# World-sheet Duality for Superspace $\sigma$-Models

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Opening Conference: "New Perspectives in String Theory"

Galileo Galilei Institute, Florence

Based on arXiv:0809.1046 (with V. Mitev and V. Schomerus)

and work in progress (with Candu, Mitev, Saleur and Schomerus)

SIXTH FRAMEWORK PROGRAMME

[This research receives funding from an Intra-European Marie-Curie Fellowship]

# Flux backgrounds

**Crucial problem: Quantization of strings in flux backgrounds**

- AdS/CFT correspondence [Maldacena] [...]

  String theory on AdS-space $\Leftrightarrow$ Gauge theory on boundary

- String phenomenology [Kachru,Kallosh,Linde,Trivedi] [...]

  Moduli stabilization through fluxes

# The pure spinor formalism

## Ingredients

- Superspace $\sigma$-model encoding geometry and fluxes
- Pure spinors: $\lambda \in \mathrm{SO}(10)/\mathrm{U}(5)$
- BRST procedure

[Berkovits at al] [Grassi et al] [...]

## Features

- Manifest target space supersymmetry
- Manifest world sheet conformal symmetry
- Action quantizable, but quantization hard in practice

# Strings on $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$

## Spectrum accessible because of integrability

- Factorizable S-matrix
- Structure fixed (up to a phase) by $SU(2|2) \ltimes \mathbb{R}^3$-symmetry
- Bethe ansatz, Y-systems, ...

## Open issues

- String scattering amplitudes?
- 2D Lorentz invariant formulation?
- Other backgrounds? $\rightarrow$ Conifold, nil-manifolds, ...

# The standard perspective on AdS/CFT

## Overview

| Gauge theory | | String theory |
|---|---|---|
| $\mathcal{N} = 4$ Super Yang-Mills | | $AdS_5 \times S^5$ |
| $\mathcal{N} = 6$ Chern-Simons | | $AdS_4 \times \mathbb{CP}^3$ |
| S-matrix, spectrum,... | $\Longleftrightarrow$ | S-matrix, spectrum,... |
| t'Hooft coupling $\lambda$, ... | | Radius $R$, ... |

## Problem

From this perspective, both sides need to be solved separately.

# An alternative perspective on AdS/CFT

**Proposal:** Two step procedure...

**Weakly coupled gauge theory**

↕ Feynman diagram expansion

**Weakly coupled 2D theory** (Topological $\sigma$-model)

↕ "Well-established machinery"

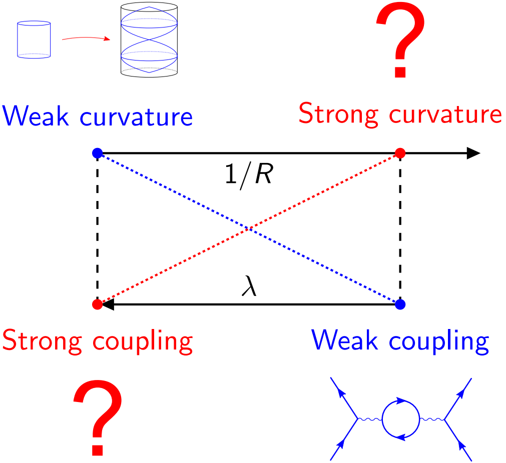
**Strongly curved $\sigma$-model**

[Berkovits] [Berkovits,Vafa] [Berkovits]

