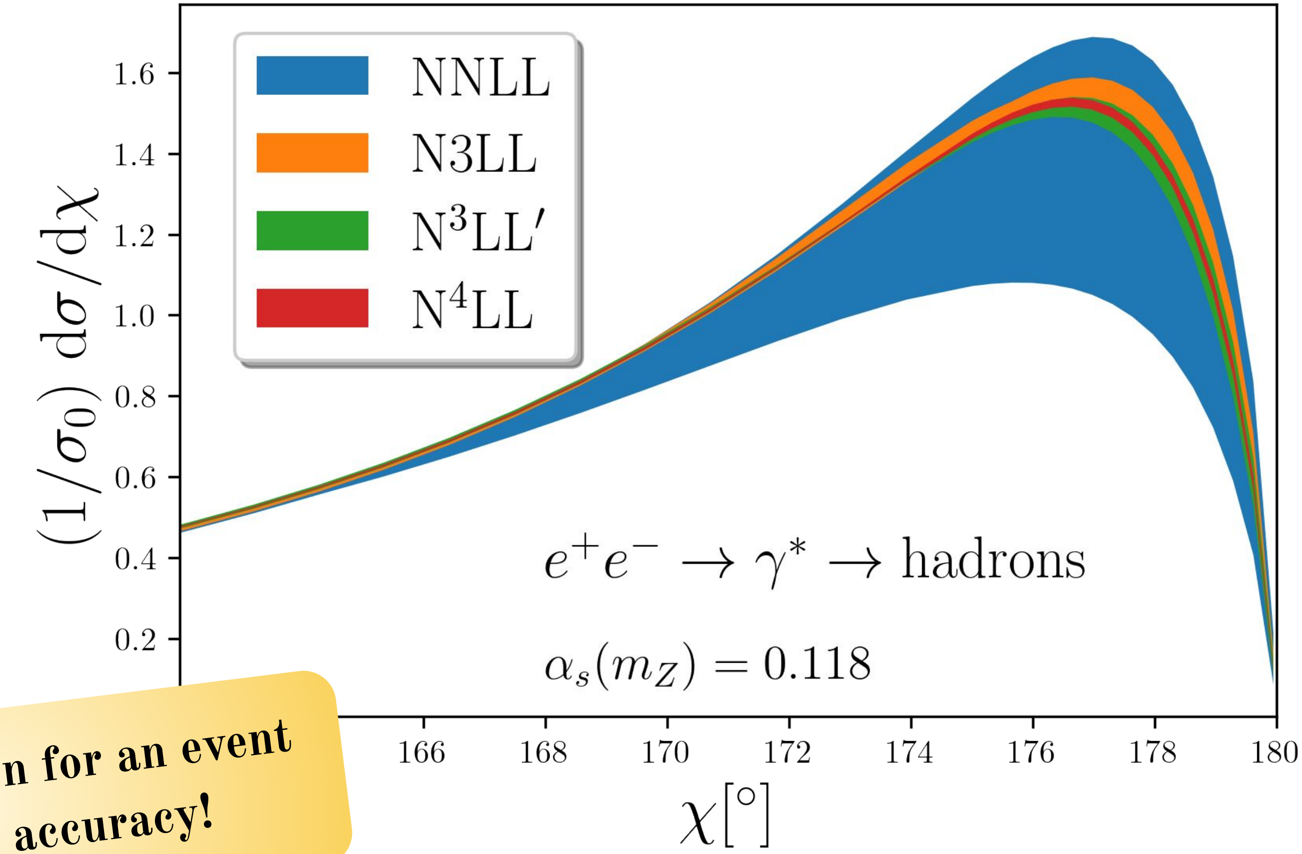
$$\times\, \mathcal{J}_q\Big(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\Big)\,\mathcal{J}_{\bar{q}}\big(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\big)\Big(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\Big)^{\boxed{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}}$$

$$\times\, \exp\left[4\int_{\mu_J}^{\mu_H}\frac{\mathrm{d}\mu'}{\mu'}\Gamma^q_{\mathrm{cusp}}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right]$$

Accuracy	H, $\mathcal{J}$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\mathrm{cusp}}(\alpha_s)$
LL	Tree level	–	–	1-loop	1-loop
NLL	Tree level	1-loop	1-loop	2-loop	2-loop
NLL$'$	1-loop	1-loop	1-loop	2-loop	2-loop
NNLL	1-loop	2-loop	2-loop	3-loop	3-loop
NNLL$'$	2-loop	2-loop	2-loop	3-loop	3-loop
N^{3}LL	2-loop	3-loop	3-loop	4-loop	4-loop
N^{3}LL$'$	3-loop	3-loop	3-loop	4-loop	4-loop
N^{4}LL	3-loop	4-loop	4-loop	5-loop	5-loop

EEC in the back to back limit to N4LL

- Implemented the resummation of this event shape at **N4LL** in new numerical framework: **pySCET**
- Nice convergence of perturbative result
- Uncertainties obtained by 15 point scale variation in SCET

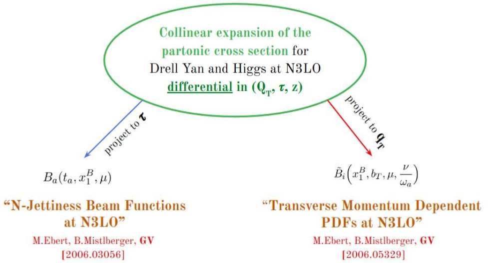


First resummation for an event shape at this accuracy!

