

Intersection Numbers

from Electromagnetism to Quantum Field Theory (and Cosmology)

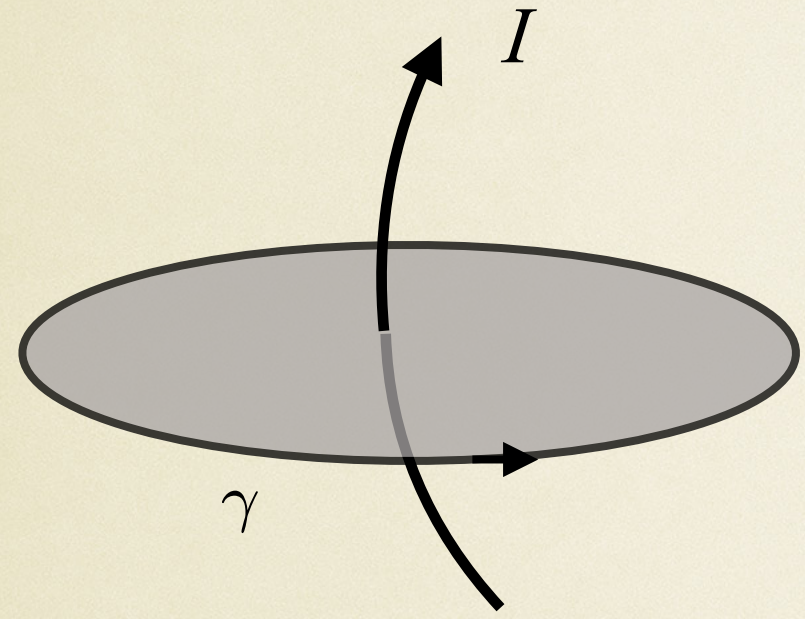
Pierpaolo Mastrolia

GGI Florence

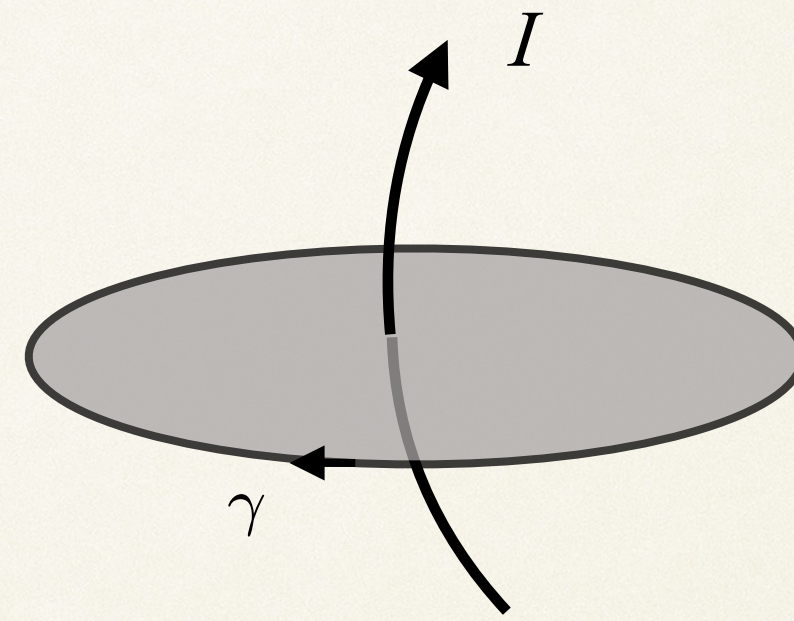
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Ampere's Law

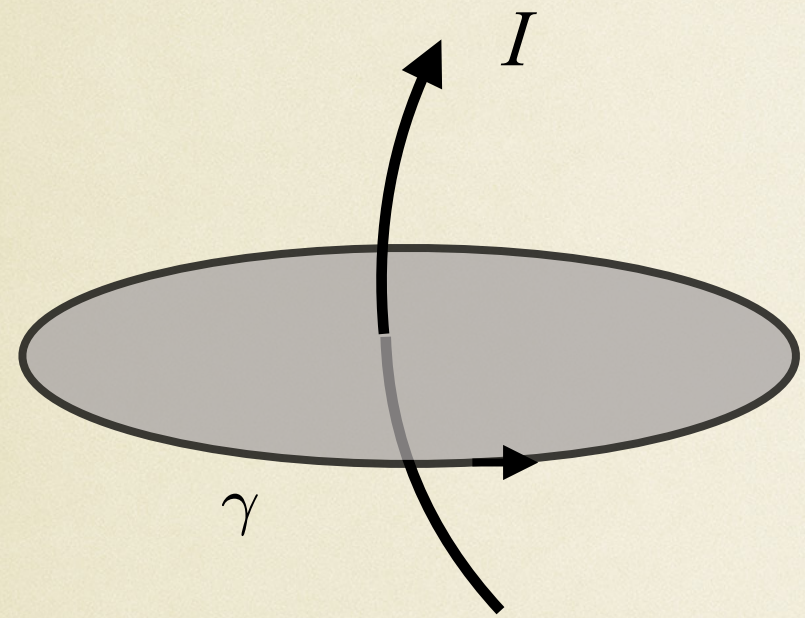


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$

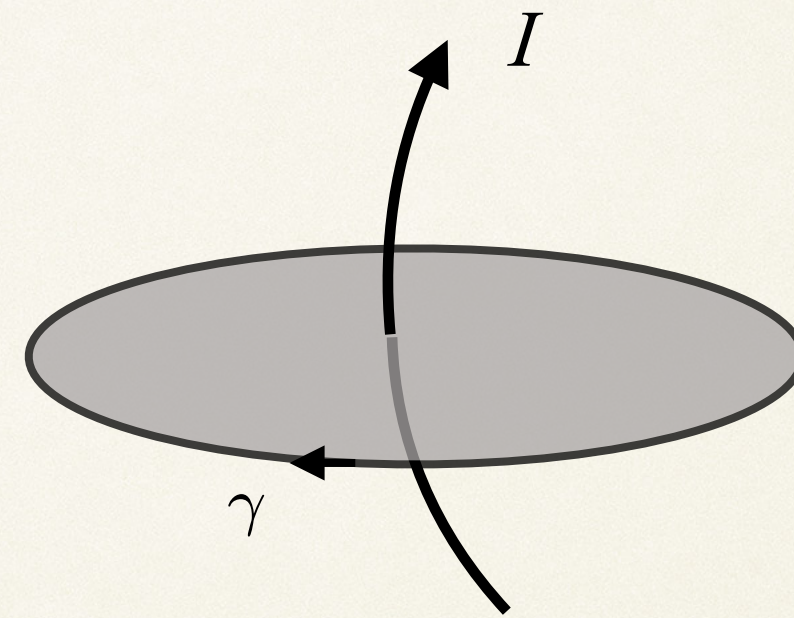


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$

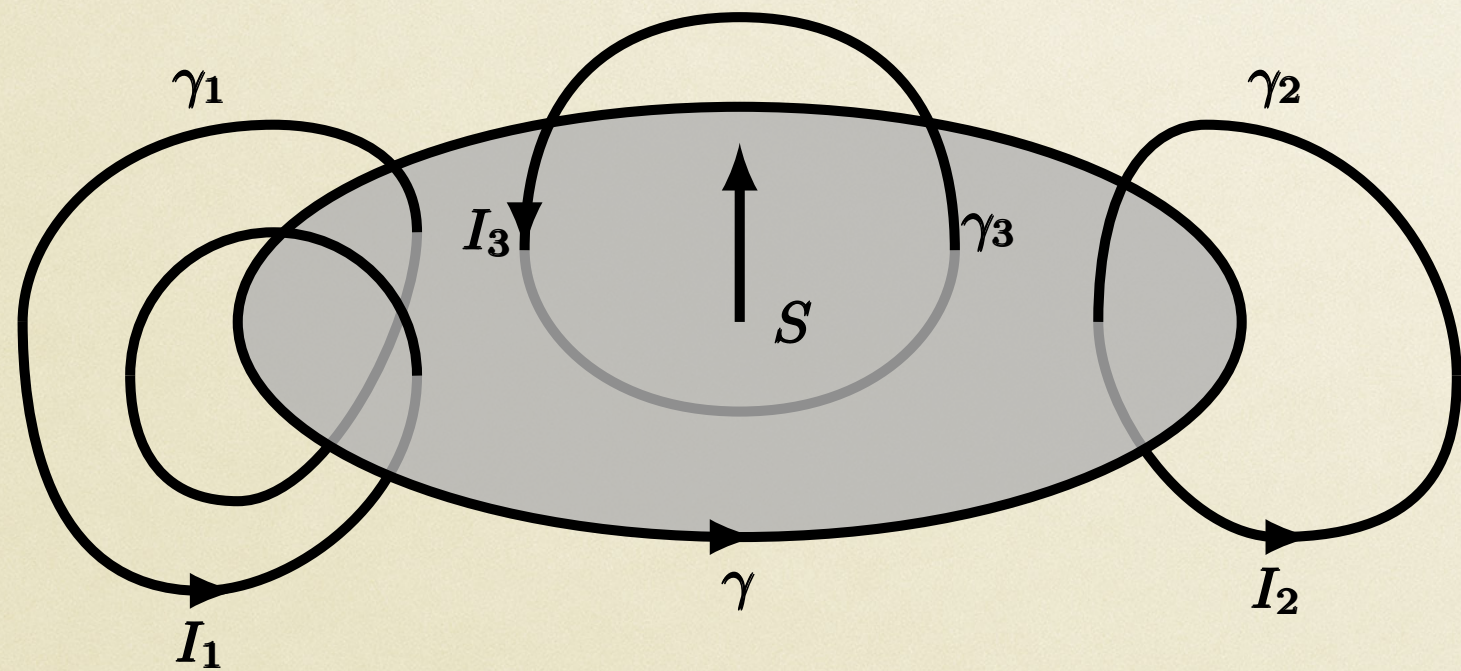
Ampere's Law



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$

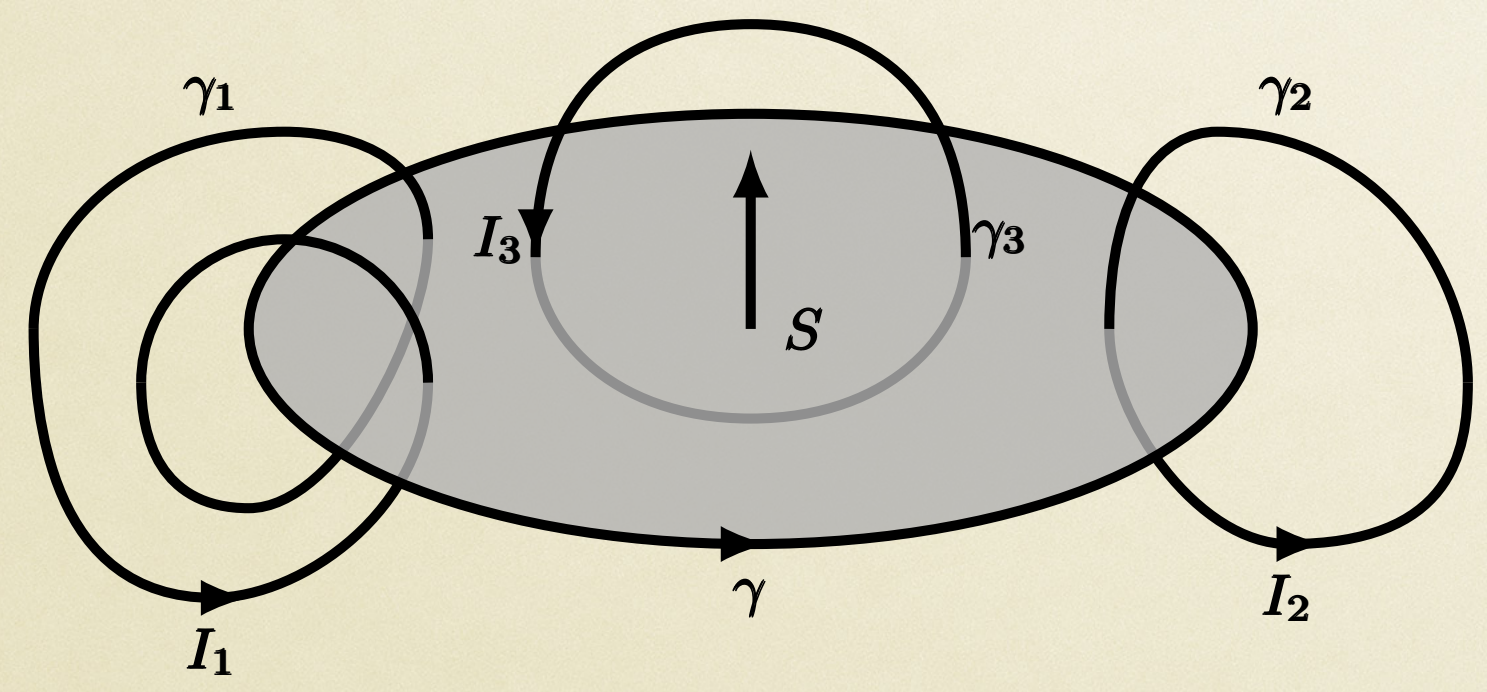
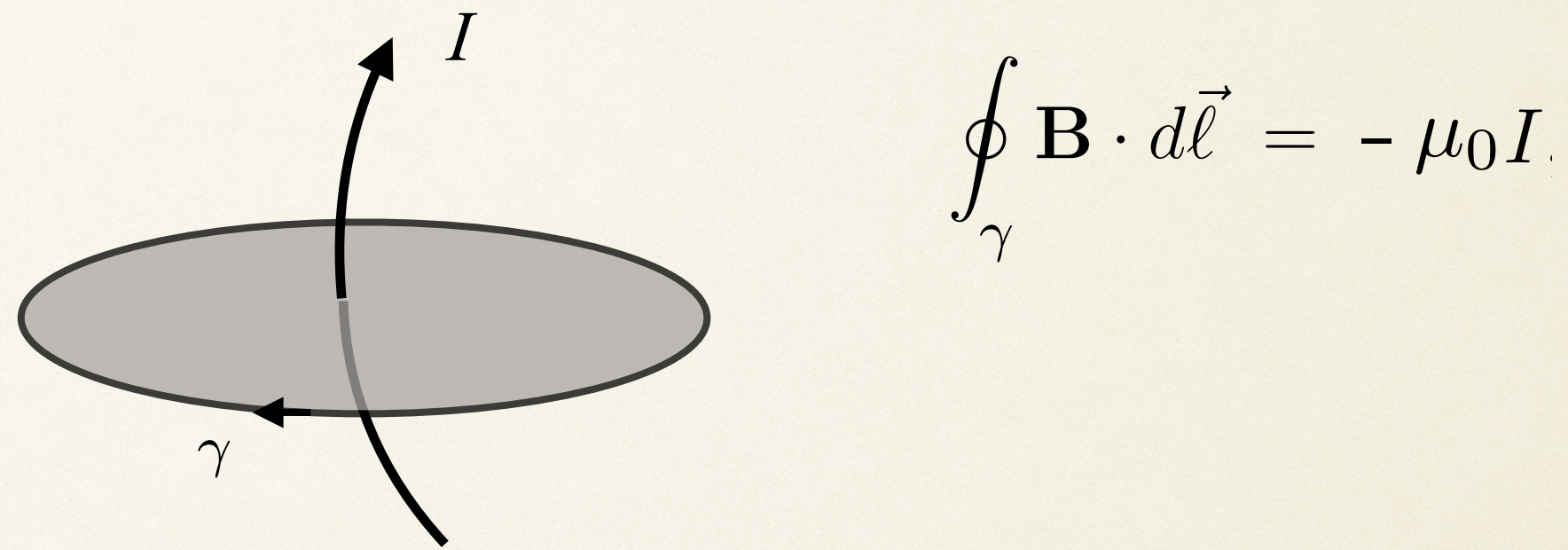
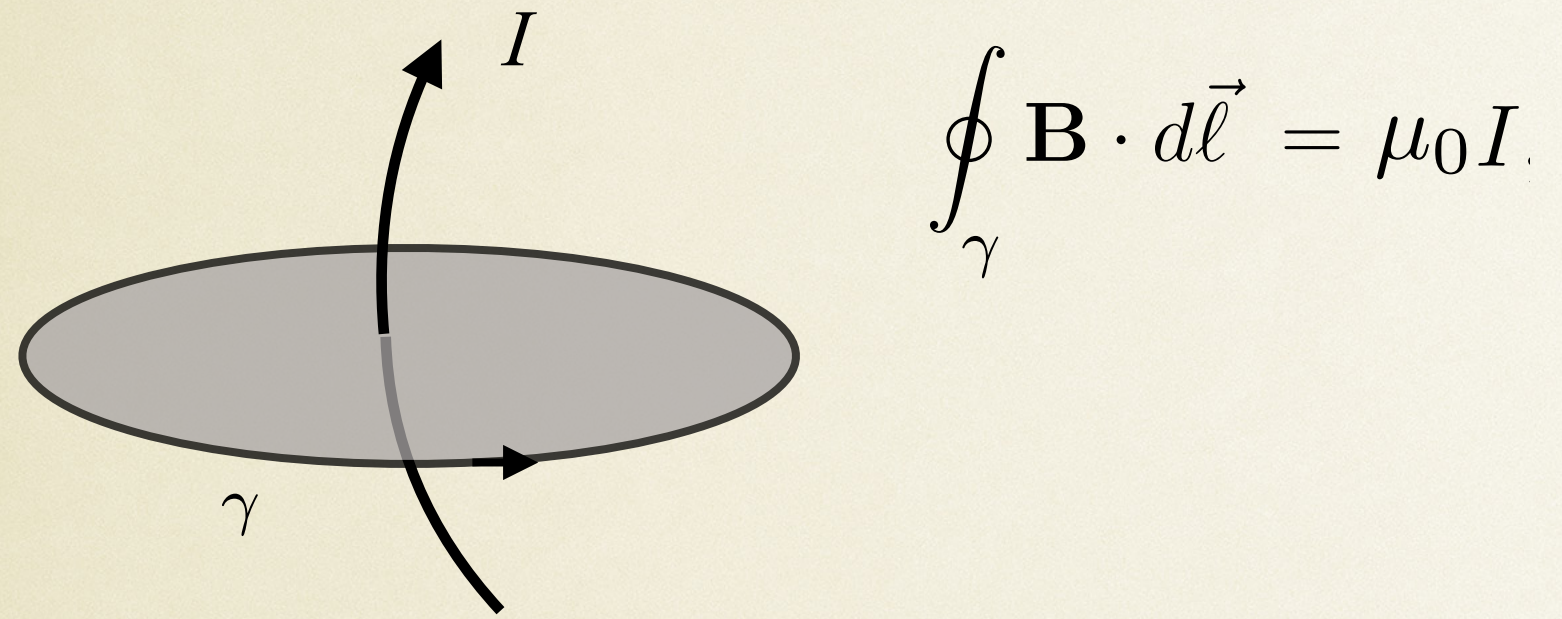


