Kaehler metrics and Yukawa couplings in magnetized brane world

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Foreword

- This talk is based on
  - P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, "Kähler metrics and Yukawa couplings in magnetized brane models", JHEP 0903:029,2009, arXiv:0810.3806 [hep-th].
  - P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, "Kähler metrics: string vs field theoretical approach" arXiv:0901:4458 [hep-th].
  - D. Cremades, L. Ibáñez and F. Marchesano, "Computing Yukawa couplings from magnetized extra dimensions", JHEP 0405 (2004) 079, arXiv:0404229 [hep-th].

Plan of the talk

Introduction

- 10-dimesional string theories contain a parameter α' of a dimension of a (*length*)^2 that acts as a physical UV cutoff $\Lambda = \frac{1}{\sqrt{\alpha'}}$ in the loop diagrams.
- For this reason one obtains a consistent quantum theory of gravity unified with gauge theories.
- The value of α' can only be determined from experiments.
- We observe only 4 and not 10 non-compact dimensions!
- We need to compactify six of them:

$$R^{1,9} \rightarrow R^{1,3} \times M_6$$

  where M_6 is a compact six-dimensional manifold.
- If we want to preserve at least $\mathcal{N} = 1$ supersymmetry, M_6 must be a Calabi-Yau manifold.
- But then the four-dimensional physics will depend not only on α', but also on the shape and the size of M_6.

- The parameters characterizing a particular compactification are called moduli.
- They cannot be arbitrarily given.
- But they are fixed by the minima of their potential: Moduli Stabilization.
- Too many consistent compactifications: Landscape Problem.
- In this seminar I will not be concerned with these problems and I will assume that the moduli can be stabilized.
- Given a certain consistent compactification, how does one compute the low energy four-dimensional effective action for the light degrees of freedom?
- I will restrict myself to compactifications that preserve $\mathcal{N} = 1$ supersymmetry.
- In order to explicitly perform the calculation of the effective action I will consider toroidal compactifications, possibly with orbifolds and orientifolds.
- In order to have chiral matter I will consider the case of magnetized D branes.

Magnetized D branes

- Assume that on the stack a (stack b) of branes there is a constant magnetic $F^{(a)}(F^{(b)})$.
- The action describing the interaction of an open string with its end-points attached to these two stacks of branes is given by:

$$S = S_{bulk} + S_{boundary}$$

$$S_{bulk} = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left[G_{ij}\partial_\alpha X^i \partial_\beta X^j \eta^{\alpha\beta} - B_{ij}\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j \right]$$

$$\begin{aligned} S_{boundary} &= -q_a \int d\tau A_i^{(a)} \partial_\tau X^i|_{\sigma=0} + q_b \int d\tau A_i^{(b)} \partial_\tau X^i|_{\sigma=\pi} \\ &= \frac{q_a}{2} \int d\tau F_{ij}^{(a)} X^j \dot{X}^i|_{\sigma=0} - \frac{q_b}{2} \int d\tau F_{ij}^{(b)} X^j \dot{X}^i|_{\sigma=\pi} \end{aligned}$$

- The two gauge field strengths are constant:

$$A_i^{(a,b)} = -\frac{1}{2} F_{ij}^{(a,b)} x^j \ .$$

- The data of the torus $\mathcal{T}^2$, called moduli, are included in the constant G_{ij} and B_{ij}.
- They are the complex and Kähler structures of the torus:

$$U \equiv U_1 + iU_2 = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{11}} \quad ; \quad T \equiv T_1 + iT_2 = B_{12} + i\sqrt{G}$$

by

$$G_{ij} = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \quad \text{and} \quad B_{ij} = \begin{pmatrix} 0 & -T_1 \\ T_1 & 0 \end{pmatrix}$$

They are the closed string moduli.
- F is constrained by the fact that its flux is an integer:

$$\int Tr\left(\frac{qF}{2\pi}\right) = m \Longrightarrow 2\pi\alpha' qF_{12} = \frac{m}{n}$$

They are the open string moduli.
- The D brane is wrapped n times on the torus and the flux of F, on a compact space as T^2, must be an integer m (magnetic charge).

