Universal BPS Structure of stationary supergravity solutions

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Stationary solutions and timelike dimensional reduction

The search for supergravity solutions with assumed Killing symmetries can be recast as a Kaluza-Klein problem. Consider a $D = 4$ theory with a nonlinear bosonic symmetry $\bar{G}$ (*e.g.* E_7 for maximal $N = 8$ supergravity). Scalar fields take their values in a target space $\bar{\Phi} = \bar{G}/\bar{H}$, where $\bar{H}$ is the corresponding linearly realized subgroup, generally the maximal compact subgroup of $\bar{G}$ (*e.g.* $\mathrm{SU}(8)$ for $N = 8$ SG).

Searching for stationary solutions to such a theory amounts to assuming further that a solution possesses a timelike Killing vector field $\kappa_\mu(x)$.

- We assume that the solution spacetime is asymptotically flat or asymptotically Taub-NUT and that there is a 'radial' function r which is divergent in the asymptotic region, $g^{\mu\nu}\partial_\mu r \partial_\nu r \sim 1 + \mathcal{O}(r^{-1})$.
- The Killing vector κ will be assumed to have $W := -g_{\mu\nu}\kappa^\mu\kappa^\nu \sim 1 + \mathcal{O}(r^{-1})$.

- We also assume asymptotic hypersurface orthogonality, $\kappa^\nu(\partial_\mu \kappa_\nu - \partial_\nu \kappa_\mu) \sim \mathcal{O}(r^{-2})$.
- In any vielbein frame, the curvature will fall off as $R_{abcd} \sim \mathcal{O}(r^{-3})$.
- Lie derivatives with respect to κ are assumed to vanish on all fields.

The $D = 3$ theory dimensionally reduced with respect to the timelike Killing vector κ will have an Abelian principal bundle structure, with a metric

$$ds^2 = -W(dt + \hat{B}_i dx^i)^2 + W^{-1}\gamma_{ij} dx^i dx^j$$

where t is a coordinate adapted to the Killing vector κ and γ is the metric on the 3-dimensional hypersurface Σ_3 at constant t. If the $D = 4$ theory has Abelian vector fields $\mathcal{A}_\mu$, they similarly reduce to $D = 3$ as

$$4\sqrt{4\pi G}\mathcal{A}_\mu dx^\mu = U(dt + \hat{B}_i dx^i) + \hat{A}_i dx^i$$

Comparison to spacelike dimensional reductions

The timelike $D = 3$ reduced theory will have a G/H^* coset space structure similar to the G/H coset space structure of a $D = 3$ theory similarly reduced on a *spacelike* Killing vector. Thus, for a spacelike reduction of maximal supergravity one obtains an $\mathrm{E}_8/\mathrm{SO}(16)$ theory continuing on in the sequence of dimensional reductions originating in $D = 11$. Julia As for the analogous spacelike reduction, the $D = 3$ theory has the possibility of exchanging $D = 3$ Abelian vector fields for scalars by dualization, contributing to the appearance of an enlarged $D = 3$ bosonic 'duality' symmetry. The resulting $D = 3$ theory contains $D = 3$ gravity coupled to a G/H^* nonlinear sigma model.

- However, although the numerator group G is the same for a timelike reduction to $D = 3$ as that obtained for a spacelike reduction, the divisor group H^* is a *noncompact* form of the spacelike divisor group H. Breitenlohner, Gibbons & Maison 1988
- The origin of this $H \to H^*$ change is the appearance of *negative-sign* kinetic terms for scalars descending from $D = 4$ vectors under the timelike reduction.

Some examples of G/H^* and G/H theories in $D = 3$

G/H	G/H^*	$\bar{G}/\bar{H}$	3+1 dimensional theory
$\frac{SL(n+2)}{SO(n+2)}$	$\frac{SL(n+2)}{SO(n,2)}$	$GL(n)/SO(n)$	$n+4$ dimensional Einstein gravity with n Killing vectors
$\frac{SU(2,1)}{S(U(2)\times U(1))}$	$\frac{SU(2,1)}{S(U(1,1)\times U(1))}$	$U(1)/U(1)$	Einstein-Maxwell ($N=2$ supergravity)
$\frac{SO(8,2)}{SO(8)\times SO(2)}$	$\frac{SO(8,2)}{SO(6,2)\times SO(2)}$	$\frac{SO(6)\times SO(2,1)}{SO(6)\times SO(2)}$	$N=4$ supergravity
$\frac{SO(8,8)}{SO(8)\times SO(8)}$	$\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$	$\frac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$	$N=4$ supergravity +supersym. Maxwell (10 dim. supergravity)
$E_{8(+8)}/SO(16)$	$E_{8(+8)}/SO^*(16)$	$E_{7(+7)}/SU(8)$	$N=8$ supergravity (11 dim. supergravity)