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = ?$$

Ampere's Law



$\text{Link}(\gamma_1, \gamma) = +2, \text{Link}(\gamma_2, \gamma) = -1, \text{and } \text{Link}(\gamma_3, \gamma) = 0$

• Integral decomposition by geometry

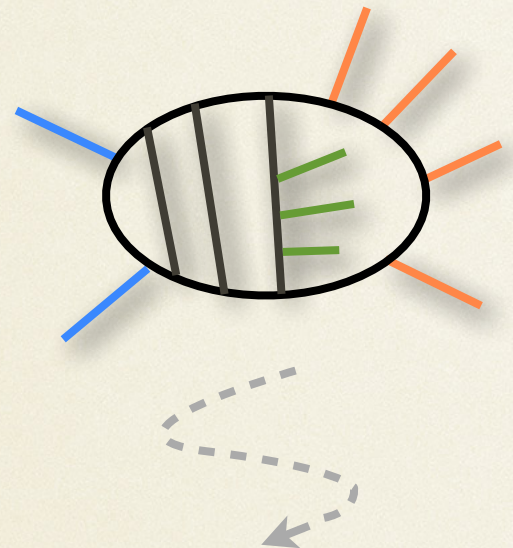
$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \sum_k (\pm n_k) \oint_{\gamma_k} \mathbf{B} \cdot d\vec{\ell} = \mu_0 \sum_k (\pm n_k) I_k$$

Gauss' Linking Number
 $n_k = \text{Link}(\gamma_k, \gamma)$

Master Contributions

Feynman Integrals

● Momentum-space Representation



$$= I_{a_1, \dots, a_N} = \int \prod_{i=1}^L d^d k_i \left(\prod_{n=1}^N \frac{1}{D_n^{a_n}} \right)$$

N-denominator
generic Integral

L loops, $E+1$ external momenta,

$N = LE + \frac{1}{2}L(L+1)$ (generalised) denominators

total number of *reducible* and *irreducible*
scalar products

't Hooft & Veltman

● Integration-by-parts Identities Tkachov; Chetyrkin & Tkachov

Laporta, Remiddi, Kuehn, Baikov, Smirnov, Melnikov, Gehrmann, Weinzierl, Anastasiou, Bonciani, &P.M. ...,

$$\int \prod_{i=1}^L d^d k_i \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^N \frac{1}{D_n^{a_n}} \right) = 0$$

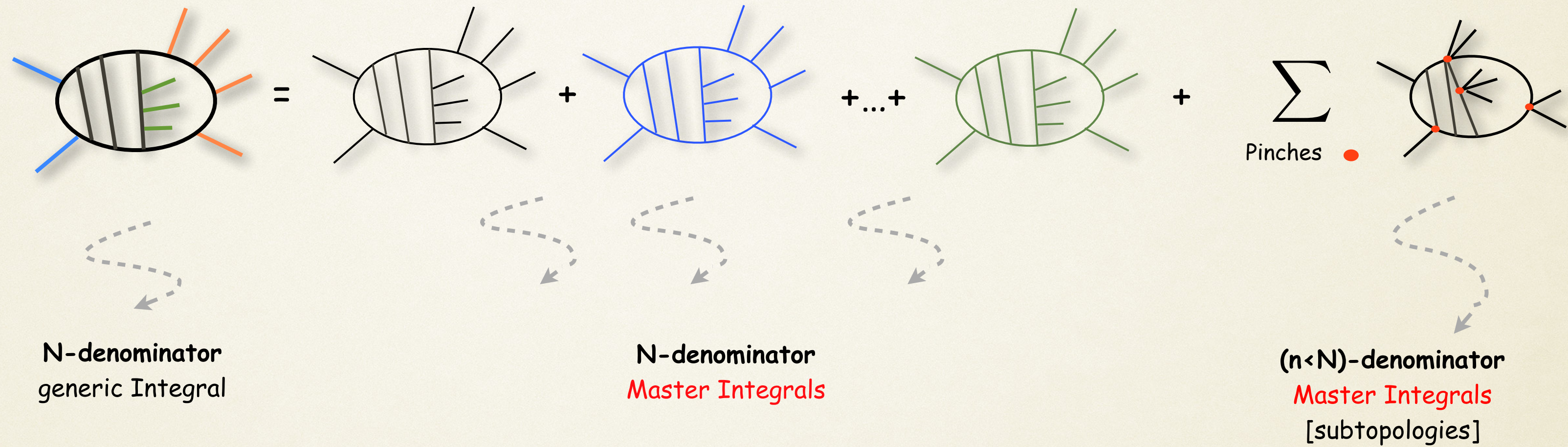
$v_\mu = v_\mu(p_i, k_j)$ arbitrary

● IBP identities

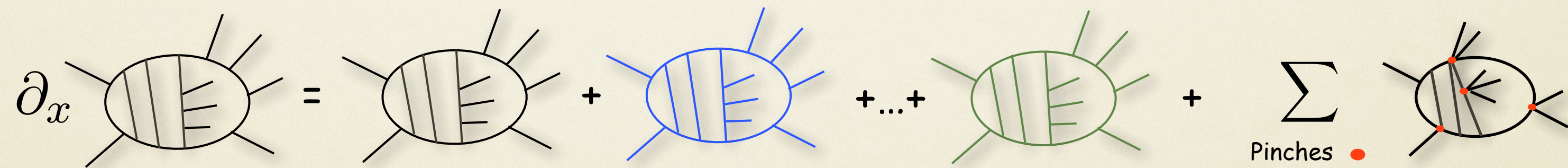
$$\sum_i b_i I_{a_1, \dots, a_i \pm 1, \dots, a_N} = 0$$

Linear relations for Feynman Integrals identities

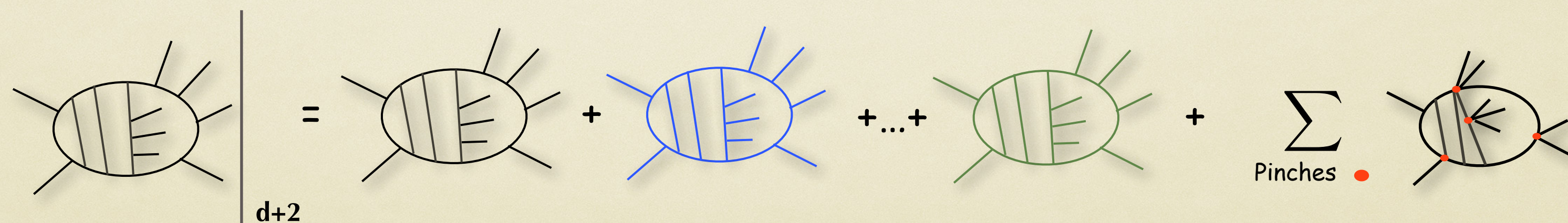
- Relations among Integrals in dim. reg.



- 1st order Differential Equations for MIs



- Dimension-Shift relations and Gram determinant relations



Outline

📌 Vector Space Structure of (Feynman, GKZ, Euler-Mellin, A-hypergeometric) twisted period Integrals

📌 Linear and Quadratic relations

📌 Intersection Numbers

📌 1-forms

📌 n-forms (I): iterative method

📌 n-forms (II): Partial Differential Equation

📌 n-forms (III): Pfaffian Systems of D-modules

📌 Applications

📌 Hypergeometric functions

📌 Feynman Integrals

📌 Matrix elements in Quantum Mechanics

📌 Wick's theorem

📌 Kontsevich-Witten tau-function

📌 AH-B-P wave function integrals

📌 Conclusions

Based on:

- **PM**, Mizera
Feynman Integral and Intersection Theory
JHEP 1902 (2019) 139 [arXiv: 1810.03818]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
Decomposition of Feynman Integrals in the Maximal Cut by Intersection Numbers
JHEP 1095 (2019) 153 [arXiv: 1901.11510]
- Frellesvig, Gasparotto, Mandal, **PM**, Mattiazzi, Mizera
Vector Space of Feynman Integrals and Multivariate Intersection Numbers
Phys. Rev. Lett. 123 (2019) 20, 201602 [arXiv 1907.02000]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
Decomposition of Feynman Integrals by Multivariate Intersection Numbers.
JHEP 03 (2021) 027 [arXiv 2008.04823]
- Chestnov, Gasparotto, Mandal, **PM**, Matsubara-Heo, Munch, Takayama
Macaulay Matrix for Feynman Integrals: linear relations and intersection numbers.
JHEP09 (2022) 187 [arXiv: 2204.12983]
- Cacciatori & **PM**,
Intersection Numbers in Quantum Mechanics and Field Theory.
2211.03729 [heo-th].
- **Brunello**, Chestnov, **Crisanti**, Frellesvig, Gasparotto & **PM**
in progress

What we have found

Vector Space Structure of Feynman [- Euler-Mellin - GKZ - A-hypergeometric] Integrals

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i J_i$$

 Master Integral = basis

ν = dimension of the vector space

- **Projections**

$$c_i = I \cdot J_i, \quad J_i \cdot J_j = \delta_{ij}$$

- **Completeness**

$$\sum_i J_i J_i = \mathbb{I}_{\nu \times \nu}$$

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The two questions:

- 1) what is the vector space dimension ν ?
- 2) what is the *scalar product* “.” between integrals ?

Basics of Intersection Theory

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,...

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$\varphi_m(\mathbf{z})$ is a differential m -form
 $u(\mathbf{z})$ is a multivalued function
 $u(\partial\mathcal{C}) = 0$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

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- **The dawn of Integration by parts identities:**

- **Equivalence Classes of DIFFERENTIAL FORMS**

- There could exist many forms φ_m that upon integration give the same result I

- **Equivalence Classes of INTEGRATION CONTOURS**

- There could exist many contours \mathcal{C} that do not alter the the result of I

Vector Space Structure of Twisted Period Integrals

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d(u \varphi_{m-1}) = \int_{\mathcal{C}} (u d\varphi_{m-1} + du \wedge \varphi_{m-1}) = \int_{\mathcal{C}} u (d + \omega \wedge) \varphi_{m-1} = \int_{\mathcal{C}} u \nabla_{\omega} \varphi_{m-1}$$

• **Covariant Derivative** $\omega \equiv d \log u$ $\nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$

• **Integrals** $I = \int_{\mathcal{C}} u \varphi_m = \int_{\mathcal{C}} u (\varphi_m + \nabla_{\omega} \varphi_{m-1}) = \int_{\mathcal{C} + \partial \Gamma} u \varphi_m$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

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• **Twisted Cohomology Group**

$$H_{\omega}^m(X) = \frac{\text{Ker}(\nabla_{\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

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• **Twisted Homology Group** $H_p^{\omega}(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{p+1} \rightarrow \mathcal{C}_p)}{\text{Im}(\partial : \mathcal{C}_p \rightarrow \mathcal{C}_{p-1})}$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d(u^{-1} \varphi_{m-1}) = \int_{\mathcal{C}} (u^{-1} d\varphi_{m-1} - u^{-2} du \wedge \varphi_{m-1}) = \int_{\mathcal{C}} u^{-1} (d - \omega \wedge) \varphi_{m-1} = \int_{\mathcal{C}} u^{-1} \nabla_{-\omega} \varphi_{m-1}$$

● **Dual Covariant Derivative**

$$\nabla_{-\omega} \equiv d - \omega \wedge \equiv u \cdot d \cdot u^{-1}$$

● **Dual Integrals**

$$\tilde{I} = \int_{\mathcal{C}} u^{-1} \phi_m = \int_{\mathcal{C}} u^{-1} (\phi_m + \nabla_{-\omega} \phi_{m-1}) = \int_{\mathcal{C} + \partial\Gamma} u^{-1} \phi_m$$

● **Dual Twisted Cohomology Group**

$$H_{-\omega}^m(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

● **Dual Twisted Homology Group**

$$H_p^{-\omega}(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{p+1} \rightarrow \mathcal{C}_p)}{\text{Im}(\partial : \mathcal{C}_p \rightarrow \mathcal{C}_{p-1})}$$

Pairings of Cycles and Co-cycles

- **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_\omega^m$$

$$| \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m$$

$$| \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^\omega$$

$$[\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

- **Integrals :: pairings of cycles and co-cycles**

$$\langle \varphi_L | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

- **Dual Integrals :: pairings of cycles and co-cycles**

$$[\mathcal{C}_L | \varphi_R \rangle \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = \tilde{I}$$

- **Intersection numbers for cycles :: pairings of cycles**

$$[\mathcal{C}_L | \mathcal{C}_R] \equiv \text{intersection number}$$

- **Intersection numbers for co-cycles :: pairings of co-cycles**

$$\langle \varphi_L | \varphi_R \rangle \equiv \int_{\mathcal{C}} \iota(\varphi_L) \wedge \varphi_R$$

Identity Resolution

$$\dim H_{\pm\omega}^m = \dim H_m^{\pm\omega} \equiv \nu$$

• **Bases** $\{\langle e_i | \}_{i=1, \dots, \nu} \in H_{\omega}^n$ and $\{|h_i\rangle\}_{i=1, \dots, \nu} \in H_{-\omega}^n$

• **Forms** $\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle (\mathbf{C}^{-1})_{ij} \langle e_j|$ $\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$ **Metric Matrix for Forms**

• **Contours** $\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i] (\mathbf{H}^{-1})_{ij} [\eta_j|$ $\mathbf{H}_{ij} \equiv [\eta_i | \gamma_j]$ **Metric Matrix for Contours**