# Summary: String theory/gauge theory dualities

**String theory in 10D**
**($\sigma$-model with constraints)**

**Gauge theory**

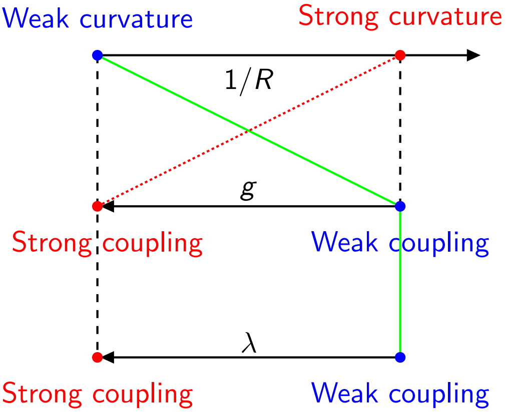


# Summary: String theory/gauge theory dualities

**String theory in 10D**
**($\sigma$-model with constraints)**

**"Some dual 2D theory"**

**Gauge theory**



# The structure of this talk

**Prospect...**

1. Introduce a class of 2D $\sigma$-models which share many conceptual features with $\sigma$-models on AdS-spaces
2. Pick one example and show how exact spectra can be derived explicitly using a duality to a non-geometric CFT

# An interesting observation

## String backgrounds as supercosets...

| Minkowski | $\mathrm{AdS}_5 \times S^5$ | $\mathrm{AdS}_4 \times \mathbb{CP}^3$ | $\mathrm{AdS}_2 \times S^2$ |
|---|---|---|---|
| $\frac{\text{super-Poincaré}}{\text{Lorentz}}$ | $\frac{\mathrm{PSU}(2,2\|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$ | $\frac{\mathrm{OSP}(6\|2,2)}{\mathrm{U}(3)\times\mathrm{SO}(1,3)}$ | $\frac{\mathrm{PSU}(1,1\|2)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ |

[Metsaev, Tseytlin] [Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach] [Arutyunov, Frolov]

## Definition of the cosets

$$G/H \;=\; \left\{ g \in G \,\middle|\, gh \sim g,\, h \in H \right\}$$

Geometric realization of supersymmetry: $\quad g \mapsto hg$

# Generalized symmetric spaces

Let $G$ be a Lie (super)group, $\Omega : G \to G$ an automorphism of finite order, $\Omega^L = \mathrm{id}$. Let $H = \mathrm{Inv}_\Omega(G) = \{h \in G | \Omega(h) = h\}$ be the invariant subgroup. Then the coset $G/H$ is called a **generalized symmetric space**.

**Theorem**

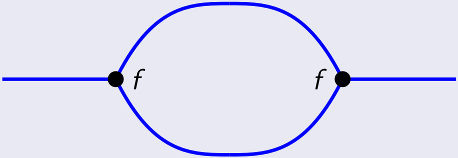
*If $G$ has vanishing Killing form then the coset $G/H$ is classically integrable* and *quantum conformally invariant, at least to the lowest non-trivial order in perturbation theory.* [Young] [Kagan, Young]

**Examples:** Cosets of $\mathrm{PSU}(N|N)$, $\mathrm{OSP}(2S+2|2S)$, $\mathrm{D}(2,1;\alpha)$.

# Two compact examples

## Compact symmetric spaces with vanishing Killing form

| Superspheres $S^{2S+1|2S}$ | Projective superspaces $\mathbb{CP}^{S-1|S}$ |
|---|---|
| $\frac{\mathrm{OSP}(2S+2|2S)}{\mathrm{OSP}(2S+1|2S)}$ | $\frac{\mathrm{U}(S|S)}{\mathrm{U}(1)\times\mathrm{U}(S-1|S)}$ |




$= 0$

**Remark:** One can write $\mathbb{CP}^{S-1|S} = S^{2S-1|2S}/U(1)$.

# Relation to AdS $\sigma$-models

## Similarities

- Family of CFTs with continuously varying exponents
- Completely new type of 2D conformal field theory

**Standard methods do not apply!**

- Target space supersymmetry
- Symmetric superspaces
- Integrability

## Differences

- Compactness
- No string constraints imposed

# Superspheres

## Realization of $S^{M|2N}$ as a submanifold of flat superspace $\mathbb{R}^{M+1|2N}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \vec{\eta}_1 \\ \vec{\eta}_2 \end{pmatrix} \qquad \text{with} \qquad \vec{X}^2 = \vec{x}^2 + 2\vec{\eta}_1\vec{\eta}_2 = R^2$$