The $D = 3$ classification of extended supergravity stationary solutions *via* timelike reduction generalizes the $D = 3$ supergravity systems obtained from spacelike reduction. de Wit, Tollsten & Nicolai

Stationary Maxwell-Einstein solutions

Consider an initial theory comprising just $D = 4$ gravity together with an Abelian $\mathrm{U}(1)$ vector field, *i.e.* $D = 4$ Maxwell-Einstein theory. Search for stationary spherically symmetric solutions, with an isometry group $\mathrm{SO}(3)$. Using polar coordinates, the $D = 3$ metric on Σ_3 can then be parametrized as $ds^2 = \gamma_{ij} dx^i dx^j = dr^2 + f(r)^2 (d\vartheta^2 + \sin^2\vartheta d\phi^2)$. The reduced $D = 3$ equations of motion become in this case

$$f^{-2}\frac{d}{dr}\left(f^2\frac{d\phi^i}{dr}\right)+\Gamma^i_{jk}(\phi)\frac{d\phi^j}{dr}\frac{d\phi^k}{dr}=0,$$

$$R_{rr}\equiv -2f^{-1}\frac{d^2f}{dr^2}=\gamma_{ij}(\phi)\frac{d\phi^i}{dr}\frac{d\phi^j}{dr},$$

$$R_{\varphi\varphi}=R_{\vartheta\vartheta}\equiv f^{-2}\left(\frac{d}{dr}f\frac{df}{dr}-1\right)=0.$$

- The third equation has the general solution $f(r)^2 = (r - r_0)^2 + c^2$.
- Introducing $\tau(r) := -\int_r^\infty f^{-2}(s)ds$, which is a harmonic function on Σ_3 equipped with the metric γ_{ij}, the first equation above becomes

$$\frac{d^2\hat{\phi}^i}{d\tau^2} + \Gamma^i_{jk}(\phi)\frac{d\hat{\phi}^j}{d\tau}\frac{d\hat{\phi}^k}{d\tau} = 0$$

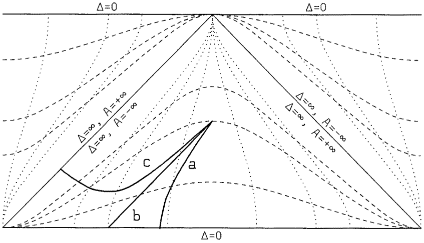
  with $\hat{\phi}^i(r) = \hat{\phi}^i(\tau(r))$.
- This is the equation for a *geodesic* in the symmetric space $G/H^* = \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(1,1) \times \mathrm{U}(1))$, with signature $(++--)$. The decomposition of $\phi : \Sigma_3 \to G/H^*$ into a harmonic map $\tau : \Sigma_3 \to \mathbb{R}$ and a geodesic $\hat{\phi} : \mathbb{R} \to G/H^*$ is in accordance with a general theorem on harmonic maps Eels & Sampson, 1964 according to which the composition of a harmonic map with a totally geodesic one is again harmonic.
- Such factorization into geodesic and harmonic maps is also characteristic of higher-dimensional p-brane supergravity solutions. Neugebauer & Kramer 1964; Clement & Gal'tsov 1996; Gal'tsov & Rychkov 1998

- Restricting attention to subspace of static solutions with electric charge only (magnetic charge can be removed by a duality transformation), the relevant sigma-model structure simplifies to $(G/H^*)_{\rm st} = {\rm SO}(2,1)/{\rm SO}(1,1)$. The line element in this two-dimensional target space is $ds^2 = \frac{d\Delta^2}{2\Delta^2} - \frac{2dA^2}{\Delta}$, where Δ and A are respectively the gravitational and electric potentials. (This is actually the metric for two-dimensional de Sitter space.) The corresponding geodesic equations are

$$\ddot{\Delta} - \Delta^{-1}\dot{\Delta}^2 - 2\dot{A}^2 = 0 \qquad\qquad \ddot{A} - \Delta^{-1}\dot{\Delta}\dot{A} = 0\ ;$$

  these can be explicitly solved subject to the boundary conditions $\Delta(0) = 1$, $A(0) = 0$, corresponding to asymptotic behaviour as $r \to \infty$.
- In this way, one obtains three families of Reissner-Nordstrom solutions, with solution classes separating according to the sign of the integration constant $v^2 = \gamma_{ij}\frac{d\hat{\phi}^i}{d\tau}\frac{d\hat{\phi}^j}{d\tau} = -c^2$, which characterizes the geodesic on ${\rm SO}(2,1)/{\rm SO}(1,1)$ as spacelike ($v^2 < 0$), lightlike ($v^2 = 0$) or timelike ($v^2 > 0$).