Conclusion

- ➢ Introduced motivations and techniques for theoretical predictions at N3LO

- ➢ Discussed the calculation of TMDPDF and N-Jettiness **Beam Functions at N3LO** via collinear expansion of cross sections



Collinear expansion of the partonic cross section for Drell Yan and Higgs at N3LO differential in (Q_T, τ, z)

project to $\boldsymbol{\tau}$

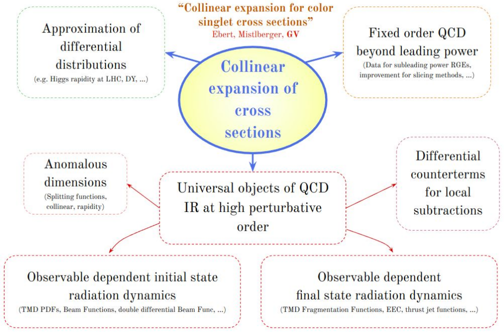
project to $\mathbf{q_T}$

$B_a(t_a, x_1^B, \mu)$

$\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$

"N-Jettiness Beam Functions at N3LO"
M.Ebert, B.Mistlberger, **GV**
[2006.03056]

"Transverse Momentum Dependent PDFs at N3LO"
M.Ebert, B.Mistlberger, **GV**
[2006.05329]



"Collinear expansion for color singlet cross sections"
Ebert, Mistlberger, **GV**

Collinear expansion of cross sections

Approximation of differential distributions
(e.g. Higgs rapidity at LHC, DY, ...)

Fixed order QCD beyond leading power
(Data for subleading power RGEs, improvement for slicing methods, ...)

Universal objects of QCD IR at high perturbative order

Anomalous dimensions
(Splitting functions, collinear, rapidity)

Differential counterterms for local subtractions

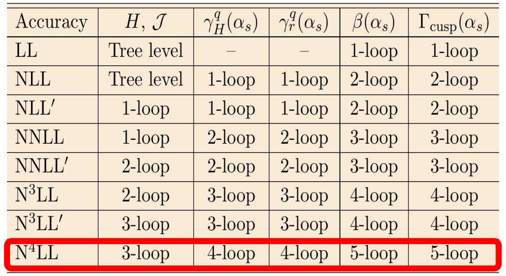
Observable dependent initial state radiation dynamics
(TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics
(TMD Fragmentation Functions, EEC, thrust jet functions, ...)

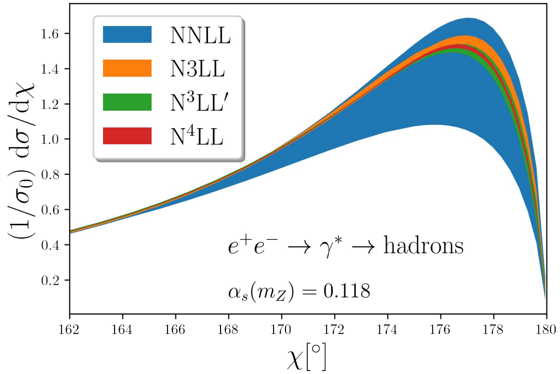
- ➢ Presented computation of quark and gluon **Rapidity Anomalous Dimension at N4LO**

$$
\begin{aligned}
\gamma^i_{r,4} = {} & C_A^3 C_R \Big(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
& - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6} \Big) + C_R n_f^3 \left(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187} \right) \\
& + C_R C_A^2 n_f \Big(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
& - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3} \Big) + C_R C_F n_f^2 \left(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243} \right) \\
& + \frac{C^4_{AR}}{d_R} \Big(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 \\
& - 384 + 4b_{4,d4AF} \Big) + C_A C_R n_f^2 \left(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} - \frac{898033}{2916} \right) \\
& + \frac{C^4_{FR}}{d_R} n_f \left(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}} \right) \\
& + C_A C_F C_R n_f \left(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486} \right) \\
& + C_F^2 C_R n_f \left(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54} \right)
\end{aligned}
$$

- ➢ Illustrated first results for **Resummation at N4LL** on event shapes



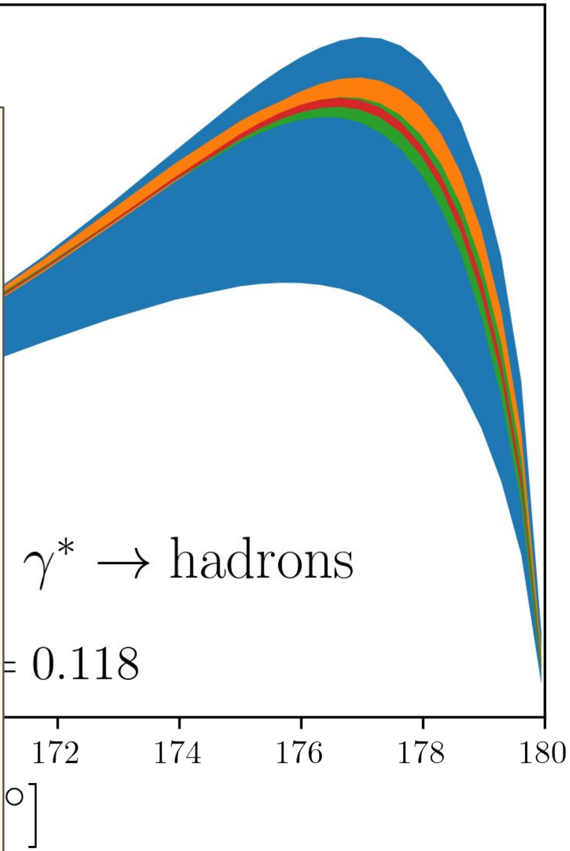
Accuracy	$H, \mathcal{J}$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\rm cusp}(\alpha_s)$
LL	Tree level	–	–	1-loop	1-loop
NLL	Tree level	1-loop	1-loop	2-loop	2-loop
NLL′	1-loop	1-loop	1-loop	2-loop	2-loop
NNLL	1-loop	2-loop	2-loop	3-loop	3-loop
NNLL′	2-loop	2-loop	2-loop	3-loop	3-loop
N^{3}LL	2-loop	3-loop	3-loop	4-loop	4-loop
N^{3}LL′	3-loop	3-loop	3-loop	4-loop	4-loop
N^{4}LL	3-loop	4-loop	4-loop	5-loop	5-loop