## Realization as a symmetric space

$$S^{M|2N} = \frac{\mathrm{OSP}(M+1|2N)}{\mathrm{OSP}(M|2N)}$$

# Superspheres: Conformal invariance

## Analogy to O($K$) models

$$S^{M|2N} \quad \longleftrightarrow \quad S^{M-2N}$$

- Similarity to $\mathrm{O}(K) = \mathrm{O}(M+1-2N)$ $\sigma$-models
- There is no topological Wess-Zumino term
- Self-duality $\mathrm{O}(K) \Leftrightarrow \mathrm{O}(4-K)$

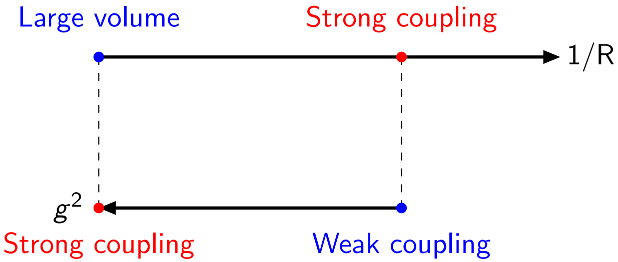
**In this talk:** Focus on $S^{3|2} = \frac{\mathrm{OSP}(4|2)}{\mathrm{OSP}(3|2)}$

**Question:** How can this theory be quantized?

[Read,Saleur] [Mann,Polchinski] [Candu,Saleur] [Mitev,TQ,Schomerus]

# A world-sheet duality for superspheres?

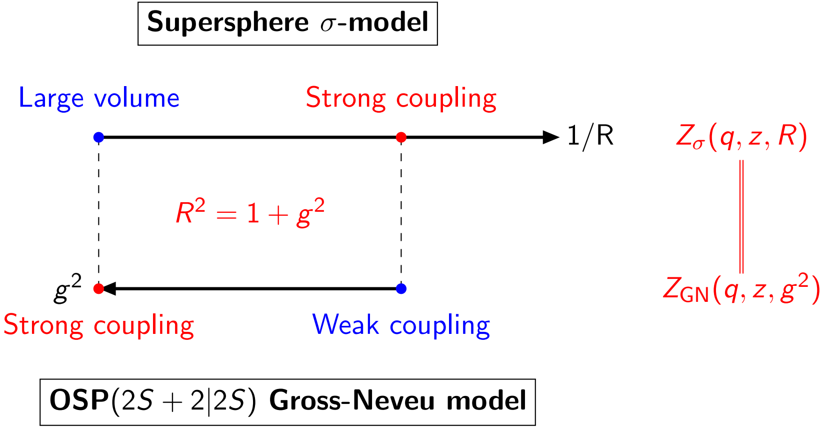
**Supersphere $\sigma$-model**




**OSP**$(2S+2|2S)$ **Gross-Neveu model**

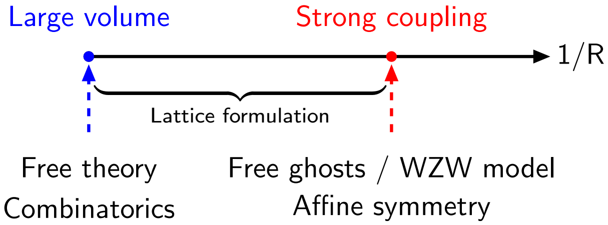
[Candu,Saleur]$^2$ [Mitev,TQ,Schomerus]

# A world-sheet duality for superspheres?



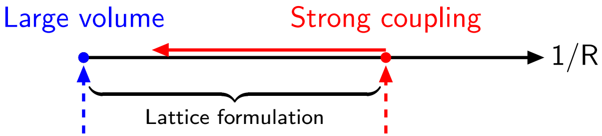

[Candu,Saleur]$^2$ [Mitev,TQ,Schomerus]

# Evidence for the duality




[Candu,Saleur]$^2$ [Mitev,TQ,Schomerus]