Reissner-Nordstrom solution families



Carter-Penrose diagram for two-dimensional de Sitter space. Curves *a*, *b* & *c* are examples of timelike, lightlike and spacelike geodesics.

- The timelike geodesic with $v^2 > 0$ corresponds to a non-extremal Reissner-Nordstrom solution.
- The lightlike geodesic with $v^2 = 0$ corresponds to an extremal Reissner-Nordstrom solution.
- The spacelike geodesic with $v^2 > 0$ corresponds to an over-extremal Reissner-Nordstrom solution with a naked singularity where $\Delta = \infty$.

Breitenlohner, Gibbons & Maison 1988

Charges

Define the Komar two-form $K \equiv \partial_\mu \kappa_\nu dx^\mu \wedge dx^\nu$. This is invariant under the action of the timelike isometry and, by the asymptotic hypersurface orthogonality assumption, is asymptotically horizontal. This condition is equivalent to a requirement that the scalar field B dual to the Kaluza-Klein vector arising by dimensional reduction out of the metric vanish like $\mathcal{O}(r^{-1})$ as $r \to \infty$. In this case, one can define the Komar mass and NUT charge by (where s^* indicates a pull-back to a section) Bossard, Nikolai & K.S.S.

$$m \equiv \frac{1}{8\pi} \int_{\partial\Sigma} s^* \star K \qquad n \equiv \frac{1}{8\pi} \int_{\partial\Sigma} s^* K$$

The Maxwell field also defines charges. Using the Maxwell field equation $d \star \mathcal{F} = 0$, where $\mathcal{F} \equiv \delta\mathcal{L}/\delta F$ is a linear combination of the two-form field strengths F depending on the four-dimensional scalar fields, and using the Bianchi identity $dF = 0$ one obtains conserved electric and magnetic charges

$$q \equiv \frac{1}{2\pi} \int_{\partial\Sigma} s^* \star \mathcal{F} \qquad p \equiv \frac{1}{2\pi} \int_{\partial\Sigma} s^* F$$

Now consider these charges from the three-dimensional point of view in order to clarify their transformation properties under the three dimensional duality group G (in our Maxwell-Einstein example, $G = \mathrm{SU}(2,1)$).

The three-dimensional theory is described in terms of a coset representative $\mathcal{V} \in G/H^*$. The Maurer–Cartan form $\mathcal{V}^{-1}d\mathcal{V}$ decomposes as

$$\mathcal{V}^{-1}d\mathcal{V} = Q + P \quad , \qquad Q \equiv Q_\mu dx^\mu \in \mathfrak{h}^* \;\; , \quad P \equiv P_\mu dx^\mu \in \mathfrak{g} \ominus \mathfrak{h}^*$$

Then the three-dimensional equations of motion can be rewritten as $d \star \mathcal{V}P\mathcal{V}^{-1} = 0$, so the $\mathfrak{g}$-valued Noether current is $\star\mathcal{V}P\mathcal{V}^{-1}$.

Since the three-dimensional theory is Euclidean, one cannot properly speak of a conserved charge. Nevertheless, since $\star\mathcal{V}P\mathcal{V}^{-1}$ is d-closed, the integral of this 2-form on a given homology cycle does not depend on the representative of the cycle.

As a result, for stationary solutions, the integral of this three-dimensional current, over any space-like closed surface containing in its interior all the singularities and topologically non-trivial subspaces of a solution, defines a $\mathfrak{g} \ominus \mathfrak{h}^*$-valued charge matrix $\mathscr{C}$

$$\mathscr{C} \equiv \frac{1}{4\pi} \int_{\partial\Sigma} \star \mathcal{V} P \mathcal{V}^{-1}$$

This transforms in the adjoint representation of G according to the standard non-linear action. For asymptotically flat solutions, $\mathcal{V}$ goes to the identity matrix asymptotically and the charge matrix $\mathscr{C}$ in that case is given by the asymptotic value of the one-form P:

$$P = \mathscr{C} \, \frac{dr}{r^2} + \mathcal{O}(r^{-2})$$

Now express this in terms suitable for more general cases than our simple Maxwell-Einstein example.