- The most general motion of an open string in this constant background can be determined and the theory can be explicitly quantized.
- One gets a string extension of the motion of an electron in a constant magnetic field on a torus (Landau levels).
- The ground state is degenerate and the degeneracy is given by the number of Landau levels.
- When $\alpha' \to 0$ one goes back to the problem of an electron in a constant magnetic field.
- The mass spectrum of the string states can be exactly determined:

$$\alpha' M^2 = N_4^X + N_4^\psi + N_{comp.}^X + N_{comp}^\psi + \frac{x}{2}\sum_{r=1}^{3} \nu_r - \frac{x}{2}$$

$x = 0$ for fermions (R sector) and $x = 1$ for bosons (NS sector)

$$N_4^X = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n \;\; ; \;\; N_4^\psi = \sum_{n=\frac{x}{2}}^{\infty} n \psi_n^\dagger \cdot \psi_n$$

$$N^X_{comp} = \sum_{r=1}^{3} \left[\sum_{n=0}^{\infty} (n+\nu_r) a^{\dagger r}_{n+\nu_r} a^r_{n+\nu_r} + \sum_{n=1}^{\infty} (n-\nu_r) \bar{a}^{\dagger r}_{n-\nu_r} \bar{a}^r_{n-\nu_r} \right]$$

$$N^\psi_{comp} = \sum_{r=1}^{3} \left[\sum_{n=\frac{x}{2}}^{\infty} (n+\nu_r) \psi^{\dagger r}_{n+\nu_r} \psi^r_{n+\nu_r} + \sum_{n=1-\frac{x}{2}}^{\infty} (n-\nu_r) \bar{\psi}^{\dagger r}_{n-\nu_r} \bar{\psi}^r_{n-\nu_i} \right]$$

- where

$$\nu_r = \nu_r^a - \nu_r^b \quad ; \quad \tan \pi \nu_r^{a,b} = \frac{m_r^{(a,b)}}{n_r^{(a,b)} T_2^{(r)}}$$

$T_2^{(r)}$ is the volume of the r-th torus.

- In the fermionic sector the lowest state is the vacuum state.
- It is a 4-dimensional massless chiral spinor!!

- For generic values of ν_1, ν_2, ν_3 there is no massless state in the bosonic sector.
- In general the original 10-dim supersymmetry is broken.
- The lowest bosonic states are

$$\psi^{\dagger r}_{\frac{1}{2}-\nu}|0> \quad ; \quad \alpha' M^2 = \frac{1}{2}\sum_{s=1}^{3} \nu_s - \nu_r \ ; \quad r = 1,2,3$$

$$\psi^{\dagger 1}_{\frac{1}{2}-\nu_1}\psi^{\dagger 2}_{\frac{1}{2}-\nu_2}\psi^{\dagger 3}_{\frac{1}{2}-\nu_3}|0> \quad ; \quad \alpha' M^2 = \frac{2-\nu_1-\nu_2-\nu_3}{2}$$

- One of these states becomes massless if one of the following identities is satisfied:

$$\nu_1 = \nu_2 + \nu_3 \ ; \ \nu_2 = \nu_1 + \nu_3 \ ; \ \nu_3 = \nu_1 + \nu_2 \ ; \ \nu_1 + \nu_2 + \nu_3 = 2$$

- In each of these cases four-dimensional $\mathcal{N} = 1$ supersymmetry is restaured!

- In general the ground state for the open strings, having their end-points respectively on stacks a and b, is degenerate.
- Its degeneracy is given by the number of Landau levels as in the case of a point-like particle:

$$I_{ab} = \prod_{r=1}^{3} \left\{ n_r^{(a)} n_r^{(b)} \int \left[\frac{q_a F_r^{(a)} - q_b F_r^{(b)}}{2\pi} \right] \right\} = \prod_{i=1}^{3} \left[m_r^{(a)} n_r^{(b)} - m_r^{(b)} n_r^{(a)} \right]$$

  that gives the number of families in the phenomenological applications.
- It corresponds to the number of intersections in the case of intersecting branes.

From 10-dim to 4-dim

- Starting from a 10-dim string theory with D branes and given a certain compactification, how do we compute the 4-dim low-energy effective action that should be compared with experiments?
- If the theory preserves $\mathcal{N} = 1$ supersymmetry, then one would like to determine the parameters of the general action:

$$S = -\frac{3}{\kappa^2} \int d^2\theta d^2\bar{\theta} E \mathrm{e}^{-\frac{1}{3}\kappa^2 K(\Phi, \bar{\Phi}; V)} + \\ + \frac{1}{4} \int d^2\theta \mathcal{E} \sum_a f_a(\Phi) (W_\alpha W^\alpha)_a + \int d^2\theta W(\Phi) + h.c.$$

namely the quantities: $K(\Phi, \bar{\Phi}; V)$, $f_a(\Phi)$, $W(\Phi)$, where

$$K(\Phi, \bar{\Phi}; V) = \frac{1}{\kappa^2} K(m, \bar{m}) + Z_{IJ}(m, \bar{m}) \bar{Q}^I \mathrm{e}^{2V} Q^J + \ldots$$

Z_{IJ} is the Kähler metric of the charged fields.