Identity Resolution

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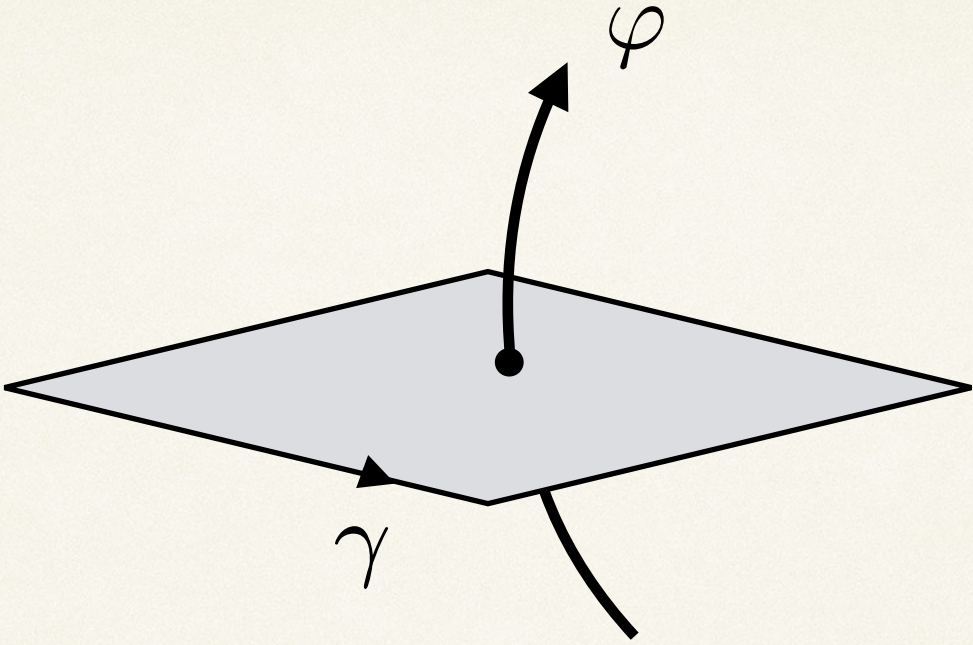
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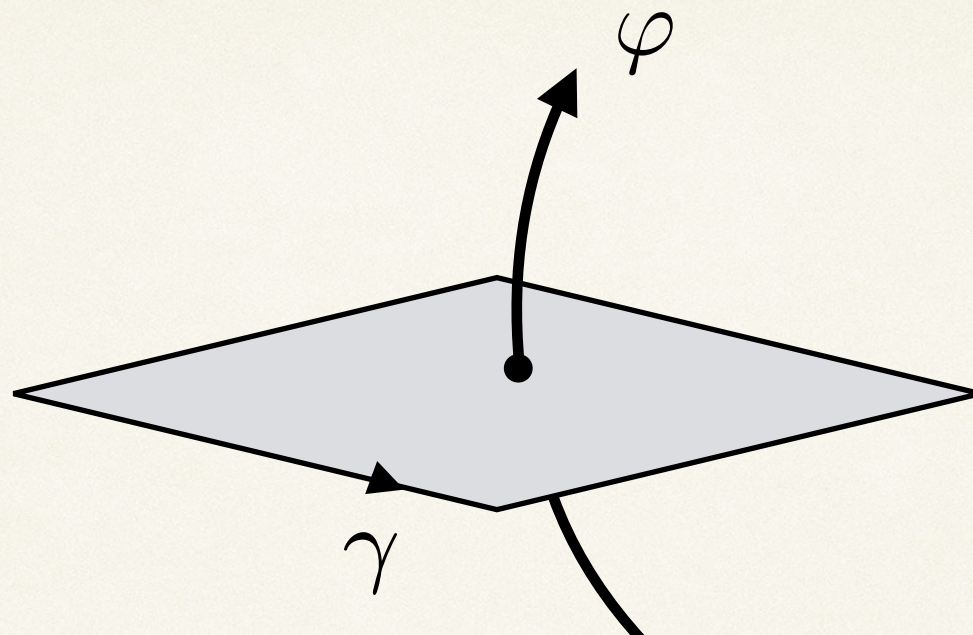
Linear Relations

Flux Decomposition

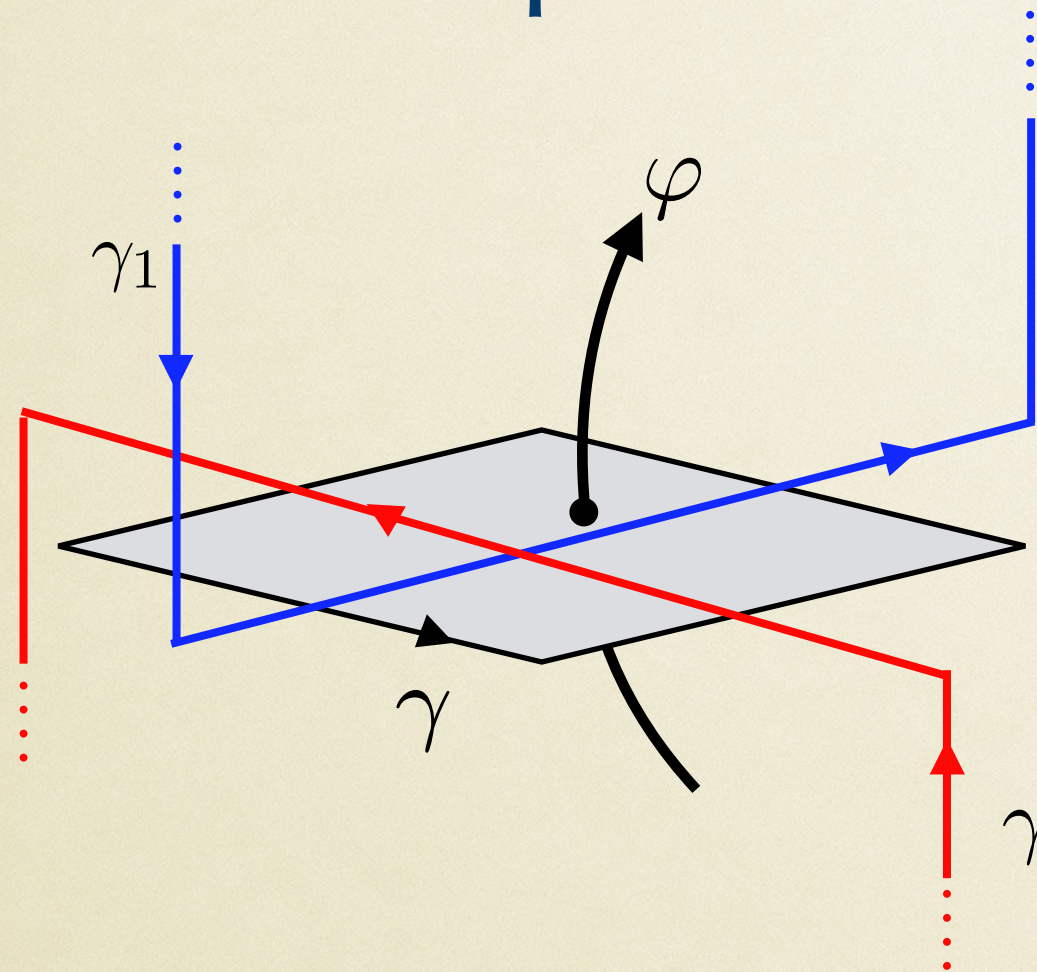
$$\int_{\gamma} \varphi = \langle \varphi | \gamma \rangle$$


The diagram illustrates the flux decomposition of a surface integral. It features a light blue diamond-shaped surface with a central black dot. A curved arrow labeled γ indicates a path on the surface. A vector labeled φ originates from the central dot and points upwards and to the right. The equation $\int_{\gamma} \varphi = \langle \varphi | \gamma \rangle$ is shown to the left and right of the diagram, indicating that the integral of φ over the path γ is equal to the inner product of φ and γ .

Flux Decomposition

$$\int_{\gamma} \varphi = \langle \varphi | \gamma \rangle$$
A diamond-shaped surface is shown in perspective. A black arrow labeled γ points along the bottom edge of the diamond. A black arrow labeled φ points upwards from a central point on the surface.

● Contour decomposition

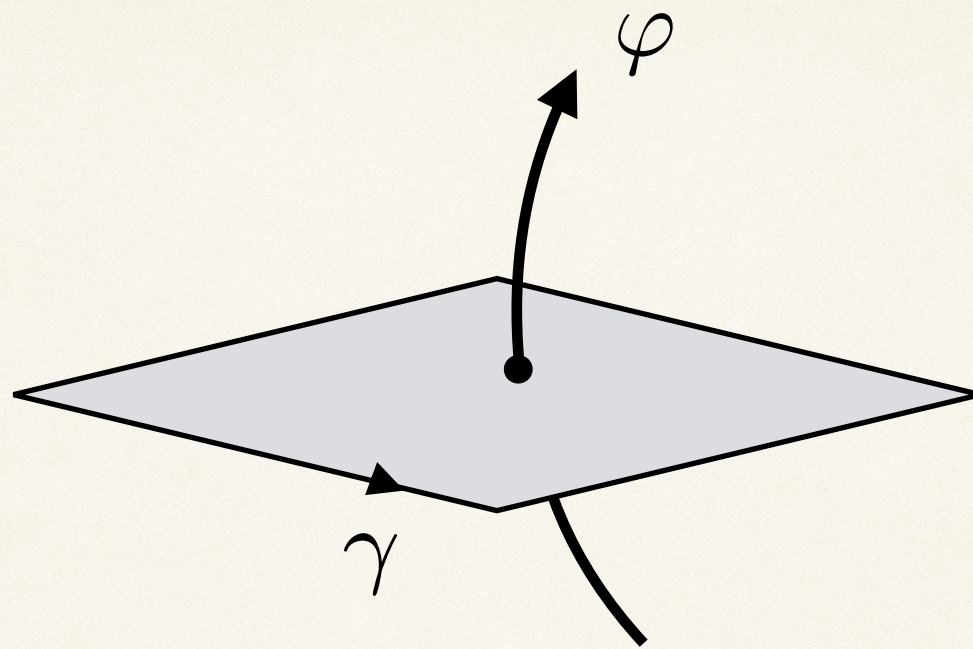
A diamond-shaped surface is shown in perspective. A black arrow labeled γ points along the bottom edge. Two other contours are shown: a blue line labeled γ_1 and a red line labeled γ_2 . Both γ_1 and γ_2 are vertical lines that cross the diamond. A black arrow labeled φ points upwards from a central point on the surface. The equation $\int_{\gamma} \varphi = \sum_{i=1} a_i \int_{\gamma_i} \varphi$ is written to the right of the diagram.
$$\int_{\gamma} \varphi = \sum_{i=1} a_i \int_{\gamma_i} \varphi$$

$$|\gamma] = \sum_i a_i |\gamma_i]$$

● Coefficients are **Intersection Numbers (contours)**

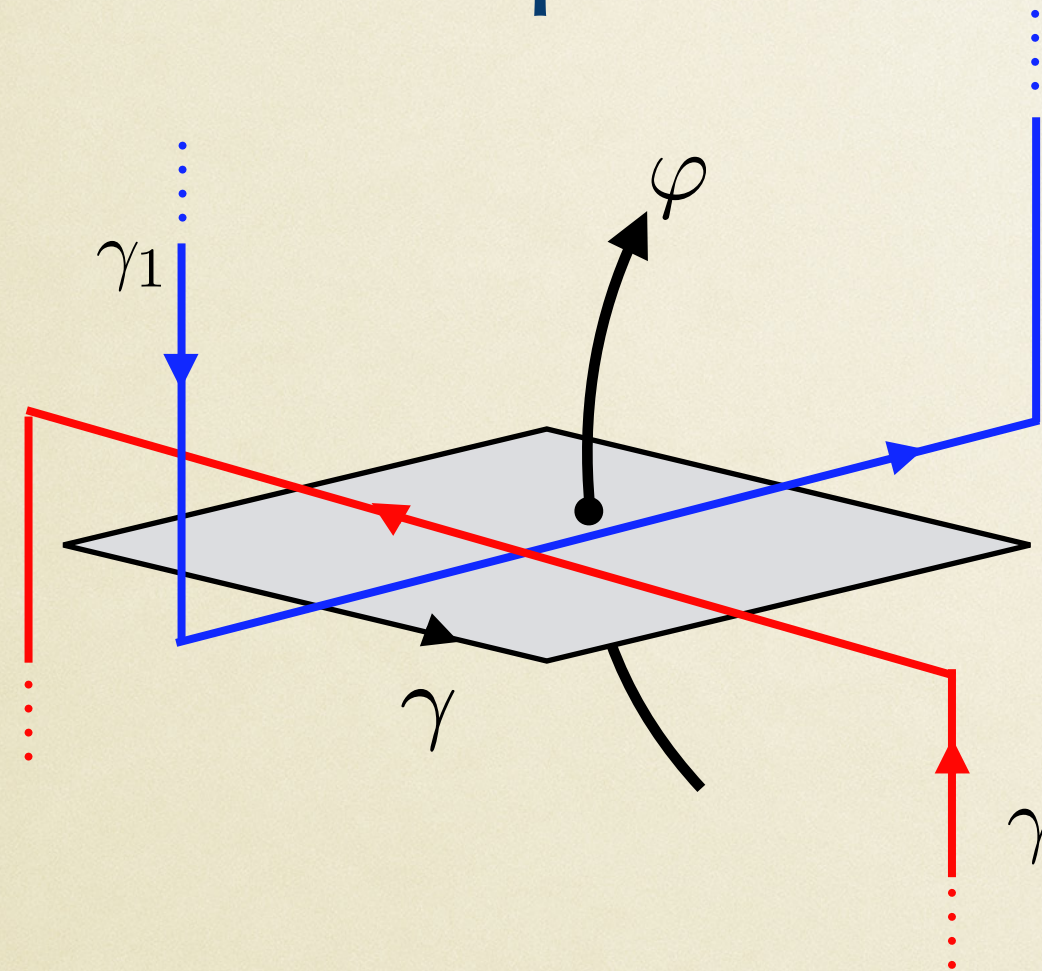
$$a_i = [\gamma_i | \gamma], \quad [\gamma_i | \gamma_j] = \delta_{ij}$$

Flux Decomposition

$$\int_{\gamma} \varphi = \langle \varphi | \gamma \rangle$$


A diamond-shaped surface is shown with a contour γ indicated by an arrow on its bottom edge. A vector φ originates from a point on the surface and points upwards and to the right.