# Evidence for the duality




Free theory
Combinatorics

Free ghosts / WZW model
Affine symmetry

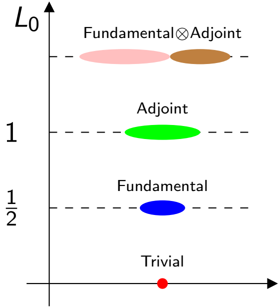
Certain partition functions can be determined **exactly for all R**!

$$Z_\sigma(q,z,R) \;=\; \sum_\Lambda \psi^\sigma_\Lambda(q,R)\,\chi_\Lambda(z)$$

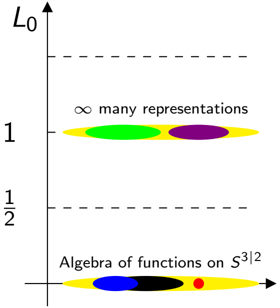
[Mitev,TQ,Schomerus]

# Interpolation of the spectrum

We will show that the following two spectra are continuously connected by a marginal deformation:




**WZW model ($R = 1$)**




**$\sigma$-model at $R \to \infty$**

# Outline

## Outline of the talk

1. Definition of the OSP(4|2) Gross-Neveu model
    - Construction of a brane spectrum at $g^2 = 0$
    - Perturbation theory (to all orders)

    $\Rightarrow$ Full spectrum of anomalous dimensions

2. Definition of the supersphere $\sigma$-model
    - Combinatorial construction of a brane spectrum at $R \to \infty$

    $\Rightarrow$ Full agreement

# The strong coupling limit: OSP(4|2) Gross-Neveu model

## Field content

Four fermions $\psi^i$ Two ghosts $\beta, \gamma$

- All fields have conformal weight $h = 1/2$
- The fields form the fundamental OSP(4|2)-multiplet $V$

# The strong coupling limit: OSP(4|2) Gross-Neveu model

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## Quartic interactions

$$\mathcal{S}_{\mathrm{GN}} = \mathcal{S}_{\mathrm{free}} + g^2\,\mathcal{S}_{\mathrm{int}} \qquad \begin{cases} \mathcal{S}_{\mathrm{free}} = \int \left[\psi\bar{\partial}\psi + 2\beta\bar{\partial}\gamma + h.c.\right] \\ \mathcal{S}_{\mathrm{int}} = \int \left[\psi\bar{\psi} + \beta\bar{\gamma} - \gamma\bar{\beta}\right]^2 \end{cases}$$

# Reformulation as a WZW model

## The OSP(4|2) Gross-Neveu model has a convenient reformulation

- At $g = 0$ there is an affine $\widehat{\text{OSP}}(4|2)_{-1/2}$ symmetry
- There is a "bosonic" realization as an orbifold

$$\widehat{\text{OSP}}(4|2)_{-1/2} \cong \left[\widehat{SU}(2)_{-1/2} \times \widehat{SU}(2)_1 \times \widehat{SU}(2)_1\right]/\mathbb{Z}_2$$

- The interaction is of current-current type

$$\mathcal{S}_{\text{int}} \sim \int J^a \bar{J}_a$$

- Vanishing Killing form ⇒ exact marginality

[Berkovits,Vafa,Witten] [Götz,TQ,Schomerus] [TQ,Schomerus,Creutzig]

# A D-brane spectrum

## A specific D-brane in the OSP(4|2) WZW model...

- Use outer automorphism to define twisted gluing conditions
- The associated spectrum is

$$Z_{\mathrm{GN}}(g^2 = 0) \;=\; \underbrace{\chi_{\{0\}}(q,z)}_{\text{vacuum}} + \underbrace{\chi_{\{1/2\}}(q,z)}_{\text{fundamental}}$$

## The problem

Organize this into representations of OSP(4|2)!