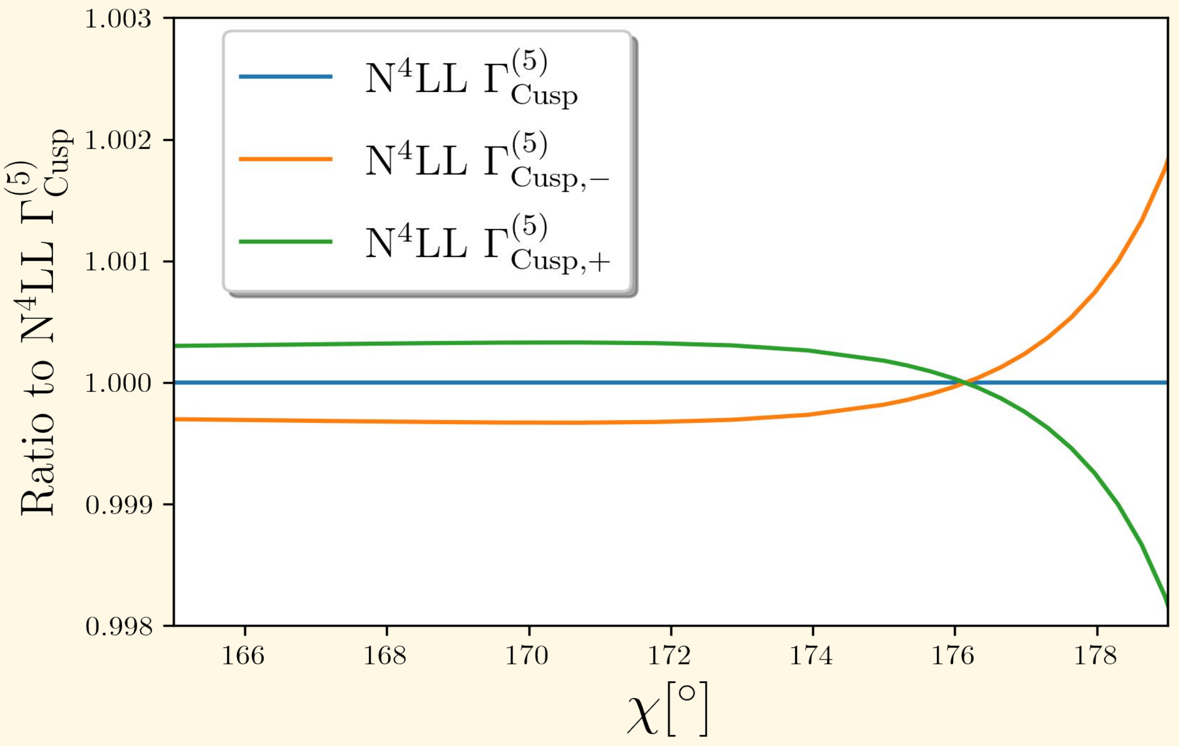
Backup

EEC in the back to back limit to N4LL

- Implemented the resummation of this event shape at **N4LL** in

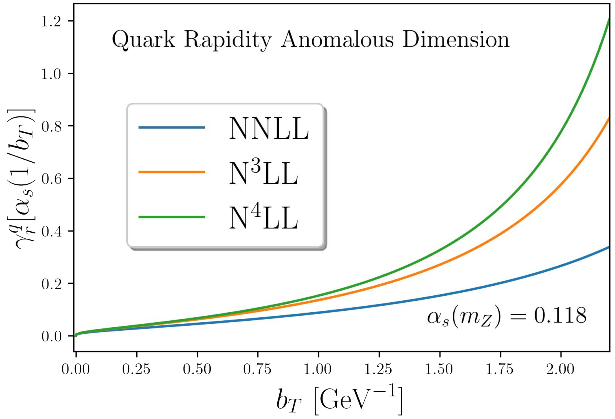


Impact of uncertainty from 5-loop Cusp is negligible



Rapidity Anomalous Dimension to Four Loops

- Obtained results at N4LO
- Quark and gluon related by **generalized casimir scaling**
- Well behaved series (stable coefficients) (see also [Moult, Zhu, Zhu])



$$\gamma_r^q(n_f\!=\!5) = 0.53929\alpha_s^2 + 0.68947\alpha_s^3 + \left(0.61753 \pm 5\cdot 10^{-5}\right)\alpha_s^4$$