● Contour decomposition



$$= \sum_{i=1} a_i \int_{\gamma_i} \varphi$$

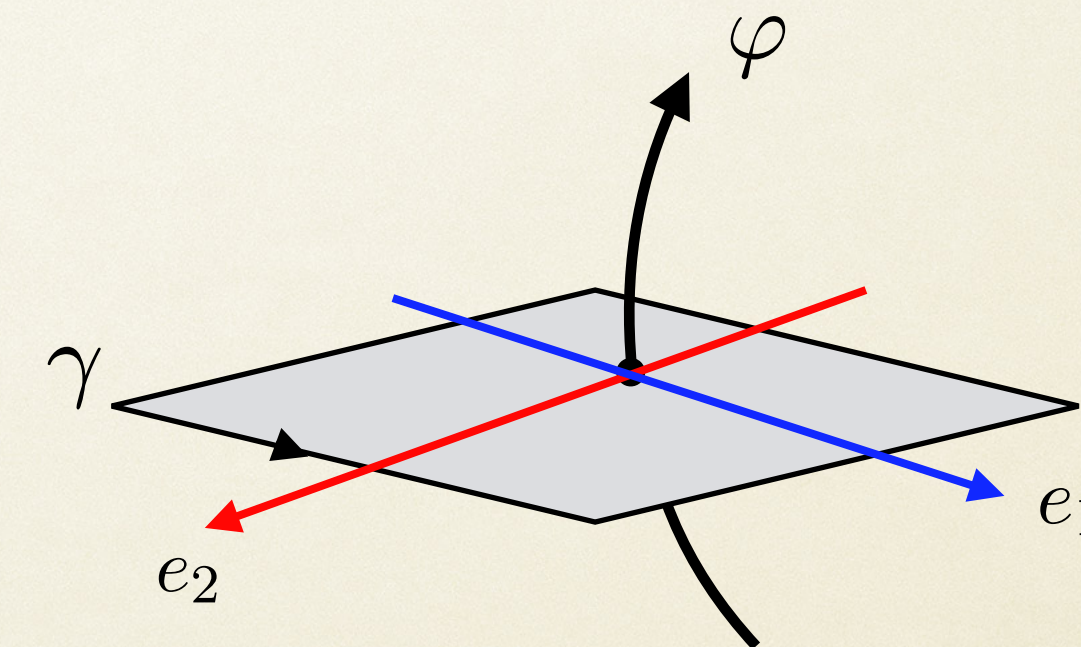
The diagram shows the same diamond-shaped surface with a contour γ (black arrow) and a flux vector φ (black arrow). Several other contours are shown in blue and red, labeled $\gamma_1, \gamma_2, \dots$. The contours are oriented in various directions across the surface.

$$|\gamma\rangle = \sum_i a_i |\gamma_i\rangle$$

● Coefficients are Intersection Numbers (contours)

$$a_i = [\gamma_i | \gamma], \quad [\gamma_i | \gamma_j] = \delta_{ij}$$

● Form decomposition



$$= \sum_i c_i \int_{\gamma} e_i$$

The diagram shows the diamond-shaped surface with a contour γ (black arrow) and a flux vector φ (black arrow). Two basis forms are shown as red and blue arrows on the surface, labeled e_1 and e_2 .

$$\langle \varphi | = \sum_i c_i \langle e_i |$$

● Coefficients are Intersection Numbers (forms)

$$c_i = \langle \varphi | e_i \rangle, \quad \langle e_i | e_j \rangle = \delta_{ij}$$

Linear Relations / IBPs identity / Gauss contiguity relations

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z}) e_i(\mathbf{z}) = \langle e_i | \mathcal{C} \rangle, \quad i = 1, \dots, \nu,$$

- **Integral decomposition**

$$I = \langle \varphi_L | \mathcal{C}_R \rangle = \sum_{i=1}^{\nu} c_i J_i.$$

Linear Relations / IBPs identity / Gauss contiguity relations

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- **Integral decomposition**

$$I = \langle \varphi_L | \mathcal{C}_R \rangle = \sum_{i=1}^{\nu} c_i J_i.$$

- **Decomposition of differential forms.**

- **Master Decomposition Formula**

$$\langle \varphi_L | = \langle \varphi_L | \mathbb{I}_{\mathcal{C}} = \sum_{i=1}^{\nu} c_i \langle e_i |, \quad \text{with} \quad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

Quadratic Relations

Twisted Riemann Periods Relations (TRPR)

Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi_L | \mathbb{I}_h | \varphi_R \rangle = \sum_{i,j=1}^{\nu} \langle \varphi_L | \gamma_i \rangle \left(\mathbf{H}^{-1} \right)_{ij} [\eta_j | \phi_R] = \left(\mathbf{P}_\omega \cdot \mathbf{H}^{-1} \cdot \mathbf{P}_{-\omega} \right)_{LR}$$

$$[C_L | C_R] = [C_L | \mathbb{I}_c | C_R] = \sum_{i,j=1}^{\nu} [C_L | h_i] \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j | C_R \rangle = \left(\mathbf{P}_{-\omega} \cdot \mathbf{C}^{-1} \cdot \mathbf{P}_\omega \right)_{LR}$$

Vector Space Structure of Feynman Integrals

Vector Space Dimensions

● Space Dimensions = Number of Master Integrals

ν = number of independent *master* integrals

Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

= is finite

Smirnov, Petuckhov (2010)

= number of critical points of graph polynomials

Lee, Pomeranski (2013)

= is related to Euler characteristics χ_E

Aluffi, Marcolli (2008)

Bitoun, Bogner, Klausen, Panzer (2018)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

= number of independent integration contours

Lee, Pomeranski (2013)

Bosma, Sogaard, Zhang (2017)

Primo, Tancredi (2017)

= number of independent forms

Mizera & P.M. (2018)

= $\dim H_{\pm\omega}^m$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

= $\dim\left(\mathbb{C}[\mathbf{z}]/\langle \hat{\omega}_1, \dots, \hat{\omega}_n \rangle\right) = \dim\left(\mathbb{C}[\mathbf{z}]/\langle \mathcal{G} \rangle\right)$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2020)

Parametric Representation(s)

- Upon a change of integration variables

$$= I_{a_1, \dots, a_N} = \int_{\mathcal{C}} u(\mathbf{z}) \varphi_N(\mathbf{z})$$

N-denominator
generic Integral

$$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N \mathbf{z} \quad \text{differential } N\text{-form}$$

$$d^N \mathbf{z} = dz_1 \wedge \dots \wedge dz_N$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^\gamma$$

$$\mathcal{P}(\mathbf{z}) = \text{graph-Polynomial}$$

$$\gamma(d) = \text{generic exponent}$$

- **Integration-by-parts:** two situations may occur

$$\int_{\mathcal{C}} d(u(\mathbf{z}) \varphi_N(\mathbf{z})) \quad \begin{cases} \neq 0, \\ = 0, \end{cases} \quad u(\partial \mathcal{C}) = 0.$$

- Schwinger representation, Lee-Pomeranski repr'n

- Baikov representation, or other repr'ns

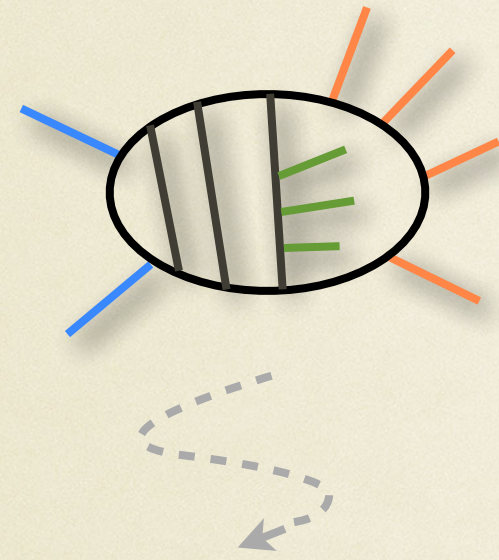
- **IBP identities**

$$\sum_i b_i I_{a_1, \dots, a_i \pm 1, \dots, a_N} = 0$$

Feynman Integrals :: Baikov Representation

- **Denominators as integration variables** Baikov (1996)

$$\{D_1, \dots, D_N\} \rightarrow \{z_1, \dots, z_N\} \equiv \mathbf{z}$$



N-denominator
generic Integral

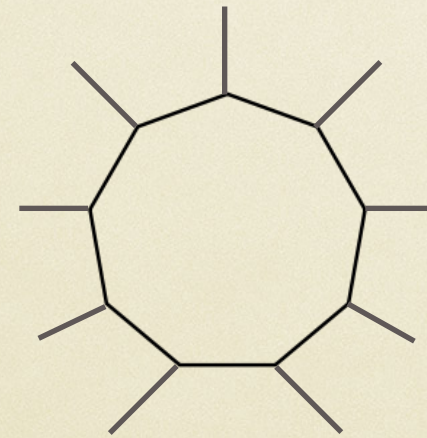
$$= I_{a_1, \dots, a_N} \equiv \int_{\mathcal{C}} d\mathbf{z} B(\mathbf{z})^\gamma \prod_{i=1}^N \frac{1}{z_i^{a_i}}$$

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \dots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \dots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

Gram determinant

$$\gamma \equiv (d - E - L - 1)/2$$

- **1-loop Nonagon**

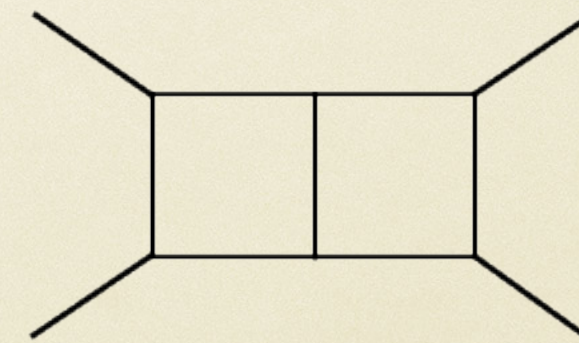


$$N = LE + \frac{1}{2}L(L + 1)$$

$$\int_{\mathcal{C}} dz_1 \wedge \dots \wedge dz_9 \frac{B(\mathbf{z})^\gamma}{z_1^{n_1} \dots z_9^{n_9}}$$

$B(\mathbf{z}), \mathcal{C}, \gamma$ depend on the graph.

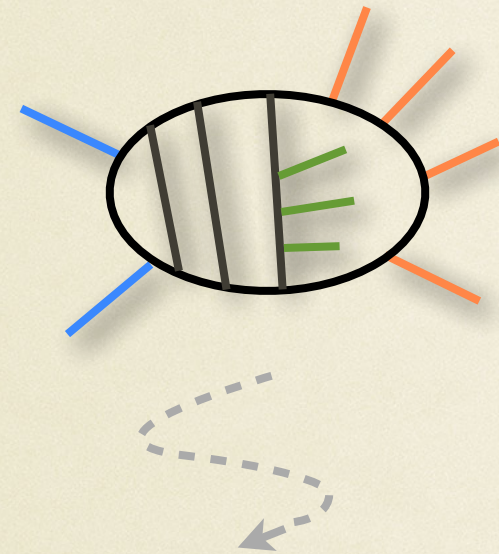
- **2-loop Box**



Feynman Integrals :: Baikov Representation

- **Denominators as integration variables** Baikov (1996)

$$\{D_1, \dots, D_N\} \rightarrow \{z_1, \dots, z_N\} \equiv \mathbf{z}$$



N-denominator
generic Integral

$$= I_{a_1, \dots, a_N} \equiv \int_{\mathcal{C}} d\mathbf{z} B(\mathbf{z})^\gamma \prod_{i=1}^N \frac{1}{z_i^{a_i}}$$

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \dots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \dots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

Gram determinant

$$\gamma \equiv (d - E - L - 1)/2$$

- **Integration-by-parts Identities**

Zhang, Larsen; Lee; Frellesvig, Papadopoulos

$$B(\partial\mathcal{C}) = 0$$

Fundamental property

$$\int_{\mathcal{C}} d \left(B(\mathbf{z})^\gamma \prod_{i=1}^N \frac{1}{z_i^{a_i}} \right) = 0$$

Three special applications:

i) Dimensional Recurrence Relation

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- MIs in $(d+2n)$ dimensions

$$J_i^{(d+2n)} \equiv K(d+2n) E_i^{(d+2n)} \quad E_i^{(d+2n)} \equiv \langle B^n e_i | \mathcal{C} \rangle = \int_{\mathcal{C}} u(B^n e_i), \quad u = B^\gamma, \quad \gamma \equiv (d-E-L-1)/2$$

- Master Decomposition Formula @ special basis choice

$$\langle B^\nu e_i | = \sum_{n=0}^{\nu-1} c_n \langle B^n e_i | \quad n = 0, 1, \dots, \nu - 1$$

- Recurrence Relations for Master Forms

$$\sum_{n=0}^{\nu} c_n \langle B^n e_i | = 0, \quad c_\nu \equiv -1$$

- Recurrence Relations for Master Integrals

$$\sum_{n=0}^{\nu} \alpha_n J_i^{(d+2n)} = 0 \quad \alpha_n \equiv c_n / K(d+2n)$$

ii) Differential Equations

- External Derivative

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C} \rangle = \partial_x \int_{\mathcal{C}} u \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma) \varphi | \mathcal{C} \rangle$$

- External (connection) dLog-form

$$\nabla_{x,\sigma} \equiv \partial_x + \sigma \quad \sigma = \partial_x \log u$$

- Derivative of Master Forms

$$\partial_x \langle e_i | = \langle (\partial_x + \sigma \wedge) e_i | = \langle (\partial_x + \sigma \wedge) e_i | h_k \rangle \underbrace{(\mathbf{C}^{-1})_{kj}}_{=1} \langle e_j | = \Omega_{ij} \langle e_j |$$

- System of DEQ for Master Forms

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |, \quad \Omega = \Omega(d, x)$$

An analogous System of DEQ can be derived for dual forms: $u \rightarrow u^{-1} \implies \nabla_{x,\sigma} \rightarrow \nabla_{x,-\sigma}$

iii) Secondary Equation

Matsubara-Heo, Takayama (2019)

Weinzierl (2020)

Chestnov, Gasparotto, Munch Matsubara-Heo, Takayama & P.M. (2022)

- DEQ for forms

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle (\mathbf{C}^{-1})_{kj}$$

- DEQ dual-forms

$$\partial_x |h_i\rangle = \tilde{\Omega}_{ji} |h_j\rangle$$

$$\tilde{\Omega}_{ji} = (\mathbf{C}^{-1})_{jk} \langle e_k | (\partial_x - \sigma_x) h_i \rangle$$

- Secondary Equation for the Intersection Matrix

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

$$\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}, \quad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$$

Intersection Numbers for 1-forms

Intersection Numbers :: 1-forms

Cho and Matsumoto (1998)

● **1-form** $\langle \varphi | \equiv \hat{\varphi}(z) dz$ $\hat{\varphi}(z)$ rational function

● **Zeroes and Poles of ω**

$$\omega \equiv d \log u$$

$$\nu = \{ \text{the number of solutions of } \omega = 0 \}$$

$$\mathcal{P} \equiv \{ z \mid z \text{ is a pole of } \omega \}$$

\mathcal{P} can also include the pole at infinity if $\text{Res}_{z=\infty}(\omega) \neq 0$.

● **Intersection Numbers**

1-forms φ_L and φ_R

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} \left(\psi_p \varphi_R \right)$$

ψ_p is a function (0-form), solution to the differential equation $\nabla_{\omega} \psi = \varphi_L$, around p

Intersection Numbers for n-forms :: Iterative Method

Intersection Numbers for *Logarithmic* n-Forms

Matsumoto (1998), Mizera (2017)

If $\langle \varphi_L |$ and $\langle \varphi_R |$ are dLog n -forms (hence contain only simple poles)

$$\langle \varphi_L | \varphi_R \rangle = \int dz_1 \cdots dz_n \delta(\omega_1) \cdots \delta(\omega_n) \hat{\varphi}_L \hat{\varphi}_R =$$

$$= \sum_{(z_1^*, \dots, z_n^*)} \det^{-1} \begin{bmatrix} \frac{\partial \omega_1}{\partial z_1} & \cdots & \frac{\partial \omega_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega_n}{\partial z_1} & \cdots & \frac{\partial \omega_n}{\partial z_n} \end{bmatrix} \hat{\varphi}_L \hat{\varphi}_R \Big|_{(z_1, \dots, z_n) = (z_1^*, \dots, z_n^*)}$$

[Global Residue Theorem]

(z_1^*, \dots, z_n^*) *critical points*, namely the solutions of the system $\omega_i = 0, \quad i = 1, \dots, n.$

In the 1-variate case: $\langle \varphi_L | \varphi_R \rangle = \text{Res}_{z \in \mathcal{P}_{\omega_1}} \left(\frac{\hat{\varphi}_L \hat{\varphi}_R}{\omega} \right) = \int dz_1 \delta(\omega_1) \hat{\varphi}_L \hat{\varphi}_R = \sum_{(z_1^*)} \frac{\hat{\varphi}_L \hat{\varphi}_R}{\partial \omega_1 / \partial z_1}$ [Residue Theorem]

● Efficiently implemented also *via Companion Matrix* credit Salvatori

Nested Integrations

- **Multivariate integral decomposition**

$$I = \int dz_n \dots \int dz_3 \int dz_2 \int dz_1 f(z_n, \dots, z_3, z_2, z_1)$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

- **Independent (Master) Integrals**

$$J_i \equiv \int dz_n \dots \int dz_3 \int dz_2 \int dz_1 f_i(z_n, \dots, z_1)$$

● Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)$$

● Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)}_{\exists \nu^{(2)} \text{ master integrals in } z_2}$$

$$I = \int dz_n \dots \int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \dots, z_3) J_{i_2}(z_n, \dots, z_3)$$

● Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)}_{\exists \nu^{(2)} \text{ master integrals in } z_2}$$

$$I = \int dz_n \dots \underbrace{\int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \dots, z_3) J_{i_2}(z_n, \dots, z_3)}_{\exists \nu^{(3)} \text{ master integrals in } z_3}$$

⋮

$$I = \underbrace{\int dz_n \sum_{i_n=1}^{\nu^{(n-1)}} c_{i_n}(z_n) J_{i_n}(z_n)}_{\exists \nu \text{ master integrals in } z_n}$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

Multivariate Intersection Numbers (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

● by *Induction*:

● (n-1)-form Vector Space: known!

$$\nu_{\mathbf{n}-1} \quad \langle e_i^{(\mathbf{n}-1)} | \quad | h_i^{(\mathbf{n}-1)} \rangle \quad (\mathbf{C}_{(\mathbf{n}-1)})_{ij} \equiv \nu_{\mathbf{n}-1} \langle e_i^{(\mathbf{n}-1)} | h_j^{(\mathbf{n}-1)} \rangle$$

● n-form decomposition: $\mathbf{n} = (\mathbf{n}-1) + (n)$

$$\langle \varphi_L^{(\mathbf{n})} | = \sum_{i=1}^{\nu_{\mathbf{n}-1}} \langle e_i^{(\mathbf{n}-1)} | \wedge \langle \varphi_{L,i}^{(n)} | ,$$

$$\langle \varphi_{L,i}^{(n)} | = \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ji} ,$$

$$\langle \varphi_{L,i}^{(n)} | (\mathbf{C}_{(\mathbf{n}-1)})_{ij} = \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle$$

$$| \varphi_R^{(\mathbf{n})} \rangle = \sum_{i=1}^{\nu_{\mathbf{n}-1}} | h_i^{(\mathbf{n}-1)} \rangle \wedge | \varphi_{R,i}^{(n)} \rangle ,$$

$$| \varphi_{R,i}^{(n)} \rangle = (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ij} \langle e_j^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle ,$$

$$(\mathbf{C}_{(\mathbf{n}-1)})_{ij} | \varphi_{R,j}^{(n)} \rangle = \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle$$

Multivariate Intersection Numbers (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

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$$\nu_{\mathbf{n}-1} \quad \langle e_i^{(\mathbf{n}-1)} | \quad | h_i^{(\mathbf{n}-1)} \rangle \quad (\mathbf{C}_{(\mathbf{n}-1)})_{ij} \equiv \nu_{\mathbf{n}-1} \langle e_i^{(\mathbf{n}-1)} | h_j^{(\mathbf{n}-1)} \rangle$$

• n-form decomposition: $\mathbf{n} = (\mathbf{n}-1) + (n)$

$$\langle \varphi_L^{(n)} | = \sum_{i=1}^{\nu_{\mathbf{n}-1}} \langle e_i^{(\mathbf{n}-1)} | \wedge \langle \varphi_{L,i}^{(n)} | ,$$

$$\langle \varphi_{L,i}^{(n)} | = \langle \varphi_L^{(n)} | h_j^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ji} ,$$

$$\langle \varphi_{L,i}^{(n)} | (\mathbf{C}_{(\mathbf{n}-1)})_{ij} = \langle \varphi_L^{(n)} | h_j^{(\mathbf{n}-1)} \rangle$$


$$| \varphi_R^{(n)} \rangle = \sum_{i=1}^{\nu_{\mathbf{n}-1}} | h_i^{(\mathbf{n}-1)} \rangle \wedge | \varphi_{R,i}^{(n)} \rangle ,$$

$$| \varphi_{R,i}^{(n)} \rangle = (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ij} \langle e_j^{(\mathbf{n}-1)} | \varphi_R^{(n)} \rangle ,$$

$$(\mathbf{C}_{(\mathbf{n}-1)})_{ij} | \varphi_{R,j}^{(n)} \rangle = \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(n)} \rangle$$

• Intersection Numbers for n-forms :: Recursive Formula

$$\begin{aligned} \langle \varphi_L^{(n)} | \varphi_R^{(n)} \rangle &= \sum_{i,j} \langle \varphi_L^{(n)} | h_j^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)})_{ji}^{-1} \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(n)} \rangle \\ &= \sum_{i,j} \langle \varphi_{L,i}^{(n)} | (\mathbf{C}_{(\mathbf{n}-1)})_{ij} \varphi_{R,j}^{(n)} \rangle \end{aligned}$$



$$\partial_{z_n} \psi_i^{(n)} + \psi_j^{(n)} \hat{\Omega}_{ji}^{(n)} = \hat{\varphi}_{L,i}^{(n)} ,$$

$\hat{\Omega}^{(n)}$ is a $\nu_{\mathbf{n}-1} \times \nu_{\mathbf{n}-1}$ matrix, whose entries are given by

$$\hat{\Omega}_{ji}^{(n)} = \langle (\partial_{z_n} + \hat{\omega}_n) e_j^{(\mathbf{n}-1)} | h_k^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ki}$$

Multivariate Intersection Numbers (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

- **Property of Intersection Number**

invariance under differential forms redefinition within the same equivalence classes,

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi'_L | \varphi'_R \rangle, \quad \varphi'_L = \varphi_L + \nabla_{\omega} \xi_L, \quad \varphi'_R = \varphi_R + \nabla_{-\omega} \xi_R$$

- **Global Residue Thm** Weinzierl (2020)

choose ξ_L and ξ_R , to build φ'_L and φ'_R that contain only simple poles, and if $\hat{\Omega}^{(n)}$ is reduced to Fuchsian form



the computation of multivariate intersection number can benefit of the evaluation of intersection numbers for dlog forms at each step of the iteration.

- **Special dual basis choice** CaronHuot Pokraka (2019-2021)

Relative Dirac-delta basis elements trivialise the evaluation of the intersection numbers

- **Multi-pole ansatz** Fontana Peraro (2022)

Solving $\nabla_{\omega} \psi = \varphi_L$, bypassing the pole factorisation, and using FF reconstruction methods.
(avoiding irrational functions which would disappear in the intersection numbers)

Contiguity relations for Special Functions

Hypergeometric ${}_3F_2$

$$u(\mathbf{z}) = ((1 - z_1)z_1(1 - z_2)z_2(1 - xz_1z_2))^\gamma; \quad \omega \equiv d \log u(\mathbf{z}) = \sum_{i=1}^2 \hat{\omega}_i dz_i;$$

a. *Number of MIs* :: I choose the ordering as $\{z_1, z_2\}$.

$$\nu_{12} = 3, \quad \{\omega_1 = 0, \omega_2 = 0\} \quad e^{(12)} = \left\{ \frac{1}{z_2(z_1 - \frac{1}{x})}, \frac{1}{(z_1 - 1)(z_2 - 1)}, \frac{1}{z_1(z_2 - xz_1)} \right\}$$

b. *Choice of bases* ::

$$\nu_2 = 2, \quad \{\omega_2 = 0\} \quad e^{(2)} = \left\{ \frac{1}{z_2}, \frac{1}{z_2 - 1} \right\}$$

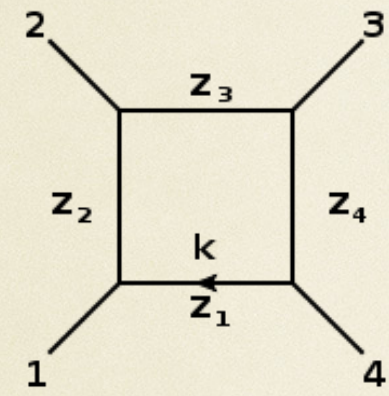
$$\partial_x \langle \hat{e}_i^{(12)} | = \langle \partial_x \hat{e}_i^{(12)} + \sigma \hat{e}_i^{(12)} | = \Omega_{ij} \langle \hat{e}_j^{(12)} |$$

$$\sigma = \frac{d \text{Log} u}{dx} = \frac{\gamma z_1 z_2}{x z_1 z_2 - 1}$$

$$\Omega = \gamma \begin{pmatrix} -\frac{x-2}{4(x-1)x} & \frac{3x+10}{20(x-1)x} & \frac{13x-10}{20(x-1)x} \\ \frac{3}{4(x-1)x} & \frac{20x+19}{20(x-1)x} & \frac{9}{20(x-1)x} \\ 0 & 0 & \frac{1}{x} \end{pmatrix} \quad \bullet \text{ Canonical}$$

Feynman Integrals Decomposition

Example: 1-Loop Box Integrals



$$u(\mathbf{z}) = \left((st - sz_4 - tz_3)^2 - 2tz_1(s(t + 2z_3 - z_2 - z_4) + tz_3) + s^2 z_2^2 + t^2 z_1^2 - 2sz_2(t(s - z_3) + z_4(s + 2t)) \right)^{\frac{d-5}{2}}$$

● Integral Decomposition

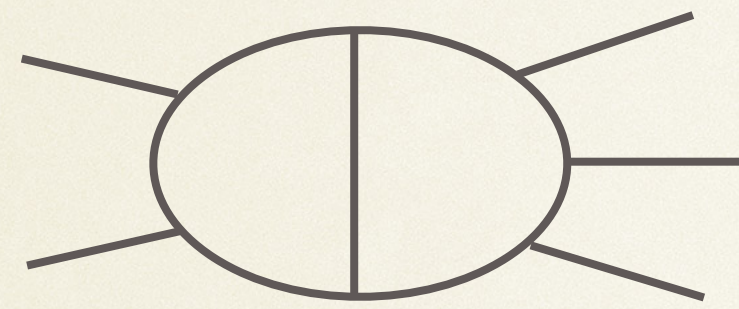
$$\langle \text{Box} \rangle = c_1 \langle \text{Box} \rangle + c_2 \langle \text{Bubble} \rangle + c_3 \langle \text{Bubble} \rangle$$

$$(c_1, c_2, c_3) = \left(\langle \text{Box} | \text{Box} \rangle, \langle \text{Box} | \text{Bubble} \rangle, \langle \text{Box} | \text{Bubble} \rangle \right) \begin{pmatrix} \langle \text{Box} | \text{Box} \rangle & \langle \text{Box} | \text{Bubble} \rangle & \langle \text{Box} | \text{Bubble} \rangle \\ \langle \text{Bubble} | \text{Box} \rangle & \langle \text{Bubble} | \text{Bubble} \rangle & \langle \text{Bubble} | \text{Bubble} \rangle \\ \langle \text{Bubble} | \text{Box} \rangle & \langle \text{Bubble} | \text{Bubble} \rangle & \langle \text{Bubble} | \text{Bubble} \rangle \end{pmatrix}^{-1}$$

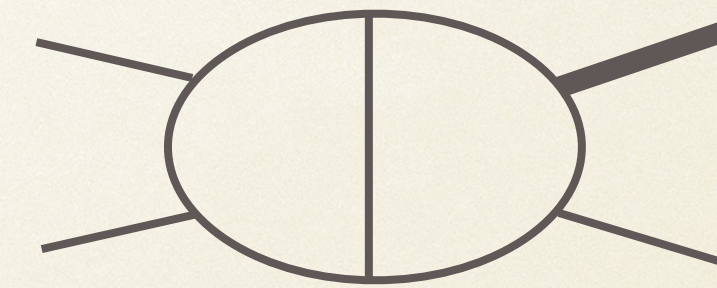
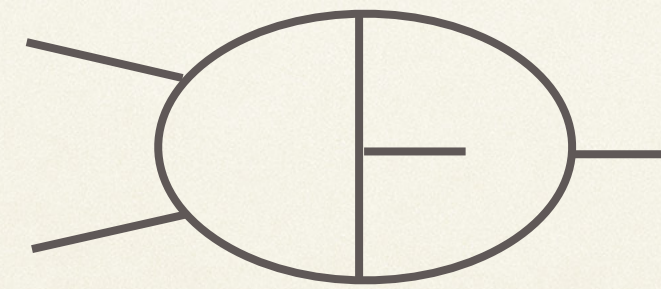
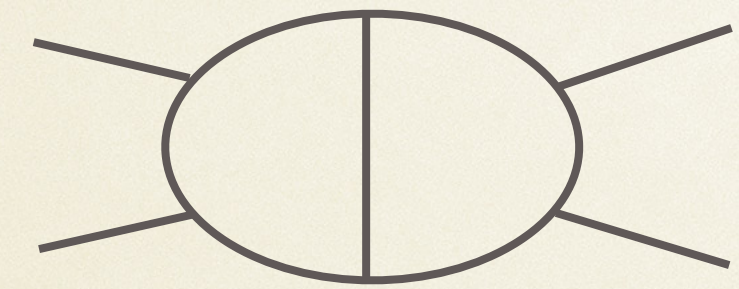
Recent Applications

Brunello, Chestnov, Crisanti, Frellesvig, Gasparotto, Mandal & P.M. (in progress)

● 2-loop 5-point integrals

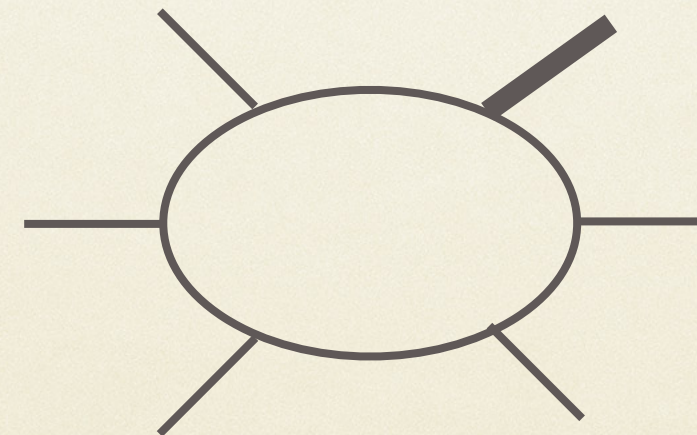
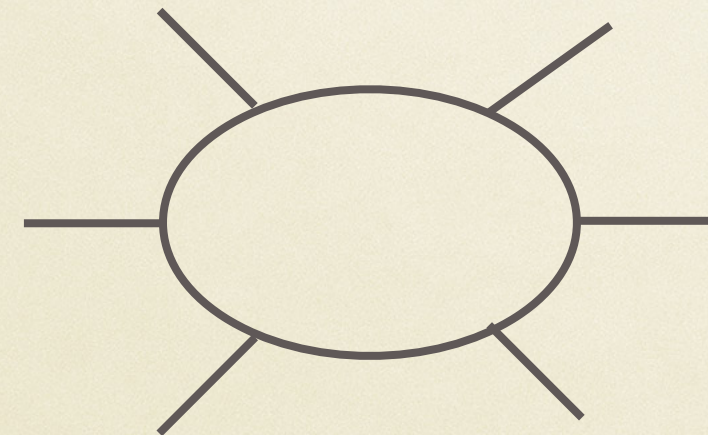


● 2-loop 4-point integrals



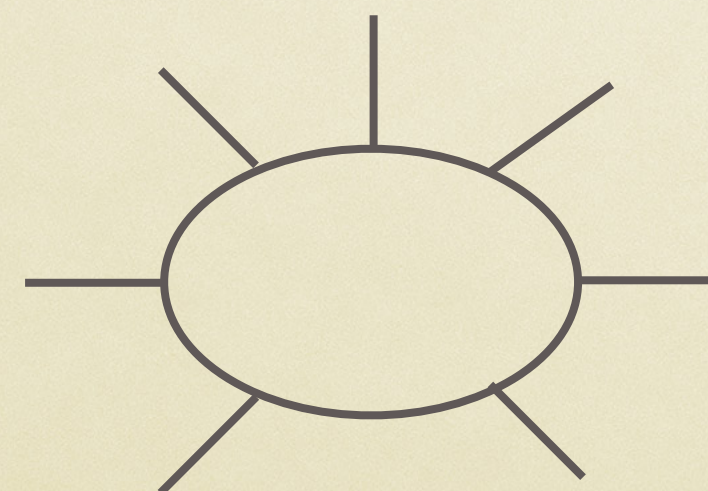
credit Brunello

● 1-Loop 6-point



credit Brunello

● 1-Loop 7-point



credit Brunello

Intersection Numbers for n-forms :: nPDE

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

$$\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = (2\pi i)^{-n} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})})$$