- Compute string amplitudes involving both open and closed strings and from them one can extract the parameters of the low-energy effective action.
- This cannot be explicitly done for an arbitrary Calabi-Yau compactification, but in the framework of the magnetized branes on a torus everything can be computed.
- By computing a three-point amplitude with two twisted (chiral) open strings and a closed string modulus the dependence of the Kähler metric of those strings on the magnetization was computed:

$$Z_{IJ}(m, \bar{m}) \sim \left[\frac{\Gamma(1-\nu_1^{ab})}{\Gamma(\nu_1^{ab})} \frac{\Gamma(\nu_2^{ab})}{\Gamma(1-\nu_2^{ab})} \frac{\Gamma(\nu_3^{ab})}{\Gamma(1-\nu_3^{ab})} \right]^{1/2} \delta_{IJ}$$

[Lüst et al. (2004) and Bertolini et al. (2005)]
in the supersymmetric case $\nu_1^{ab} = \nu_2^{ab} + \nu_3^{ab}$.

- On the other hand, by instanton calculations requiring the holomorphicity of the superpotential, one obtained:

$$Z(m,\bar{m}) = (s_2)^{-\frac{1}{4}} \left(t_2^{(1)} t_2^{(2)} t_2^{(3)}\right)^{-\frac{1}{4}} \left(u_2^{(1)} u_2^{(2)} u_2^{(3)}\right)^{-\frac{1}{2}} \times$$
$$\times \left[\frac{\Gamma(1-\nu_1^{ab})}{\Gamma(\nu_1^{ab})} \frac{\Gamma(\nu_2^{ab})}{\Gamma(1-\nu_2^{ab})} \frac{\Gamma(\nu_3^{ab})}{\Gamma(1-\nu_3^{ab})} \right]^{1/2} \mathcal{Z}_\Phi$$

[Akerblom et al. , Billó et al. , Blumenhagen et al., (2007)]

- $\mathcal{Z}_\Phi$ is restricted to satisfy an equation.
- The holomorphic variables in supergravity are related to those of string theory by:

$$s_2 = \mathrm{e}^{-\phi_{10}} \prod_{r=1}^{3} T_2^{(r)} \;\; ; \;\; t_2^{(r)} = \mathrm{e}^{-\phi_{10}} T_2^{(r)} \;\; ; \;\; u_2^{(r)} = U_2^{(r)}$$

- How do we get the extra dependence on the moduli?
- It is likely that the methods used to compute string amplitudes with magnetized branes are not complete yet.
- The oscillator modes are treated correctly, but the effect of the zero modes is probably not fully incorporated.
- If we are not interested in the string corrections to the supergravity quantities we do not need to perform a complete string calculation.
- We can use just the low-energy brane effective action.
- This will also help to understand the field theory limit of the magnetized D branes.

KK reduction of super Yang-Mills theory in D=10

- Low-energy limit of the DBI action for M D9 branes:

$$S = \frac{1}{g^2} \int d^{10} X \mathrm{Tr} \Big(- \frac{1}{4} F_{MN} F^{MN} + \frac{i}{2} \bar{\lambda} \Gamma^M D_M \lambda \Big),$$

$$g^2 = 4\pi \mathrm{e}^{\phi_{10}} (2\pi\sqrt{\alpha'})^6 \quad ; \quad g_s \equiv \mathrm{e}^{\phi_{10}} = \text{string coupling const.}$$

$$F_{MN} = \nabla_M A_N - \nabla_N A_M - i[A_M, A_N] \;\; ; \;\; D_M \lambda = \nabla_M \psi - i[A_M, \lambda]$$

λ is a ten dimensional Weyl-Majorana spinor.