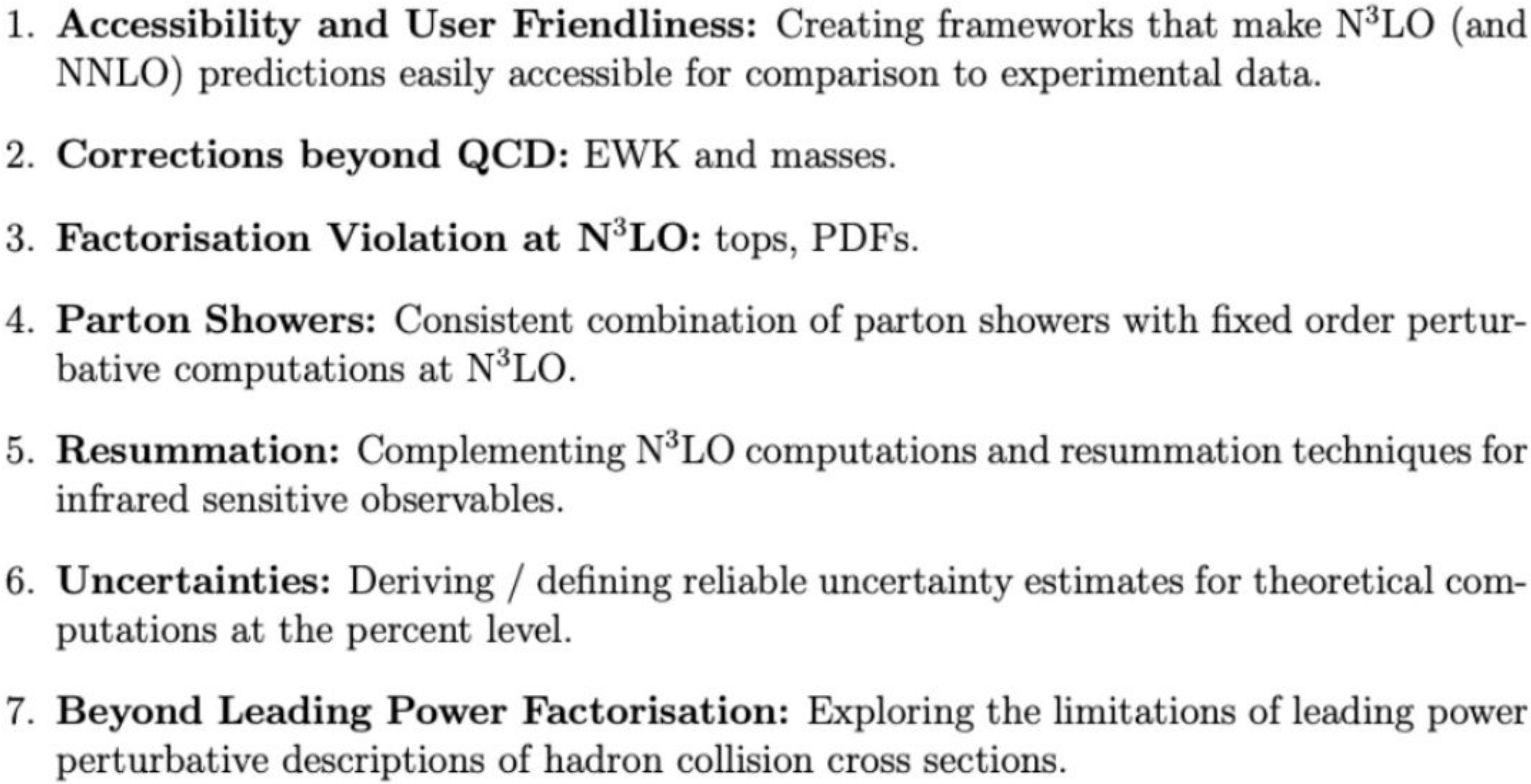
$$\gamma_r^g(n_f\!=\!5) = 1.21341\alpha_s^2 + 1.55130\alpha_s^3 + \left(1.6041 \pm 5\cdot 10^{-4}\right)\alpha_s^4$$

- 4 coefficients are not known analytically but only numerically (very well)

$$\begin{aligned}
\gamma^i_{r,4} = {} & C_A^3 C_R \bigg(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 \\
& + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
& - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6}\bigg) \\
& + C_A C_R n_f^2 \bigg(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} \\
& - \frac{898033}{2916}\bigg) + C_R n_f^3 \bigg(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187}\bigg) \\
& + C_R C_A^2 n_f \bigg(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} \\
& + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
& - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3}\bigg) \\
& + C_R C_F n_f^2 \bigg(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243}\bigg) \\
& + \frac{C^4_{AR}}{d_R} \bigg(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} \\
& + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 \\
& - 384 + 4b_{4,d4AF}\bigg) \\
& + \frac{C^4_{FR}}{d_R} n_f \bigg(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} \\
& + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}}\bigg) \\
& + C_A C_F C_R n_f \bigg(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} \\
& + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486}\bigg) \\
& + C_F^2 C_R n_f \bigg(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 \\
& + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54}\bigg)
\end{aligned}$$

More things towards percent level predictions…

$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 + \mathcal{O}(\Lambda^2/Q^2)$$

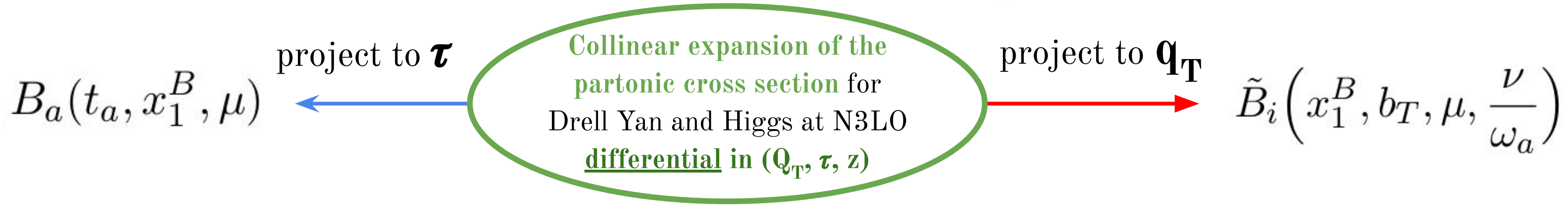


1. **Accessibility and User Friendliness:** Creating frameworks that make N^{3}LO (and NNLO) predictions easily accessible for comparison to experimental data.
2. **Corrections beyond QCD:** EWK and masses.
3. **Factorisation Violation at N^{3}LO:** tops, PDFs.
4. **Parton Showers:** Consistent combination of parton showers with fixed order perturbative computations at N^{3}LO.
5. **Resummation:** Complementing N^{3}LO computations and resummation techniques for infrared sensitive observables.
6. **Uncertainties:** Deriving / defining reliable uncertainty estimates for theoretical computations at the percent level.
7. **Beyond Leading Power Factorisation:** Exploring the limitations of leading power perturbative descriptions of hadron collision cross sections.