# Decomposition into representations of OSP(4|2)

Plugging in concrete expressions, one obtains

$$Z_{\mathrm{GN}}(g^2=0) = \frac{\eta(q)}{\theta_4(z_1)}\left[\frac{\theta_2(q^2,z_2^2)\theta_2(q^2,z_3^2)}{\eta(q)^2}+\frac{\theta_3(q^2,z_2^2)\theta_3(q^2,z_3^2)}{\eta(q)^2}\right]$$

$$= \sum \psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q)\,\chi_{[j_1,j_2,j_3]}(z)$$

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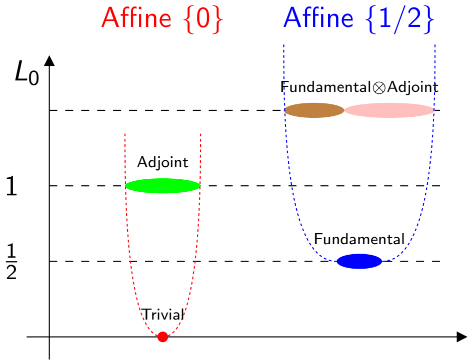
$$= \sum \psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q)\,\chi_{[j_1,j_2,j_3]}(z)$$

$$\psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q) \;=\; \frac{1}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{n+m}q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}-\frac{1}{8}}$$

$$\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)$$

# What did we achieve so far?




**WZW model ($R = 1$)**

# Deformation of the spectrum I

## Quasi-abelian perturbation theory

- Vanishing Killing form $\Rightarrow$ the perturbation is abelian (for the purposes of calculating anomalous dimensions)

[Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig]

- An OSP(4|2) representation $\Lambda$ is shifted according to

$$\delta h_\Lambda(g^2) = -\frac{1}{2}\frac{g^2 C_\Lambda}{1+g^2} = -\frac{1}{2}\left(1-\frac{1}{R^2}\right)C_\Lambda$$

- Geometric series in $g^2$
- Deformation only depends on quadratic Casimir $C_\Lambda$

# Deformation of the spectrum II

## The full spectrum of anomalous dimensions

- As a consequence one has

$$\psi^{\sigma}_{\Lambda}(q,R) \;:=\; q^{-\frac{1}{2}\left(1-\frac{1}{R^2}\right)C_\Lambda}\,\psi^{\mathrm{WZW}}_{\Lambda}(q)$$

- For $R \to \infty$ one obtains

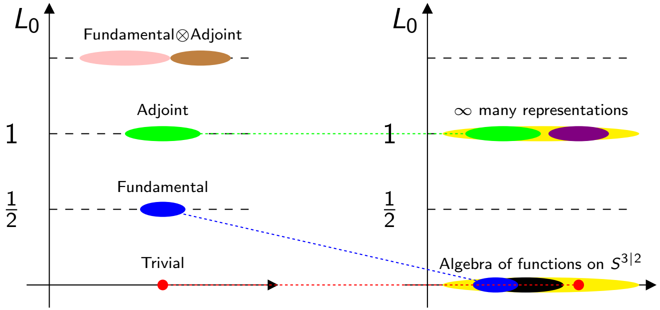
$$\psi^{\sigma}_{\Lambda}(q,R_\infty) \;=\; q^{-C_\Lambda/2}\,\psi^{\mathrm{WZW}}_{\Lambda}(q)$$

- This should be reproduced by the semi-classical analysis

# Interpolation of the spectrum

A sketch of our current achievements...

**Prediction!**




**WZW model ($R = 1$)**

**$\sigma$-model at $R \to \infty$**

# The supersphere $\sigma$-model

## Action functional

$$\mathcal{S}_\sigma = \int \partial\vec{X} \cdot \bar{\partial}\vec{X} \qquad \text{with} \qquad \vec{X}^2 = R^2$$

## Properties of this $\sigma$-model

- There is no topological term
- Conformal invariance for each value of $R$
- Central charge: $c = 1$
- Non-unitarity

[Read,Saleur] [Polchinski,Mann] [Candu,Saleur]$^2$ [Mitev,TQ,Schomerus]

# The large volume limit

## For $R \to \infty$ one has a **free field theory**...

- Coordinates $\vec{X} \to$ fields $\vec{X}(z)$
- Partition function is pure combinatorics
- Symmetry

$$\mathrm{OSP}(4|2) \;\to\; \mathrm{SP}(2) \times \mathrm{SO}(4) \;\cong\; \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$$