● nPDE

$$\nabla_{\omega_1} \nabla_{\omega_2} \cdots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}$$

Intersection Numbers for n-forms: Pfaffian systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & **P.M.** (2022)

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**



Gelfand Kapranov Zelevinsky (1990)

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● **Thm : Isomorphism**

nth-Cohomology group \cong

GKZ D-module

Gelf'and Kapranov Zelevinsky (1990)

Euler-Mellin Integrals

Differential Operators
(w.r.t. external variables)

Pfaffian Systems: for **Master Integrals** (alias **Master forms**)

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

$$\Omega = \Omega(d, x) \quad \bullet \text{ Pfaffian Matrix}$$

Basis of the Cohomology group

Integral decomposition (IBP/InterX)

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**

nth-Cohomology group \cong

GKZ D-module

Gelf'and Kapranov Zelevinsky (1990)

Euler-Mellin Integrals

Differential Operators
(w.r.t. external variables)

Pfaffian Systems: for **Master Integrals** (alias **Master forms**) & for **D-operators** (alias **Std mon's**)

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

Basis of the D-Operators

$$\Omega = \Omega(d, x) \quad \bullet \text{ Pfaffian Matrix}$$

Macaulay Matrix method

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

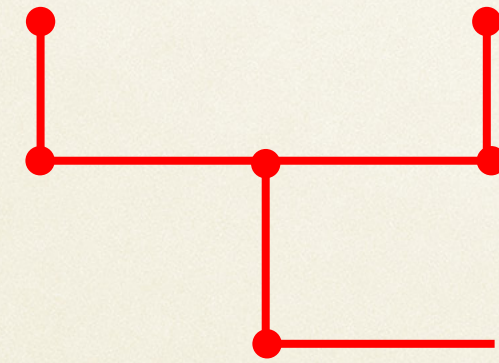
$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● Thm : Isomorphism

n th-Cohomology group \cong GKZ D-module

● Secondary Equations

$$\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}, \quad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$$



1) Build them from Macaulay Matrix for D-module

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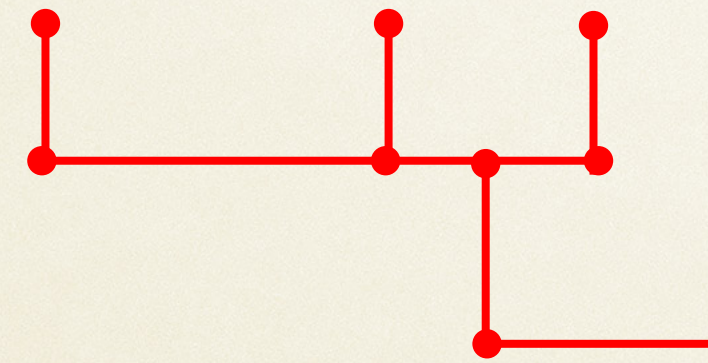
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2) Rational Solutions of Secondary Equations
[integrable connections]

Barkatou et al. @ MAPLE

Direct determination of Intersection Matrices

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

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$$\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}, \quad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$$

• **Master Decomposition**
$$\langle \varphi | = \sum_{\lambda=1}^r c_\lambda \langle e_\lambda |,$$

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{bmatrix} = C^{\text{aux}} \cdot C^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{bmatrix} \implies C^{\text{aux}} \cdot C^{-1} = \left[\begin{array}{c|c} \text{id}_{r-1} & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array} \right]$$

↑
Coefficients from matrix multiplication

Intersections Numbers @ QM and QFT

Cacciatori & P.M. (2022)

(Special) Applications of Intersection Numbers for 1-forms

- Looking at a known landscapes with new eyes

1. Identify a univariate twisted period integral $\int_{\Gamma} \mu \varphi$

If μ is not multivalued, replace it with the regulated twist $u = u(\rho)$ by introducing a regulator ρ , so that, for a suitable value ρ_0 , $u(\rho_0) = \mu$.

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2. After choosing the bases of forms $e_i \equiv \hat{e}_i dz$ and dual forms $h_i \equiv \hat{h}_i dz$, with $\hat{h}_i = \hat{e}_i$, such that $\hat{e}_1 = \hat{h}_1 = 1$, decompose φ

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3. Translate the decomposition of φ to the one of the corresponding integral, (eventually, taking the $\rho \rightarrow \rho_0$ limit)

$$\int_{\Gamma} \mu \varphi = c_1 E_1 + c_2 E_2 + \dots + c_v E_v, \quad \text{with} \quad E_1 \equiv \int_{\Gamma} \mu dz, \quad \text{and} \quad E_j = \int_{\Gamma} \mu e_j, \quad (j \neq 1),$$

and compare the result with the literature.

Orthogonal Polynomials and Matrix Elements in QM

Case i)
$$I_{nm} \equiv \int_{\Gamma} P_n(z) P_m(z) f(z) dz,$$

Case ii)
$$I_{nm} \equiv \langle n | \mathcal{O} | m \rangle = \int_{\Gamma} \psi_n^*(z) \mathcal{O}(z) \psi_m(z) f(z) dz$$

- **Master Decomposition formula**

For the considered cases, we obtain:

$$\varphi = c_1 e_1,$$

in terms of just one basic form, $e_1 = dz$

corresponding to:

$$I_{nm} = c_1 E_1$$

(one master integral)

i) Orthogonal Polynomials

Laguerre $L_n^{(\rho)}$, Legendre P_n , Tchebyshev T_n , Gegenbauer $C_n^{(\rho)}$, and Hermite H_n polynomials:

$$I_{nm} \equiv \int_{\Gamma} \mu P_n P_m dz = f_n \delta_{nm} = \int_{\Gamma} \mu \varphi = c_1 E_1 \quad \varphi \equiv P_n P_m dz$$

Type	u	ν	\hat{e}_i	C-matrix	ρ_0	E_1	c_1
$L_n^{(\rho)}$	$z^\rho \exp(-z)$	1	1	ρ	-	$\Gamma(1 + \rho)$	$(\rho + 1)(\rho + 2) \cdots (\rho + n)/n!$
P_n	$(z^2 - 1)^\rho$	1	1	$2\rho/(4\rho^2 - 1)$	0	2	$1/(2n + 1)$
T_n	$(1 - z^2)^\rho$	1	1	$2\rho/(4\rho^2 - 1)$	-1/2	π	1/2
$C_n^{(\rho)}$	$(1 - z^2)^{\rho-1/2}$	1	1	$(1 - 2\rho)/(4\rho(\rho - 1))$	-	$\sqrt{\pi}\Gamma(1/2 + \rho)/\Gamma(1 + \rho)$	$\rho(2\rho(2\rho + 1) \cdots (2\rho + n - 1))/((n + \rho)n!)$
H_n	$z^\rho \exp(-z^2)$	2	1, 1/z	diagonal(1/2, 1/ ρ)	0	$\sqrt{\pi}$	$2^n n!$

Let us observe that, in the case of Hermite polynomials, $\nu = 2$, yielding $\varphi = c_1 e_1 + c_2 e_2$, but $c_2 = 0$, due to the adopted basis

ii) Matrix Elements in QM

Harmonic Oscillator. (for unitary mass and pulsation, $m = 1 = \omega$)

$$\langle z|n\rangle = \psi_n(z) = e^{-\frac{z^2}{2}} W_n(z), \quad \text{with} \quad W_n(z) \equiv N_n H_n(z), \quad N_n \equiv 1/\sqrt{(2^n n! \sqrt{\pi})}$$

● **Position operator**

$$\langle m|z^k|n\rangle = \int_{-\infty}^{\infty} dz \psi_m(z) z^k \psi_n(z) = \int_{\Gamma} \mu \varphi = c_1 E_1, \quad \text{with} \quad \mu \equiv e^{-z^2}, \quad \text{and} \quad \varphi \equiv W_m(z) z^k W_n(z) dz.$$

Type	u	ν	\hat{e}_i	C-matrix	ρ_0	E_1
W_n	$z^\rho \exp(-z^2)$	2	1, 1/z	diagonal(1/2, 1/ ρ)	0	$\sqrt{\pi}$

$$\begin{aligned} \langle n|m\rangle &= \delta_{nm}, \\ \langle n|z^{2k+1}|n\rangle &= 0, \\ \langle n|z^4|n\rangle &= \frac{3}{4}(2n^2 + 2n + 1), \\ \langle n|z^3|n-3\rangle &= \sqrt{n(n-1)(n-2)/8}, \\ \langle n|z^3|n-1\rangle &= \sqrt{9n^3/8}. \end{aligned}$$

● **Hamiltonian operator**

$$\langle n|H|n\rangle = (n + 1/2) \quad H \equiv (1/2)(-\nabla^2 + z^2) \quad \varphi = \sum_{k=0}^n b_k z^{2k}$$

ii) Matrix Elements in QM

Hydrogen Atom. (for unitary Bohr radius $a_0 = 1$)

$$\langle z|n, \ell\rangle = R_{n, \ell}(z) = e^{-\frac{z}{n}} W_{n, \ell}(z), \quad \text{with} \quad W_{n, \ell}(z) \equiv N_{n\ell} \left(\frac{2z}{n}\right)^\ell L_{(n-\ell-1)}^{2\ell+1} \left(\frac{2z}{n}\right) \quad N_{n\ell} = (2/n)^{3/2} \sqrt{(n-\ell-1)!/(2n(n+\ell)!)}$$

• Position operator

$$\langle n_1, \ell|z^k|n_2, \ell\rangle = \int_0^\infty dz z^2 R_{n_1, \ell}(z) z^k R_{n_2, \ell}(z) = \int_\Gamma \mu \varphi = c_1 E_1, \quad \text{with} \quad \mu \equiv z^2 e^{-z\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \quad \text{and} \quad \varphi \equiv W_{n_1, \ell}(z) z^k W_{n_2, \ell}(z)$$

Type	u	v	\hat{e}_i	C-matrix	ρ_0	E_1
$W_{n, \ell}$	$z^{\rho+2} \exp(-z(n_1 + n_2)/(n_1 n_2))$	1	1	$(n_1 n_2 / (n_1 + n_2))^2 (2 + \rho)$	0	$2(n_1 n_2 / (n_1 + n_2))^3$

$$\langle n_1, \ell|n_2, \ell\rangle = \delta_{n_1 n_2},$$

$$\langle n, \ell|z|n, \ell\rangle = \frac{1}{2} [3n^2 - \ell(\ell + 1)],$$

$$\langle n, \ell|z^{-1}|n, \ell\rangle = \frac{1}{n^2},$$

$$\langle n, \ell|z^{-2}|n, \ell\rangle = \frac{2}{n^3(2\ell + 1)},$$

$$\langle n, \ell|z^{-3}|n, \ell\rangle = \frac{2}{n^3 \ell(\ell + 1)(2\ell + 1)}$$

Green's Function and Kontsevich-Witten tau-function

Case i)
$$G_n \equiv \frac{\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp[-S_E]}{\int \mathcal{D}\phi \exp[-S_E]}$$

Weinzierl (2020)

Case ii)
$$Z_{KW} \equiv \frac{\int d\Phi \exp \left[-\text{Tr} \left(-\frac{i}{3!} \Phi^3 + \frac{\Lambda}{2} \Phi^2 \right) \right]}{\int d\Phi \exp \left[-\text{Tr} \left(\frac{\Lambda}{2} \Phi^2 \right) \right]}$$

$$c_1 = \frac{\int_{\Gamma} \mu \varphi}{\int_{\Gamma} \mu e_1}, \quad \text{equivalently rewritten as} \quad \int_{\Gamma} \mu \varphi = c_1 E_1$$

● **Master Decomposition formula**

● **Toy models univariate integrals**

i) Green's Function

Single field, ϕ^4 -theory

real scalar field $\phi(x)$ $S_E \equiv S_0 + \varepsilon S_1$, with $S_0 = (\gamma/2) \phi^2(x)$, and $S_1 = \phi^4(x)$

$$\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{-S_E} = G_n \int \mathcal{D}\phi e^{-S_E}$$

$$\int_{\Gamma} \mu \varphi = G_n E_1, \quad \text{with} \quad \mu \equiv e^{-S_E}, \quad \varphi \equiv \phi(x_1) \cdots \phi(x_n) \mathcal{D}\phi, \quad E_1 \equiv \int_{\Gamma} \mu e_1, \quad \text{and} \quad e_1 \equiv \mathcal{D}\phi$$

Free theory. The n -point Green's function $G_n^{(0)}$ $\phi(x) \equiv z$ $\mu \equiv e^{-S_0}$ $\varphi = z^n dz$

Type	u	v	\hat{e}_i	C-matrix	ρ_0	E_1	c_1
$G_n^{(0)}$	$z^\rho \exp(-\gamma z^2/2)$	2	1, 1/z	diagonal(1/γ, 1/ρ)	0	not needed	$(n-1)!!/\gamma^{n/2}$

for even n

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for even n

• **2-point function: the propagator** $G_2^{(0)} = 1/\gamma$

Perturbation Theory. The n -point correlation function G_n in the full theory can be computed perturbatively, in the small coupling limit, $\epsilon \rightarrow 0$, and expressed in terms of $G_n^{(0)}$. For example, the determination of the next-to-leading order (NLO) corrections to the 2-point function, proceeds as follows,

$$\begin{aligned} G_2 &= \frac{\int dz z^2 e^{-S_0 - \epsilon S_1}}{\int dz e^{-S_0 - \epsilon S_1}} = \frac{\int dz z^2 e^{-S_0} (1 - \epsilon S_1 + \dots)}{\int dz e^{-S_0} (1 - \epsilon S_1 + \dots)} = \left(G_2^{(0)} - \epsilon G_6^{(0)} + \dots \right) \left(1 + \epsilon G_4^{(0)} + \dots \right) = G_2^{(0)} + \epsilon \left(G_2^{(0)} G_4^{(0)} - G_6^{(0)} \right) + \mathcal{O}(\epsilon^2) \\ &= \frac{1}{\gamma} \left(1 - 12\epsilon \frac{1}{\gamma^2} \right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

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real scalar field $\phi(x)$ $S_E \equiv S_0 + \epsilon S_1$, with $S_0 = (\gamma/2) \phi^2(x)$, and $S_1 = \phi^4(x)$

Exact theory. $\phi(x) \equiv z$ $\mu \equiv e^{-S_E}$ $\varphi = z^n dz$

$$u \equiv z^\rho \mu \quad \nu = 4,$$

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\} = \{1, 1/z, z, z^2\},$$

$$\{\hat{h}_i\}_{i=1}^4 = \{\hat{e}_i\}_{i=1}^4,$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4\gamma} \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & \frac{1}{4\gamma} & 0 \\ \frac{1}{4\gamma} & 0 & 0 & -\frac{\gamma}{16\epsilon^2} \end{pmatrix}$$

For instance, let us consider the decomposition:

$$\varphi = z^4 dz = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 \quad c_1 = \frac{1}{4\epsilon}, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = -\frac{\gamma}{4\epsilon}$$

$$\int_{\Gamma} dz z^4 e^{-S_E} = c_1 \int_{\Gamma} dz e^{-S_E} + c_4 \int_{\Gamma} dz z^2 e^{-S_E}$$

$$G_4 = c_1 + c_4 G_2$$

$$G_2 = \frac{1}{\gamma} (1 - 4\epsilon G_4)$$

ii) Kontsevich-Witten tau-function

$$Z_{KW} \equiv \frac{\int d\Phi \exp \left[-\text{Tr} \left(-\frac{i}{3!} \Phi^3 + \frac{\Lambda}{2} \Phi^2 \right) \right]}{\int d\Phi \exp \left[-\text{Tr} \left(\frac{\Lambda}{2} \Phi^2 \right) \right]}$$

• Univariate Model

Itzykson-Zuber (1992)

$$Z_{KW} = \sum_{n=0}^{\infty} Z_{KW}^{(n)} \quad \int_{\Gamma} \mu \varphi = c_1 E_1 \quad c_1 = Z_{KW}^{(n)} \quad \varphi \equiv N_n z^{6n}, \quad N_n \equiv \varepsilon^{2n} \quad \varepsilon \equiv i/(3!)(\Lambda/2)^{-3/2}$$

Type	u	v	\hat{e}_i	C-matrix	ρ_0	E_1	c_1
$Z_{KW}^{(n)}$	$z^\rho \exp(-z^2)$	2	1, 1/z	diagonal(1/2, 1/ ρ)	0	not needed	$(-2/9)^n (\Lambda^{-3n}/(2n)!) \prod_{j=0}^{3n-1} (j+1/2)$

Intersections Numbers @ Cosmology

[Piementel] + **Brunello** & P.M.