Bare Beam Functions and Renormalization

N-Jettiness Beam Function

q_T Beam Function



$B_a(t_a, x_1^B, \mu)$ $\xleftarrow{\text{project to } \boldsymbol{\tau}}$ Collinear expansion of the partonic cross section for Drell Yan and Higgs at N3LO differential in (Q_T, τ, z) $\xrightarrow{\text{project to } \mathbf{q_T}}$ $\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$

- Poles in dimensional regularization (up to $1/\varepsilon^6$)
- Logs/Plus Distributions in $\boldsymbol{\tau}$
- Iterated Integrals up to weight 5, with alphabet

$$\mathcal{A} = \left\{\frac{1}{z}, \frac{1}{1-z}, \frac{1}{2-z}, \frac{1}{1+z}, \frac{1}{z}, \frac{1}{\sqrt{4-z}\sqrt{z}}\right\}$$

- Constants to weight 6

Bare Results

- Poles in dimensional regularization
- Rapidity divergences regulated by exponential regulator
- Logs/Plus Distributions in $\mathbf{b_T/q_T}$
- HPLs in z up to weight 5
- Constants to weight 6

- Coupling renormalization
- SCET_I renormalization
- IR poles subtracted via NNLO PDF counterterms

Renormalization

- Coupling renormalization
- Zero-bin subtraction via calculation of bare q_T Soft Function at N3LO
- SCET_II renormalization
- IR poles subtracted via NNLO PDF counterterms

Checks

N-Jettiness Beam Function

q_T Beam Function

$B_a(t_a, x_1^B, \mu)$ ← project to $\boldsymbol{\tau}$

Collinear expansion of the partonic cross section for Drell Yan and Higgs at N3LO differential in (Q_T, τ, z)

project to $\mathbf{q_T}$ → $\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$

Checks

- 6 orders of poles cancel in all channels
- Terms involving $\mathcal{L}_n\left(\frac{t}{\mu^2}\right)$ $n = 0, \ldots, 5$ vs RGE prediction
- Eikonal limit vs threshold consistency
  [Billis, Ebert, Michel and Tackmann]
- Generalized leading color approx
  [Behring, Melnikov, Rietkerk, Tancredi, Wever]

Confirmation of our results
in later independent calculation

(Baranowski, Behring, Melnikov,
Tancredi, Wever)
[2211.05722]

- All rapidity divergences regulated
- 3 orders of $\boldsymbol{\varepsilon}$ poles cancel for all channels
- Log terms vs RGE prediction
  [Billis, Ebert, Michel and Tackmann]
- Eikonal limit vs threshold consistency
- Quark channels vs [Luo, Yang, Zhu, Zhu 1912.05778] (found small discrepancy)

Confirmation of our results
in later independent calculation

(Luo, Yang, Zhu, Zhu)
[2012.03256]

Slicing Power Corrections

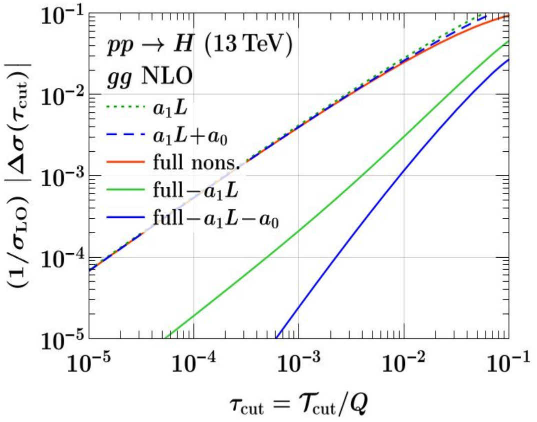
- Error due to higher order terms in slicing observable expansion

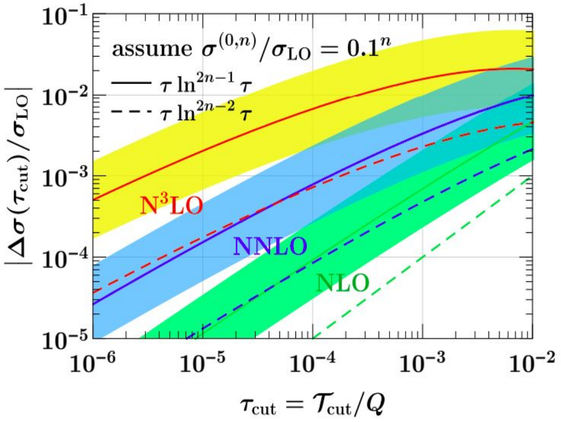
$$\Delta\sigma(X, q_{T\,\mathrm{cut}}) \equiv \sum_{i>0} \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}\tau \, \frac{\mathrm{d}\sigma^i(X)}{\mathrm{d}q_T}$$