Let $\mathfrak{g}_4$ be the algebra of the $D = 4$ symmetry group G and let $\mathfrak{h}_4$ be the algebra of the $D = 4$ divisor group H. $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$ is the algebra of the Ehlers group (*i.e.* the $D = 3$ duality group of pure $D = 4$ gravity); $\mathfrak{so}(2)$ is the algebra of its divisor group. Let $\mathfrak{l}_4$ be the $\mathfrak{h}_4$ representation carried by the electric and magnetic charges q and p. Then $\mathscr{C}$ can be decomposed into three irreducible representations with respect to $\mathfrak{so}(2) \oplus \mathfrak{h}_4$ according to

$$\mathfrak{g} \ominus \mathfrak{h}^* \cong \big(\mathfrak{sl}(2, \mathbb{R}) \ominus \mathfrak{so}(2)\big) \oplus \mathfrak{l}_4 \oplus \big(\mathfrak{g}_4 \ominus \mathfrak{h}_4\big)$$

The metric induced by the Cartan-Killing metric of $\mathfrak{g}$ on this coset is positive definite for the first and last terms, and negative definite for $\mathfrak{l}_4$.

One associates the $\mathfrak{sl}(2, \mathbb{R}) \ominus \mathfrak{so}(2)$ component with the Komar mass and the Komar NUT charge, and one associates the $\mathfrak{l}_4$ component with the electromagnetic charges. The remaining $\mathfrak{g}_4 \ominus \mathfrak{h}_4$ charges come from the Noether current of the four-dimensional theory, which transforms in the adjoint of G.

Characteristic equation

Breitenlohner, Gibbons and Maison proved that if G is simple, all the non-extremal single-black-hole solutions of a given theory lie on the H^* orbit of a Kerr solution. Moreover, all static solutions regular outside the horizon with a charge matrix satisfying $\mathrm{Tr}\,\mathscr{C}^2 > 0$ lie on the H^*-orbit of the Schwarzschild solution. (Turning on and off angular momentum requires consideration of the $D = 2$ duality group generalizing the Geroch A_1^1 group, and will be considered in future work.)

Using Weyl coordinates, the coset representative $\mathcal{V}$ associated to the Schwarzschild solution with mass m can be written in terms of the non-compact generator $\mathbf{h}$ of $\mathfrak{sl}(2,\mathbb{R})$ only, *i.e.*

$$\mathcal{V} = \exp\left(\frac{1}{2}\ln\frac{r-m}{r+m}\,\mathbf{h}\right) \quad \rightarrow \qquad \mathscr{C} = m\mathbf{h}$$

For the maximal $N = 8$ theory with symmetry $\mathrm{E}_{8(8)}$ (and also for the exceptional 'magic' $N = 2$ supergravity Gunaydin, Sierra & Townsend with symmetry $\mathrm{E}_{8(-24)}$), one finds

$$\mathbf{h}^5 = 5\mathbf{h}^3 - 4\mathbf{h}$$

- Consequently, the charge matrix $\mathscr{C}$ satisfies in all cases

$$\boxed{\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}}$$

  where $c^2 \equiv \frac{1}{k} \operatorname{Tr} \mathscr{C}^2$ is the extremality parameter (vanishing for extremal solutions) and $k \equiv \operatorname{Tr} \mathbf{h}^2 > 0$.
- For all but the two exceptional E_8 cases, a stronger constraint is satisfied by the charge matrix $\mathscr{C}$:

$$\boxed{\mathscr{C}^3 = c^2\mathscr{C}}$$

Supersymmetry 'Dirac equation'

Extremal solutions have $c^2 = 0$, implying that the charge matrix $\mathscr{C}$ becomes nilpotent: $\mathscr{C}^5 = 0$ in the E_8 cases and $\mathscr{C}^3 = 0$ otherwise.