AH-B-P like integral

Brunello & P.M.

$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1 + y_1 + 1)(z_2 + y_2 + 1)(z_1 + z_2 + y_1 + y_2)}$$

curtesy **Piementel**

● Twisted Period Integrals

$$I = \int_{\mathcal{C}} u(z_1, z_2) \varphi(z_1, z_2)$$

$$u = (z_1 z_2)^\epsilon (D_1 D_2 D_3)^\gamma \quad D_1 = (z_1 + y_1 + 1), \quad D_2 = (z_2 + y_2 + 1), \quad D_3 = (z_1 + z_2 + y_1 + y_2)$$

γ is a regulator

$$\omega = d \log(u) = \omega_1 dz_1 + \omega_2 dz_2$$

$$\omega_1 = \frac{\gamma(2y_1 + y_2 + 2z_1 + z_2 + 1)}{(y_1 + z_1 + 1)(y_1 + y_2 + z_1 + z_2)} + \frac{\epsilon}{z_1} \quad \omega_2 = \frac{\gamma(y_1 + 2y_2 + z_1 + 2z_2 + 1)}{(y_2 + z_2 + 1)(y_1 + y_2 + z_1 + z_2)} + \frac{\epsilon}{z_2}$$

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● Number of MIs = dimH and bases choice

$$\omega_2 = 0$$

$$\nu_2 = 2$$

$$e^{(2)} = h^{(2)} = \left\{ \frac{1}{D_1}, \frac{1}{D_2} \right\}$$

● 2 MIs in the internal layer

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \end{cases}$$

$$\nu_{21} = 4$$

$$e^{(21)} = h^{(21)} = \left\{ \frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

● 4 MIs in the external layer

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Brunello & P.M.

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curtesy **Piementel**

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● 4 MIs in the external layer

● Intersection Matrix

$$C = \begin{pmatrix} \frac{(\gamma+\epsilon)^2}{\gamma(\gamma^2-1)\epsilon^2(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & \frac{1}{\gamma\epsilon-\gamma^2\epsilon} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{1}{\gamma^2} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{\gamma^2} \\ -\frac{1}{\gamma^2\epsilon+\gamma\epsilon} & \frac{1}{\gamma^2} & \frac{1}{\gamma^2} & \frac{3}{\gamma^2} \end{pmatrix}$$

AH-B-P like integral

$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1 + y_1 + 1)(z_2 + y_2 + 1)(z_1 + z_2 + y_1 + y_2)}$$

curtesy **Piementel**

Brunello & P.M.

• 4 MIs

$$e^{(21)} = \left\{ \frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

• System of Differential Equations

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

• Master Decomposition Formula

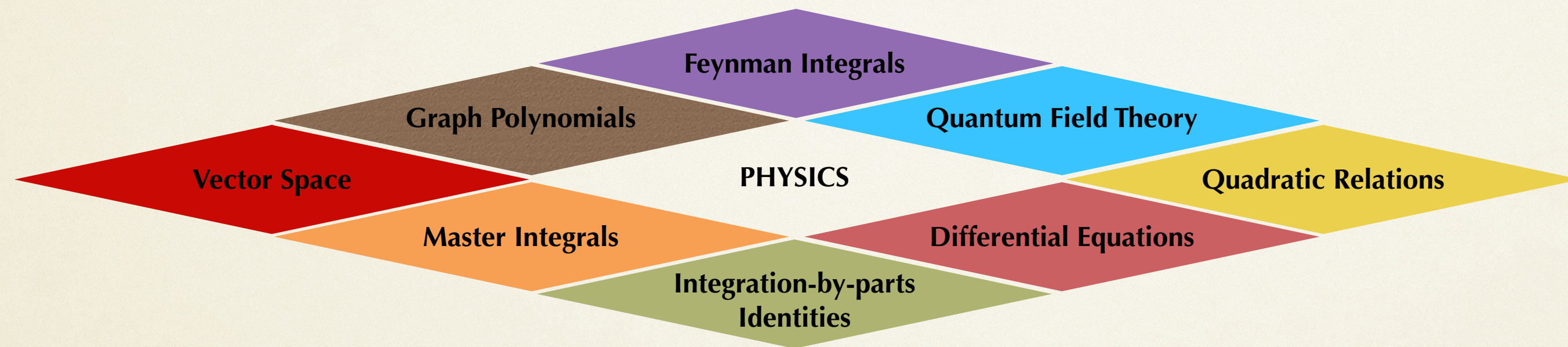
$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle (\mathbf{C}^{-1})_{kj}$$

after taking the limit $\gamma \rightarrow 0$:

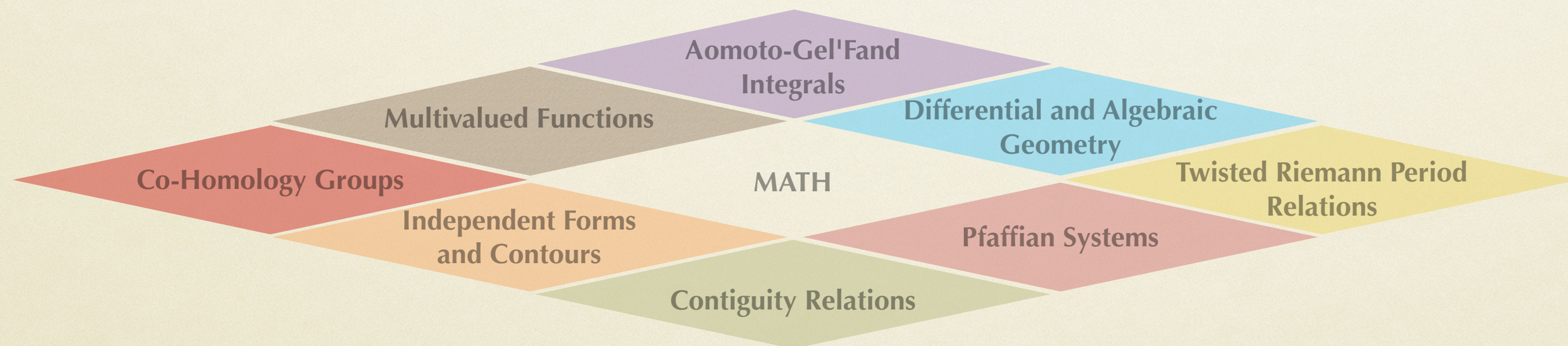
$$\Omega_{y_1} = \begin{pmatrix} \frac{2\epsilon}{y_1 + y_2 + 1} & 0 & 0 & 0 \\ -\frac{\epsilon}{y_1 + 1} & \frac{\epsilon}{y_1 + 1} & 0 & 0 \\ \frac{\epsilon}{y_1} & 0 & \frac{\epsilon}{y_1} & 0 \\ \frac{\epsilon}{y_1(y_1 + 1)} & 0 & \frac{\epsilon}{y_1(y_1 + 1)} & \frac{\epsilon}{y_1 + 1} \end{pmatrix} \quad \Omega_{y_2} = \begin{pmatrix} \frac{2\epsilon}{y_1 + y_2 + 1} & 0 & 0 & 0 \\ \frac{\epsilon}{y_2} & \frac{\epsilon}{y_2} & 0 & 0 \\ -\frac{\epsilon}{y_2 + 1} & 0 & \frac{\epsilon}{y_2 + 1} & 0 \\ \frac{\epsilon}{y_2(y_2 + 1)} & \frac{\epsilon}{y_2(y_2 + 1)} & 0 & \frac{\epsilon}{y_2 + 1} \end{pmatrix}$$

• Canonical system

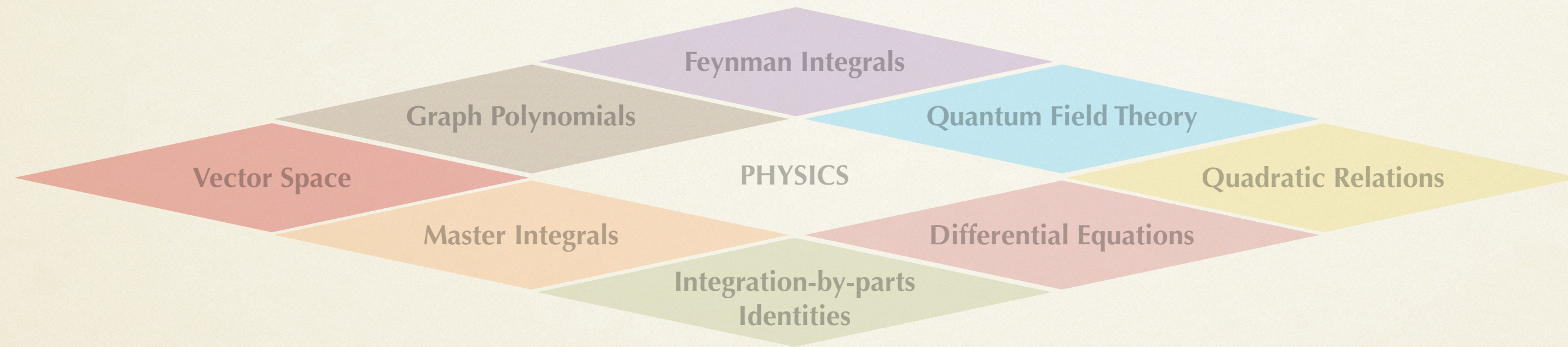
To Conclude:



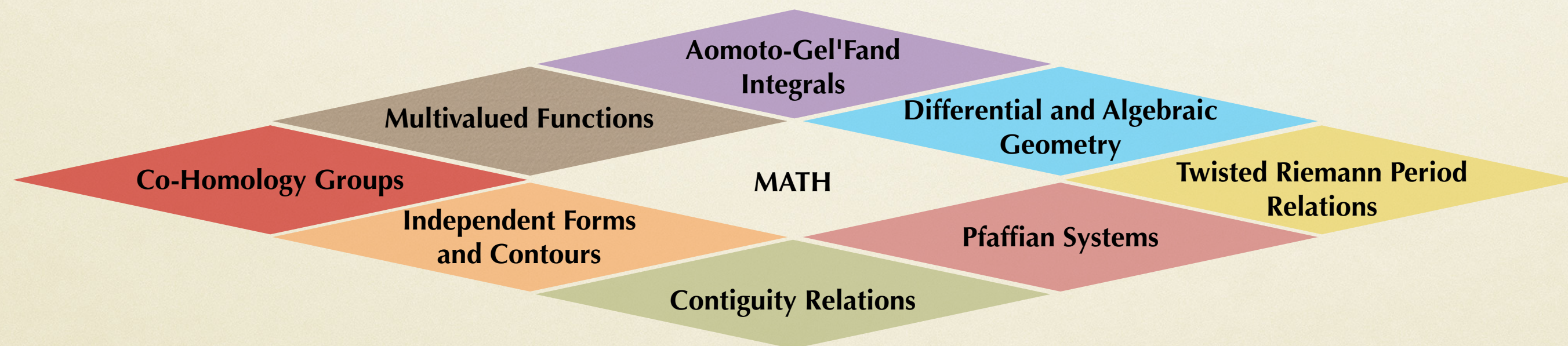
Quantum Field Theory



Twisted de Rham Theory



Quantum Field Theory



Twisted de Rham Theory

Summary

- **Novel Mathematical Structure for Quantum Field Theory Integrals came into view**

- Intersection Theory for Twisted de Rham co-homology

- Rich theory :: Differential and Algebraic Geometry, Topology, Number Theory, Combinatorics, Statistics

- **Novel Concepts: Vector Space Structures**

- Space dimensions = Dimension of co-homology group = number of independent Integrals

- Intersection Numbers ~ **Scalar Product** for Feynman (Twisted Period) Integrals

- **New Methods for Multivariate Intersection number**

- Iterative method

- Higher-Order PDE method

- Secondary equation (Pfaffians via Macaulay)

- **General algorithm for Physics and Math applications**

- key: Co-Homology Group Isomorphisms

- Feynman Integrals, Euler-Mellin Integrals, D-Module and GKZ hypergeometric theory, Orthogonal Polynomials, QM matrix elements, Correlator functions in QFT.

- **Modern Multi-Loop diagrammatic techniques and Amplitudes calculus useful beyond Particle Physics**

- Triggering interdisciplinarity

- **Emerging Picture**

- Interwintwinement between Fundamental Physics, Geometry and Statistics: fluxes ~ period integrals ~ statistical moments

- Interesting implications in QM, QFT (and Cosmology): invariance and independent moments of distributions, perturbation vs non-perturbative approaches

Definition. Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$. Different branches of physics are distinguished by the range of the variable x and by the names used for $f(x)$, $g(x)$ and for the integral. [...]

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ . We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

The unreasonable effectiveness of mathematics

E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Extra Slides

Intersection Numbers for n-forms :: nPDE

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

$$\langle \varphi_L^{(\mathbf{n})} \mid \varphi_R^{(\mathbf{n})} \rangle = (2\pi i)^{-n} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})})$$

● nPDE

$$\nabla_{\omega_1} \nabla_{\omega_2} \cdots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}$$

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

$$\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = (2\pi i)^{-n} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})})$$

• nPDE

$$\nabla_{\omega_1} \nabla_{\omega_2} \dots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}$$

Proof.

$$\eta := \bar{h}_1 \dots \bar{h}_n (u \psi) (u^{-1} \varphi_R^{(\mathbf{n})}) \quad d_{z_1} \dots d_{z_n} \eta = (u \varphi_{L,c}) \wedge (u^{-1} \varphi_R),$$

$$\bar{h}_i := 1 - h_i$$

$$h_i \equiv h(z_i) := \begin{cases} 1 & \text{for } |z_i| < \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{L,c} := \bar{h}_1 \dots \bar{h}_n \varphi_L + \dots + (-1)^n \psi dh_1 \wedge \dots \wedge dh_n \equiv \nabla_{\omega_1} \dots \nabla_{\omega_n} (\bar{h}_1 \dots \bar{h}_n \psi)$$

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

$$\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = (2\pi i)^{-n} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})})$$

• nPDE

$$\nabla_{\omega_1} \nabla_{\omega_2} \dots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}$$

Proof.

$$\eta := \bar{h}_1 \dots \bar{h}_n (u \psi) (u^{-1} \varphi_R^{(\mathbf{n})}) \quad d_{z_1} \dots d_{z_n} \eta = (u \varphi_{L,c}) \wedge (u^{-1} \varphi_R),$$

$$\bar{h}_i := 1 - h_i$$

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$$\varphi_{L,c} := \bar{h}_1 \dots \bar{h}_n \varphi_L + \dots + (-1)^n \psi dh_1 \wedge \dots \wedge dh_n \equiv \nabla_{\omega_1} \dots \nabla_{\omega_n} (\bar{h}_1 \dots \bar{h}_n \psi)$$

$$\begin{aligned} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) &= \sum_{p \in \mathbb{P}_\omega} \int_{D_p} d_{z_1} \dots d_{z_n} \eta = (-1)^n \sum_{p \in \mathbb{P}_\omega} \int_{D_p} (u \psi) dh_1 \wedge \dots \wedge dh_n \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) \\ &= \sum_{p \in \mathbb{P}_\omega} \int_{\cup_1 \wedge \dots \wedge \cup_n} \psi \varphi_R^{(\mathbf{n})} = (2\pi i)^n \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})}) \end{aligned}$$

Intersection Numbers and Pfaffian systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & **P.M.** (2022)

GKZ Hypergeometric Systems

Bernstein, Saito, Sturmfels, Takayama, Matsubara-Heo,
Agostini, Fevola, Sattelberger, Tellen,
De La Crux,...