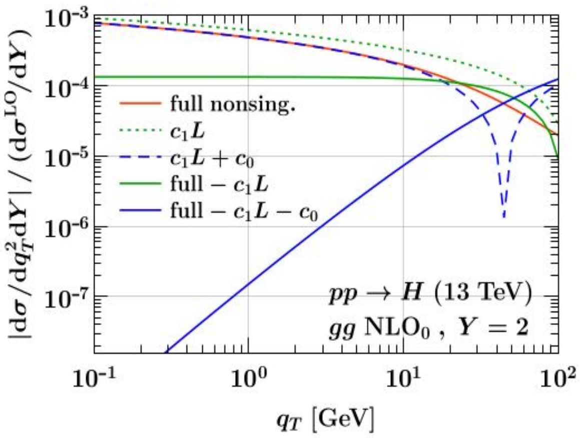
  - ○ In principle: made negligible by pushing cut to small values
  - ○ In practice: tradeoff between numerical stability and size of power corrections

- Interesting prospects of improving them by computing power corrections **analytically**

[Boughezal, Isgro', Petriello '19]
See for example: [Ebert, Moult, Stewart, Tackmann, **GV**, Zhu, 1807.10764, 1812.08189]







Analytic cross sections for collider observables

- **Important! Analytic** (not numerical) computations of **cross sections** (not amplitudes)
- Integral over **phase space** of final state particles
- Sum over all Real and Virtual corrections
- **Analytic** control of IR divergences

Trade phase space integrals for loop integrals with **reverse unitarity**

[Anastasiou, Melnikov] [Anastasiou, Dixon, Melnikov]

$$\int \mathrm{d}^d p\,\delta_+(p^2)$$

See phase space constraints as “cut” propagators

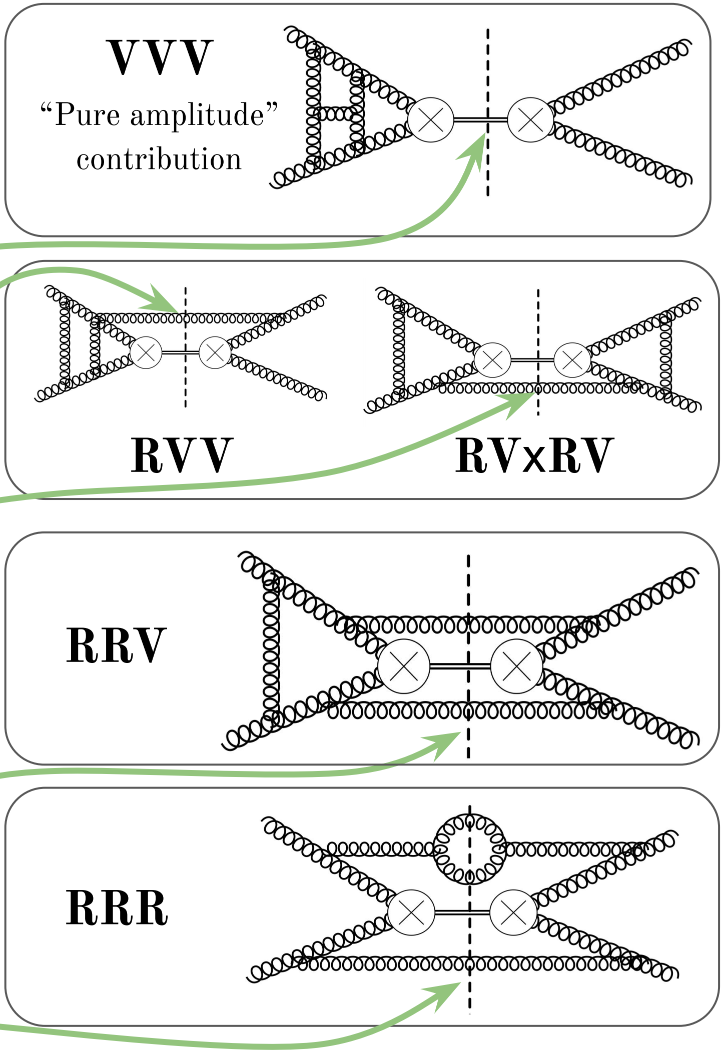
See this as a loop integral

$$\delta_+(p^2) \sim \lim_{\varepsilon\to 0}\left[\frac{1}{p^2+i\varepsilon}-\frac{1}{p^2-i\varepsilon}\right]$$

More measurements, more cut propagators, more difficult integrals

Example:

Higgs production at N3LO in gg



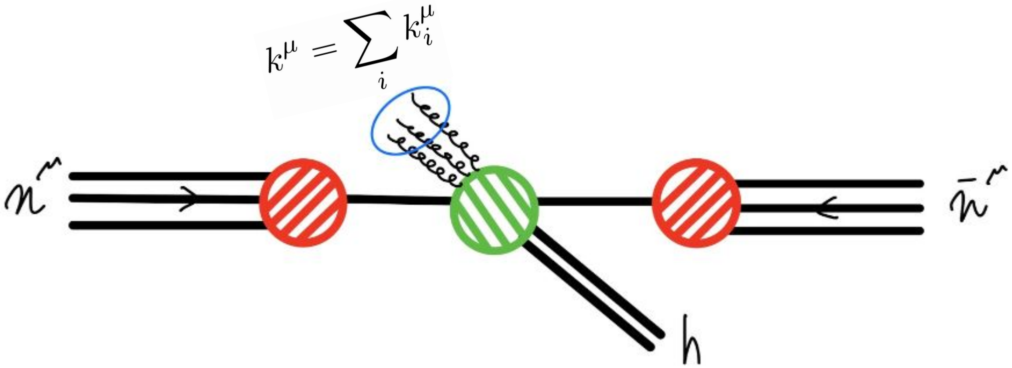
Expansion for Color Singlet Cross Sections