- Classify states according to the bosonic symmetry:

$$\vec{X} = (\vec{x}, \eta_1, \eta_2): \qquad V \;=\; \underbrace{\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)}_{\text{bosons}} \oplus \underbrace{\left(\tfrac{1}{2}, 0, 0\right)}_{\text{fermions}}$$

# The large volume partition function

## State of states (on a space-filling brane)

$$\prod X^{a_i} \prod \partial X^{b_j} \prod \partial^2 X^{c_k} \cdots \qquad \text{and} \qquad \vec{X}^2 = R^2$$

$\Rightarrow$ Products of coordinate fields and their derivatives.

## How to count? $\longrightarrow$ Want to keep representation content!

- The state space is built from (symmetrized) tensor products

$$\big[\underbrace{V \otimes \cdots \otimes V}_{\sum a_i \text{ factors}}\big]_{\text{sym}} \otimes \big[\underbrace{V \otimes \cdots \otimes V}_{\sum b_i \text{ factors}}\big]_{\text{sym}} \otimes \cdots$$

- This can be encoded in characters of $OSP(4|2)$

# Constituents of the partition function

## A useful dictionary

| **Field theoretic quantity** | **Contribution** | Representation |
|---|---|---|
| 2 Fermionic coordinates | $t\, z_1^{\pm 1}$ | $\frac{1}{2}$ |
| 4 Bosonic coordinates | $t\, z_2^{\pm 1} z_3^{\pm 1}\,,\ t\, z_2^{\pm 1} z_3^{\mp 1}$ | $\left(\frac{1}{2}, \frac{1}{2}\right)$ |
| Derivative $\partial$ | $q$ | |
| Constraint $\vec{X}^2 = R^2$ | $1 - t^2$ | |
| Constraint $\partial^n \vec{X}^2 = 0$ | $1 - t^2 q^n$ | |

# The full $\sigma$-model partition function

## Summing up all contributions...

$$Z_\sigma(R_\infty) = \lim_{t\to 1}\Bigg[q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1-t^2q^n)\times \\ \times\prod_{n=0}^{\infty}\frac{(1+z_1tq^n)(1+z_1^{-1}tq^n)}{(1-z_2z_3tq^n)(1-z_2z_3^{-1}tq^n)(1-z_2^{-1}z_3tq^n)(1-z_2^{-1}z_3^{-1}tq^n)}\Bigg]$$

## The problem (yet again...)

Organize this into representations of $OSP(4|2)$!

# Decomposition into representations of OSP(4|2)

Since the model is symmetric under OSP(4|2) the partition function may be decomposed into characters of OSP(4|2):

$$Z_\sigma(R_\infty) = \sum_{[j_1,j_2,j_3]} \psi^\sigma_{[j_1,j_2,j_3]}(q)\, \chi_{[j_1,j_2,j_3]}(z)$$

All the non-trivial information is encoded in

$$\psi^\sigma_{[j_1,j_2,j_3]}(q) = \frac{q^{-C_{[j_1,j_2,j_3]}/2}}{\eta(q)^4} \sum_{n,m=0}^{\infty} (-1)^{m+n} q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1-\frac{1}{8}} \times \left(q^{(j_2-\frac{n}{2})^2} - q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2} - q^{(j_3+\frac{n}{2}+1)^2}\right)$$

# Conclusions

## Conclusions

- Using supersymmetry we provided strong evidence for a duality between supersphere $\sigma$-models and Gross-Neveu models
- We determined the full spectrum of anomalous dimensions on a space-filling brane as a function of the radius

World sheet methods appear to be more powerful than expected!

# Open issues and outlook

## Several open issues remain...

- More points with enhanced symmetry?
- Deformation of the bulk spectrum
- Interplay with integrability (S-matrix approach)
- Correlation functions
- Path integral derivation?

## Outlook

- Projective superspaces $\mathbb{CP}^{S-1|S}$ $\rightarrow$ role of $\theta$-term...?
- Other spaces: AdS-spaces, confifold, nil-manifolds, ...