• Euler-Mellin Integral / A-Hypergeometric function

$$f_{\Gamma}(z) = \int_{\Gamma} g(z; x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{dx}{x}, \quad \frac{dx}{x} := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

$$g(z; x) = \sum_{i=1}^N z_i x^{\alpha_i}$$

$$x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}$$

$$A = (a_1 \ \dots \ a_N) \quad (n+1) \times N \text{ matrix} \quad a_i := (1, \alpha_i)$$

$$\text{Ker}(A) = \left\{ u = (u_1, \dots, u_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N u_j a_j = \mathbf{0} \right\}$$

• GKZ system of PDEs

$$E_j f_{\Gamma}(z) = 0,$$

$$\square_u f_{\Gamma}(z) = 0,$$

$$E_j = \sum_{i=1}^N a_{j,i} z_i \frac{\partial}{\partial z_i} - \beta_j, \quad j = 1, \dots, n+1$$

$$\square_u = \prod_{u_i > 0} \left(\frac{\partial}{\partial z_i} \right)^{u_i} - \prod_{u_i < 0} \left(\frac{\partial}{\partial z_i} \right)^{-u_i}, \quad \forall u \in \text{Ker}(A).$$

GKZ D-Module and De Rham Cohomology group

E_j \square_u can be regarded as elements of a Weyl algebra

$$\mathcal{D}_N = \mathbb{C}[z_1, \dots, z_N] \langle \partial_1, \dots, \partial_N \rangle \quad , \quad [\partial_i, \partial_j] = 0 \quad , \quad [\partial_i, z_j] = \delta_{ij}$$

GKZ system as the left \mathcal{D}_N -module $\mathcal{D}_N/H_A(\beta)$

$$H_A(\beta) = \sum_{j=1}^{n+1} \mathcal{D}_N \cdot E_j + \sum_{u \in \text{Ker}(A)} \mathcal{D}_N \cdot \square_u$$

• **Standard Monomials** $\text{Std} := \{\partial^k\}$ found by Groebner basis Hibi, Nishiyama, Takayama (2017)

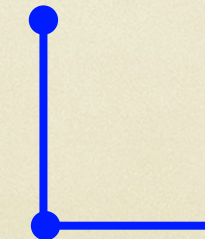
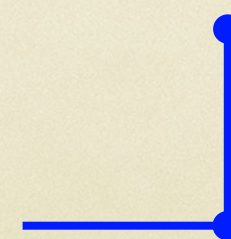
The holonomic rank equals the number of independent solutions to the system of PDEs

$$r = n! \cdot \text{vol}(\Delta_A)$$

• **Isomorphism**

$$\mathcal{D}_N/H_A(\beta) \simeq \mathbb{H}^n$$

GKZ D-module



nth-Cohomology group

Generalised Feynman Integrals

$$I(d_0, \nu; z) := c(d_0, \nu) f_\Gamma(\beta)$$

$$\beta = (\epsilon, -\epsilon\delta, \dots, -\epsilon\delta) - (d_0/2, \nu_1, \dots, \nu_n)$$

Let $0 < \epsilon, \delta \ll 1$, $d_0 \in 2 \cdot \mathbb{N}$, $L \in \mathbb{N}$ and $\nu := (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$

$$f_\Gamma(\beta) := \int_\Gamma \mathcal{G}(z; x)^{\epsilon - d_0/2} x_1^{\nu_1 + \epsilon\delta} \dots x_n^{\nu_n + \epsilon\delta} \frac{dx}{x},$$

$$c(d_0, \nu) := \frac{\Gamma(d_0/2 - \epsilon)}{\Gamma((L + 1)(d_0/2 - \epsilon) - |\nu| - n\epsilon\delta) \prod_{i=1}^n \Gamma(\nu_i + \epsilon\delta)}, \quad |\nu| := \nu_1 + \dots + \nu_n$$

Generalised Feynman Integrals

$$I(d_0, \nu; z) := c(d_0, \nu) f_\Gamma(\beta)$$

$$\beta = (\epsilon, -\epsilon\delta, \dots, -\epsilon\delta) - (d_0/2, \nu_1, \dots, \nu_n)$$


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Pfaffian Systems: for **Master Integrals** (alias **Master forms**)

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

 Basis of the Cohomology group

$$\Omega = \Omega(d, x) \quad \bullet \text{ Pfaffian Matrix}$$

 Integral decomposition (IBP/InterX)

Generalised Feynman Integrals

$$I(d_0, \nu; z) := c(d_0, \nu) f_\Gamma(\beta)$$

$$\beta = (\epsilon, -\epsilon\delta, \dots, -\epsilon\delta) - (d_0/2, \nu_1, \dots, \nu_n)$$

Let $0 < \epsilon, \delta \ll 1$, $d_0 \in 2 \cdot \mathbb{N}$, $L \in \mathbb{N}$ and $\nu := (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$

$$f_\Gamma(\beta) := \int_\Gamma \mathcal{G}(z; x)^{\epsilon - d_0/2} x_1^{\nu_1 + \epsilon\delta} \dots x_n^{\nu_n + \epsilon\delta} \frac{dx}{x},$$

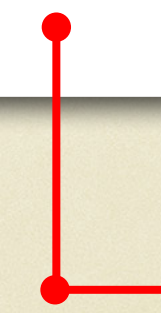
$$c(d_0, \nu) := \frac{\Gamma(d_0/2 - \epsilon)}{\Gamma((L+1)(d_0/2 - \epsilon) - |\nu| - n\epsilon\delta) \prod_{i=1}^n \Gamma(\nu_i + \epsilon\delta)}, \quad |\nu| := \nu_1 + \dots + \nu_n$$

Pfaffian Systems: for **Master Integrals** (alias **Master forms**) & for **D-operators** (alias **Std mon's**)

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

 **Basis of the D-Operators**

$$\Omega = \Omega(d, x) \quad \bullet \text{ Pfaffian Matrix}$$

 **Macaulay Matrix method**

Master Decomposition Formula & Pfaffian

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for \mathbb{H}^{n^\vee}

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

$$\langle \varphi | = \sum_{\lambda=1}^r c_\lambda \langle e_\lambda |, \quad c_\lambda = \sum_{\mu=1}^r \langle \varphi | h_\mu \rangle (C^{-1})_{\mu\lambda} \quad C_{\lambda\mu} := \langle e_\lambda | h_\mu \rangle$$

$$\begin{cases} \partial_{z_i} \langle e_\lambda | = (P_i)_{\lambda\nu} \langle e_\nu | \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle (P_i^\vee)_{\xi\mu} \end{cases} \implies \partial_{z_i} C = P_i \cdot C + C \cdot (P_i^\vee)^\text{T}$$

● Secondary Equation 1

Master Decomposition Formula & Pfaffian

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

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● Secondary Equation 1

auxiliary basis $e^{\text{aux}} := \{e_1, \dots, e_{r-1}, \varphi\}$

$$\begin{cases} \partial_{z_i} \langle e_\lambda^{\text{aux}} | = (P_i^{\text{aux}})_{\lambda\nu} \langle e_\nu^{\text{aux}} | \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle (P_i^\vee)_{\xi\mu} \end{cases} \implies \partial_{z_i} C^{\text{aux}} = P_i^{\text{aux}} \cdot C^{\text{aux}} + C^{\text{aux}} \cdot (P_i^\vee)^\text{T}$$

● Secondary Equation 2

$C_{\lambda\mu}^{\text{aux}} := \langle e_\lambda^{\text{aux}} | h_\mu \rangle$

Master Decomposition Formula & Pfaffian

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

$$\langle \varphi | = \sum_{\lambda=1}^r c_\lambda \langle e_\lambda |, \quad c_\lambda = \sum_{\mu=1}^r \langle \varphi | h_\mu \rangle (C^{-1})_{\mu\lambda} \quad C_{\lambda\mu} := \langle e_\lambda | h_\mu \rangle$$

$$\begin{cases} \partial_{z_i} \langle e_\lambda | = (P_i)_{\lambda\nu} \langle e_\nu | \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle (P_i^\vee)_{\xi\mu} \end{cases} \implies \partial_{z_i} C = P_i \cdot C + C \cdot (P_i^\vee)^\text{T}$$

● Secondary Equation 1

auxiliary basis $e^{\text{aux}} := \{e_1, \dots, e_{r-1}, \varphi\}$

$$\begin{cases} \partial_{z_i} \langle e_\lambda^{\text{aux}} | = (P_i^{\text{aux}})_{\lambda\nu} \langle e_\nu^{\text{aux}} | \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle (P_i^\vee)_{\xi\mu} \end{cases} \implies \partial_{z_i} C^{\text{aux}} = P_i^{\text{aux}} \cdot C^{\text{aux}} + C^{\text{aux}} \cdot (P_i^\vee)^\text{T}$$

● Secondary Equation 2

$$C_{\lambda\mu}^{\text{aux}} := \langle e_\lambda^{\text{aux}} | h_\mu \rangle$$

Rational Solutions of PDE
[integrable connections]

Barkatou et al. @ MAPLE

Direct determination of Intersection Matrices

Multivariate Intersection Numbers (III) from Pfaffians

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

$$\langle \varphi | = \sum_{\lambda=1}^r c_{\lambda} \langle e_{\lambda} |,$$

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{bmatrix} = C^{\text{aux}} \cdot C^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{bmatrix} \implies C^{\text{aux}} \cdot C^{-1} = \left[\begin{array}{c|c} \text{id}_{r-1} & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array} \right]$$

Coefficients from matrix multiplication

Quadratic Relations

Twisted Riemann Periods Relations (TRPR)

- Completeness for forms

$$\sum_{i,j=1}^{\nu} |e_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i| = \mathbb{I}_c \quad \mathbf{C}_{ij} \equiv \langle e_i|e_j\rangle$$

- Completeness for contours

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_j] (\mathbf{H}^{-1})_{ji} [\mathcal{C}_i| = \mathbb{I}_h \quad \mathbf{H}_{ij} \equiv [\mathcal{C}_i|\mathcal{C}_j]$$

- Riemann Twisted Period Relations Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{i,j} \langle \varphi_L | \mathcal{C}_{R,j}] [\mathcal{C}_{L,j} | \mathcal{C}_{R,i}]^{-1} [\mathcal{C}_{L,i} | \varphi_R \rangle$$

$$[\mathcal{C}_L | \mathcal{C}_R] = \sum_{i,j} [\mathcal{C}_L | \varphi_{R,j} \rangle \langle \varphi_{L,j} | \varphi_{R,i} \rangle^{-1} \langle \varphi_L | \mathcal{C}_R]$$

TRPR for Gauss Hypergeometric Function

Cho, Matsumoto (1995)

$$u = t^\alpha (1-t)^{\gamma-\alpha} (1-xt)^{-\beta}, \quad \varphi_1 = \left(\frac{dt}{t-x_1} - \frac{dt}{t-x_2} \right) = \frac{dt}{t(1-t)}, \quad \varphi_3 = \left(\frac{dt}{t-x_3} - \frac{dt}{t-x_4} \right) = \frac{-xdt}{1-xt},$$

$$P^+ = \begin{pmatrix} \int_0^1 u \varphi_1 & \int_{1/x}^\infty u \varphi_1 \\ \int_0^1 u \varphi_3 & \int_{1/x}^\infty u \varphi_3 \end{pmatrix}, \quad P^- = \begin{pmatrix} \int_0^1 u^{-1} \varphi_1 & \int_{1/x}^\infty u^{-1} \varphi_1 \\ \int_0^1 u^{-1} \varphi_3 & \int_{1/x}^\infty u^{-1} \varphi_3 \end{pmatrix}, \quad I_{ch} = 2\pi i \begin{pmatrix} 1/\alpha + 1/(\gamma-\alpha) & 0 \\ 0 & -1/\beta + 1/(\beta-\gamma) \end{pmatrix}, \quad I_h = - \begin{pmatrix} d_{12}/d_1 d_2 & 0 \\ 0 & d_{30}/d_3 d_0 \end{pmatrix},$$

$c_{jk\dots} = c_j c_k \dots, d_{jk\dots} = c_j c_k \dots - 1$
 $c_j = \exp 2\pi i \alpha_j$

$$\int_0^1 u \varphi_1 = B(\alpha, \gamma-\alpha) F(\alpha, \beta, \gamma; x),$$

$$\int_{1/x}^\infty u \varphi_1 = -(-1)^{\gamma-\alpha-\beta} x^{1-\gamma} B(\beta-\gamma+1, -\beta+1) \times F(\beta-\gamma+1, \alpha-\gamma+1, 2-\gamma; x),$$

● Riemann Twisted Period Relations

$$P^+ {}^t I_h^{-1} {}^t P^- = I_{ch}$$

(1,2)- component $F(\alpha, \beta, \gamma; x) F(1-\alpha, 1-\beta, 2-\gamma; x) = F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x) F(\gamma-\alpha, \gamma-\beta, \gamma; x)$

(1,1)-component $F(\alpha, \beta, \gamma; x) F(-\alpha, -\beta, -\gamma; x) - 1 = \frac{\alpha\beta(\gamma-\alpha)(\gamma-\beta)}{\gamma^2(\gamma+1)(\gamma-1)} F(\beta-\gamma+1, \alpha-\gamma+1, -\gamma+2; x) \times F(\gamma-\beta+1, \gamma-\alpha+1, \gamma+2; x).$

Elliot's Identity from Intersections Matsumoto & P.M.

The complete elliptic integrals \mathcal{K} and \mathcal{E} of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}} \quad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \phi} d\phi$$

● Legendre Identity

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}$$

$$\mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r') \\ r^2 + r'^2 = 1$$

Elliot's Identity from Intersections Matsumoto & P.M.

The complete elliptic integrals \mathcal{K} and \mathcal{E} of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}} \quad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \phi} d\phi$$

● Legendre Identity

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2} \quad \mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r')$$

$$r^2 + r'^2 = 1$$

● Elliot's Identity and Hypergeometric Functions

Balasubramanian, Naik, Ponnusamy, Vuorinen (2001)

$$\begin{aligned} & F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & + F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & - F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & = \frac{\Gamma(1 + \lambda + \mu)\Gamma(1 + \mu + \nu)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})\Gamma(\mu + \frac{1}{2})}. \end{aligned}$$

the choice $\lambda = \mu = \nu = 0$ gives the Legendre relation.

Elliot's Identity from Intersections Matsumoto & P.M.

● *Hypothesis: too close to RTPR to be accidental*

● *Proof*

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \left(\frac{1}{t} - \frac{1}{t-1/r}\right)dt,$$

$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \left(\frac{1}{t} - \frac{1}{t-1}\right)dt.$$

$$\gamma = (0, 1) \otimes u(t) \text{ and } \delta = (-\infty, 0) \otimes 1/u(t)$$

● *Twisted Riemann Period Relation*

$${}^t\Pi_\omega {}^tH_c^{-1}\Pi_{-\omega} = H_h.$$

$$\left(\int_0^1 u(t)\varphi_1, \int_0^1 u(t)\varphi_2\right) {}^tH_c^{-1} \begin{pmatrix} \int_{-\infty}^0 \frac{1}{u(t)}\psi_1 \\ \int_{-\infty}^0 \frac{1}{u(t)}\psi_2 \end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda} + 1}.$$

$$\left(F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu, 1 + \lambda + \mu; r\right), F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right)\right) \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \end{pmatrix} = \frac{\Gamma(\lambda + \mu + 1)\Gamma(\mu + \nu + 1)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})\Gamma(\mu + \frac{1}{2})}$$

Elliot's Identity from Intersections

Matsumoto & P.M.

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- Quadratic relations for Feynman Integrals

Broadhurst, Roberts (2018)

Lee, Pomeranski (2019)

$$\mathbf{P}_k^{\text{BR}} \cdot \mathbf{D}_k^{\text{BR}} \cdot {}^t\mathbf{P}_k^{\text{BR}} = \mathbf{B}_k^{\text{BR}}$$

Fresan, Sabbah, Yu (2020)

- String-Theory Amplitudes: **KLT relations = TRPR**

Mizera (2016/17)

$$\mathcal{A}^{\text{GR}} = \sum_{\beta, \gamma} \mathcal{A}^{\text{YM}}(\beta) m^{-1}(\beta|\gamma) \mathcal{A}^{\text{YM}}(\gamma)$$

$$\mathcal{A}^{\text{closed}} = \sum_{\beta, \gamma} \mathcal{A}^{\text{open}}(\beta) m_{\alpha'}^{-1}(\beta|\gamma) \mathcal{A}^{\text{open}}(\gamma)$$