- Consider production of a color singlet state **h** in proton-proton collision
- **Measurements**: total momentum of radiation, color singlet Q and Y

Reverse Unitarity: think of **measurements** as cut propagators!

$$n^\mu = (1,0,0,1)$$

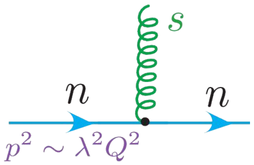
$$\bar{n}^\mu = (1,0,0,-1)$$



$$Y = \frac{1}{2}\log\left(\frac{\bar{n}\cdot p_h}{n\cdot p_h}\right)$$

$$Q^2 = p_h^2$$

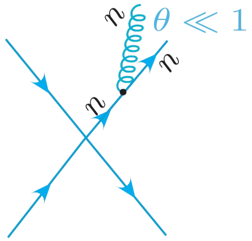
- **Limit** where total momentum of radiation is **soft** compared to Q



$$k^\mu \sim \lambda k^- \frac{n^\mu}{2} + \lambda k^+ \frac{\bar{n}^\mu}{2} + \lambda k_\perp^\mu\,, \quad \lambda \ll 1$$

Threshold expansion (very well known in literature)

- **Limit** where total momentum of radiation is **collinear** to proton axis



$$k^\mu \sim k^- \frac{n^\mu}{2} + \lambda^2 k^+ \frac{\bar{n}^\mu}{2} + \lambda k_\perp^\mu\,, \quad \lambda \ll 1$$

Collinear Expansion

Our work!

Collinear Expansion for Matrix Elements

- Kinematic limit ⟶ expansion of Feynman integrands appearing in the calculation of **partonic cross sections**

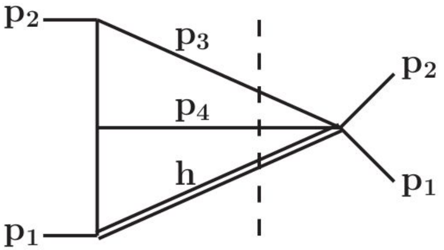
General idea has long history, see e.g. Expansion by region [Beneke, Smirnov '97]

- Take for example double real emission (RR) scalar integral

$$w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \quad w_2 = -\frac{n\cdot k}{n\cdot p_2},$$

$$x = \frac{k^2}{(\bar{n}\cdot k)(n\cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}$$

$$I_{\text{RR}} =$$



$$= \int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx} \frac{1}{(p_2+p_3)^2 (p_2+p_3+p_4)^2}$$

  - Differential double real particle phase space scales homogeneously

In the **collinear** limit:

$$\int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx} \to \lambda^{2-4\epsilon} \int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx}$$

  - Propagators can be **expanded** easily

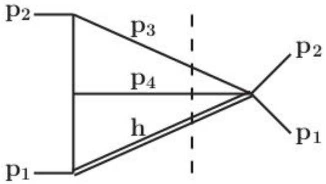
$$\frac{1}{(p_2+p_3+p_4)^2} \xrightarrow{\text{coll}} \frac{1}{2p_2\cdot(p_3+p_4) + \lambda^2\, 2p_3\cdot p_4} = \sum_{n=0}^{\infty} (\lambda^2)^n \frac{(-2p_3\cdot p_4)^n}{\left[p_2^+(p_3^- + p_4^-)\right]^{n+1}}$$

Collinear Expansion for double real graphs

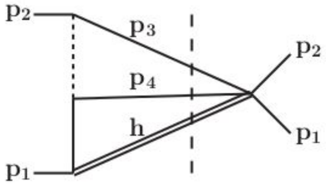
- We can perform a **collinear expansion** of the **integrand**

$$I_{\mathrm{RR}} \xrightarrow{\mathrm{coll}} \lambda^{2-4\epsilon} \int \frac{\mathrm{d}\Phi_{h+2}}{\mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}x} \left[\frac{1}{(p_2+p_3)^2 \left[p_2^+ (p_3^- + p_4^-)\right]} + \lambda^2 \frac{(-2p_3 \cdot p_4)}{(p_2+p_3)^2 \left[p_2^+ (p_3^- + p_4^-)\right]^2} + \mathcal{O}(\lambda^3) \right]$$

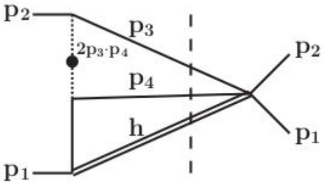
- Collinear expansion admits **diagrammatic** representation!



$\rightarrow \lambda^{2-4\epsilon} \Bigg[$



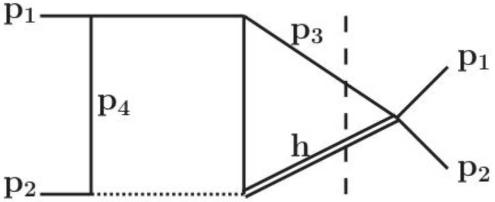
$- \lambda^2$



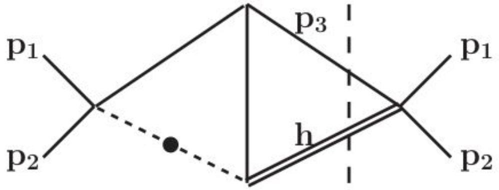
$+ \mathcal{O}(\lambda^3) \Bigg]$

- Same procedure can be applied for mixed loop/radiation integrals
(like RV integrals at NNLO)