For $\mathcal{N}$ extended supergravity theories, one finds $H^* \cong \mathrm{Spin}^*(2\mathcal{N}) \times H_0$ and the charge matrix $\mathscr{C}$ transforms as a Weyl spinor of $\mathrm{Spin}^*(2\mathcal{N})$ valued in a representation of $\mathfrak{h}_0$. Define the $\mathrm{Spin}^*(2\mathcal{N})$ fermionic oscillators

$$a_i := \frac{1}{2}\Big(\Gamma_{2i-1} + i\Gamma_{2i}\Big) \qquad a^i \equiv (a_i)^\dagger = \frac{1}{2}\Big(\Gamma_{2i-1} - i\Gamma_{2i}\Big)$$

for $i, j, \dots = 1, \dots, \mathcal{N}$. These obey standard anticommutation relations

$$\{a_i, a_j\} = \{a^i, a^j\} = 0 \quad , \qquad \{a_i, a^j\} = \delta_i^j$$

Using this oscillator basis, the charge matrix $\mathscr{C}$ can be represented as a state

$$|\mathscr{C}\rangle \equiv \Big(W + Z_{ij} a^i a^j + \Sigma_{ijkl} a^i a^j a^k a^l + \cdots\Big)|0\rangle$$

From the requirement that dilatino fields be left invariant under an unbroken supersymmetry of a BPS solution, one derives a 'Dirac equation' for the charge state vector,

$$\boxed{\left(\epsilon^i_\alpha a_i + \Omega_{\alpha\beta}\epsilon^\beta_i a^i\right)|\mathscr{C}\rangle = 0}$$

where $(\epsilon^i_\alpha, \epsilon^\alpha_i)$ is the asymptotic (for $r \to \infty$) value of the Killing spinor and $\Omega_{\alpha\beta}$ is a symplectic form on $\mathbb{C}^{2n}$ for n/N preserved supersymmetry.

This equation encapsulates all information about solutions with residual supersymmetry and allows for a complete analysis of the BPS sector. Analysis of the BPS conditions can now be reduced to calculations with the fermionic oscillators.

Note that extremal and BPS are not always synonymous conditions, although generally they coincide. $c^2 = 0$ is a *weaker* condition than the supersymmetry Dirac equation.

Almost Iwasawa decomposition

Earlier analysis of the orbits of the $D = 4$ symmetry groups $\bar{G}$ Cremmer, Lü, Pope & K.S.S. heavily used the Iwasawa decomposition

$$g = u_{(g,Z)} \exp\Big(\ln \lambda_{(g,Z)}\, \mathbf{z}\Big)\, b_{(g,Z)}$$

with $u_{(g,Z)} \in \bar{H}$ and $b_{(g,Z)} \in \mathfrak{B}_Z$ where $\mathfrak{B}_Z \subset \bar{G}$ is the 'parabolic' (Borel) subgroup that leaves the charges Z invariant up to a multiplicative factor $\lambda_{(g,Z)}$. This multiplicative factor can be compensated for by 'trombone' transformations combining Weyl scalings with compensating dilational coordinate transformations, leading to a formulation of 'active' symmetry transformations that map solutions into other solutions with unchanged asymptotic values of the spacetime metric and asymptotic scalar values.

- The $D = 4$ 'trombone' transformation finds a natural home in the parabolic subgroup of the $D = 3$ duality group G.
- However, the $D = 3$ analysis is complicated by the fact that the Iwasawa decomposition *breaks down* for noncompact divisor groups H^*.

- The Iwasawa decomposition does, however work "almost everywhere" in the $D = 3$ solution space. The places where it fails are precisely the extremal suborbits of the duality group.
- This has the consequence that G does not act transitively on its own orbits. There are G transformations which allow one to send $c^2 \to 0$, thus landing on an extremal (generally BPS) suborbit. However, one cannot then invert the map and return to a generic non-extremal solution from the extremal solution reached on a given G trajectory.

Conclusions

The understanding of duality group orbits for stationary supergravity solutions has been deepened in the following ways.

- The Noether charge matrix $\mathscr{C}$ satisfies a characteristic equation $\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}$ in the maximal E_8 cases and $\mathscr{C}^3 = c^2\mathscr{C}$ in the non-maximal cases, where $c^2 \equiv \frac{1}{k}\,\mathrm{Tr}\,\mathscr{C}^2$ is the extremality parameter.
- Extremal solutions are characterized by $c^2 = 0$, and $\mathscr{C}$ becomes nilpotent ($\mathscr{C}^5 = 0$ *viz.* $\mathscr{C}^3 = 0$) on the corresponding suborbits.
- BPS solutions have a charge matrix $\mathscr{C}$ satisfying an algebraic 'supersymmetry Dirac equation' which encodes the general properties of such solutions. This is a stronger condition than the $c^2 = 0$ extremality condition.
- The orbits of the $D = 3$ duality group G are not always acted upon transitively by G. This is related to the failure of the Iwasawa decomposition for noncompact divisor groups H^*. The Iwasawa failure set corresponds to the extremal suborbits.