$I_{\mathrm{RV}} \xrightarrow{\mathrm{coll}} \lambda^{-2-4\epsilon} \Bigg[$



$- \lambda^2$

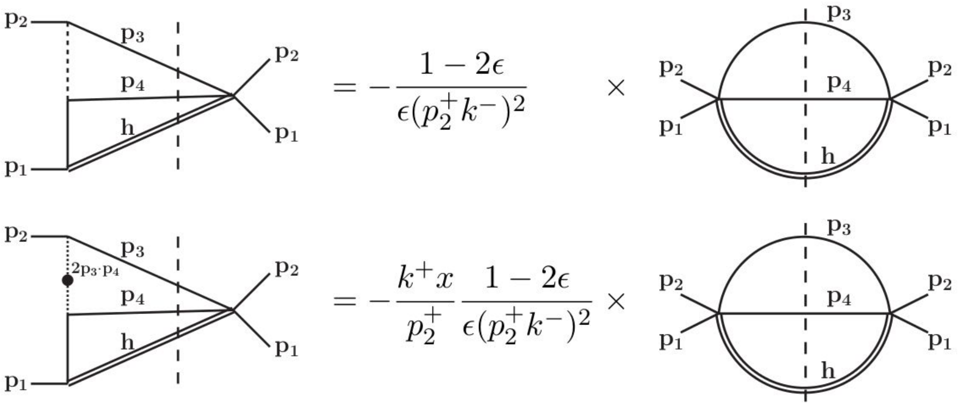


$+ \mathcal{O}(\lambda^3) \Bigg]$

Collinear Expansion and IBPs

Key Point!

Expanded diagrams admit (simplified) integration by parts (**IBPs**) identities



$$= -\frac{1-2\epsilon}{\epsilon(p_2^+ k^-)^2} \times$$

$$= -\frac{k^+ x}{p_2^+}\frac{1-2\epsilon}{\epsilon(p_2^+ k^-)^2} \times$$

- We can make use of modern technology for multiloop calculations with simplified kinematic dependence!

IBPs

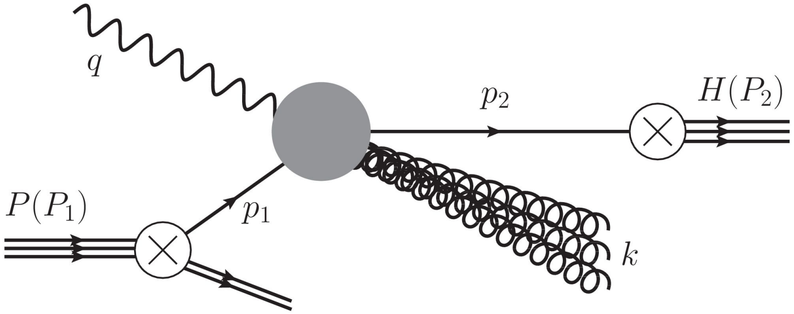
Canonical Differential Equations

Reverse Unitarity

- **Simplifications w.r.t. full kinematics are huge** and enter at each step:
  - IBPs (smaller set of MI, smaller coefficients)
  - System of DE (e.g. ~ 10 MB for differential N3LO in collinear limit vs ~ 10 GB in full kinematics)
  - Space of functions (e.g. @N3LO: Elliptic functions for inclusive color singlet production in full kinematics vs only HPL for q_T distributions in collinear limit)

SIDIS at small $\mathbf{q_T}$

- Factorization for **SIDIS at small $\mathbf{q_T}$** contains **TMD Fragmentation Functions (TMDFFs)**



$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_F \mathrm{d}^2\vec{q}_T} = (2\pi)^2 \alpha_{\mathrm{em}} \frac{x_B x_F^2}{Q^2} \sum_f H_{f\bar{f}}(q^2, \mu^2)$$

$$\times \int \frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T} \underbrace{\tilde{B}_f(x_B, \vec{b}_T, \mu, \nu/\omega_a)}_{\text{TMDPDF (q_T Beam Functions)}} \underbrace{\tilde{D}_{H/\bar{f}}(x_F, \vec{b}_T, \nu/\omega_b)}_{\text{TMD Fragmentation Function}} \tilde{S}_q(b_T, \mu, \nu)$$

Factorization for SIDIS at small $\mathbf{q_T}$

TMDPDF
($\mathbf{q_T}$ Beam Functions)

TMD Fragmentation Function

- **TMDFFs** are final state (time-like) analog of TMDPDFs
- **TMDFFs** can be OPEd onto longitudinal Fragmentation Functions (FF) for $q_T \gg \Lambda_{\mathrm{QCD}}$

$$\underbrace{D_{H/j}(x_F, q_T)}_{\text{TMDFF}} \sim \boxed{\sum_{j'} \underbrace{\mathcal{K}_{jj'}(x_F, q_T)}_{\text{Perturbative Kernel}}} \otimes_{x_F} \underbrace{d_{H/j'}(x_F)}_{\text{FF}} + \mathcal{O}(\Lambda_{\mathrm{QCD}}/q_T)$$

LP Collinear expansion of SIDIS

$$\boxed{\tilde{\mathcal{K}}_{jj'}\Big(\zeta, q_T\Big)} \sim \int_0^1 \mathrm{d}x\, \mathrm{d}w_2\, \delta\big[q_T^2 - Q^2(1-\zeta)w_2(1-x)\big] \boxed{\lim_{\text{strict coll.}} \frac{\mathrm{d}\hat{\eta}_{\bar{j}+h\to j'+X}}{\mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}x}\bigg|_{w_1=-\frac{1-\zeta}{\zeta}}}$$