- Separate the generators of the gauge group $U(M)$ into those, called U_a, that live in the Cartan subalgebra and those, called e_{ab}, that are outside of the Cartan subalgebra

$$(U_a)_{ij} = \delta_{ai}\delta_{aj}, \qquad (e_{ab})_{ij} = \delta_{ai}\delta_{bj} \quad (a \neq b).$$

- Write

$$A_M = B_M + W_M = B^a_M U_a + W^{ab}_M e_{ab}\ ;\ \lambda = \chi + \Psi = \chi^a U_a + \Psi^{ab} e_{ab}$$

- Separate the ten-dimensional coordinate $X^M = (x^\mu, y^i)$ into a four-dimensional non-compact coordinate x^μ and a six-dimensional compact coordinate y^i.
- Perform a Kaluza-Klein reduction of the Lagrangian expanding around the background fields:

$$\begin{aligned} B^a_M(x^\mu, y^i) &= \langle B^a_M \rangle (y^i) + \delta B^a_M(x^\mu, y^i) \\ W^{ab}_M(x^\mu, y^i) &= 0 + \Phi^{ab}_M(x^\mu, y^i) \end{aligned}$$

- Four-dimensional Lorentz invariance is kept by allowing a non-vanishing background value $\langle B^a_M \rangle (y^i)$ only for $M = i$, i.e. along the compact extra-dimensions.

- The presence of different background values along the Cartan subalgebra breaks the original $U(M)$ symmetry into $(U(1))^M$.
- In terms of D branes this corresponds to generate M stacks, each consisting of one D brane, having different magnetization.
- $\Phi_M^{ab}(x^\mu, y^i)$ for $M = i$ describe twisted open strings with the two end-points attached respectively to two D branes a and b having different magnetizations.
- One can rewrite the original action in terms of the fields $\delta B_M^a, \Phi^{ab}, \chi^a, \Psi^{ab}$.
- Here we limit ourselves to the terms containing the Kähler metrics and the Yukawa couplings.

- Quadratic terms for the fields $\Phi_M^{ab}(x^\mu, y^i)$:

$$S_2^{(\Phi)} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \int d^6y \sqrt{G_6} \times$$
$$\times \Phi^{jba} \left[G^i_j \left(D_\mu D^\mu + \tilde{D}_k \tilde{D}^k \right) + 2i < F^i_j >^{ab} \right] \Phi_i^{ab}$$

where

$$\tilde{D}_i \Phi_j^{ab} \equiv \nabla_i \Phi_j^{ab} - i B_i^a \Phi_j^{ab} + i \Phi_j^{ab} B_i^b$$

with

$$< F^i_j >^{ab} = (F_B^a)^i{}_j - (F_B^b)^i{}_j$$

$(F_B^a)^i{}_j$ is the (constant) field strength obtained from the background field B^a.

- Analogously for the fields $\delta B_i^a(x^\mu, y^i)$:

$$S_2^{(\delta B)} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \int d^6y \sqrt{G_6} \delta B_i^a \left(\partial_j \partial^j + D_\mu D^\mu \right) \delta B^{ai}$$

- for the fermions

$$S_2^{(\Psi)} = \frac{i}{2g^2} \int d^4x \sqrt{G_4} \int d^6y \sqrt{G_6} \bar{\Psi}^{ba} \left(\Gamma^\mu D_\mu + \Gamma^i \tilde{D}_i \right) \Psi^{ab}$$

where

$$D_\mu \Psi = \partial_\mu \Psi - i B_\mu^a \Psi^{ab} + i \Psi^{ab} B_\mu^b$$
$$\tilde{D}_i \Psi^{ab} = \partial_i \Psi^{ab} - i \langle B_i^a \rangle \Psi^{ab} + i \Psi^{ab} \langle B_i^b \rangle$$

- and for the tri-linear Yukawa couplings:

$$S_3^{\Phi} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \int d^6y \sqrt{G_6} \left(\bar{\Psi}^{ca} \Gamma^i \Phi_i^{ab} \Psi^{bc} - \bar{\Psi}^{ca} \Gamma^i \Phi_i^{bc} \Psi^{ab} \right)$$

for the twisted scalar Φ and

$$S_3^{\delta B} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \int d^6y \sqrt{G_6} \bar{\Psi}^{ab} \Gamma^i (\delta B_i{}^b - \delta B_i{}^a) \Psi^{ba}$$

for the untwisted scalar B.

- Kaluza-Klein reduction:

$$\Phi_i^{ab}(X) = \sum_n \varphi_{n,i}^{ab}(x)\phi_n^{ab}(y) \;\; ; \;\; \Psi^{ab}(X) = \sum_n \psi_n^{ab}(x) \otimes \eta_n^{ab}(y)$$

- The spectrum of KK states and their wave-functions along the compact directions are obtained by solving the eigenvalue equations for the six-dimensional Laplace and Dirac operators:

$$\begin{aligned} -\tilde{D}_k \tilde{D}^k (\phi^{ab})_n &= m_n^2 \phi_n^{ab}(y) \\ i\gamma_{(6)}^i \tilde{D}_i \eta_n^{ab} &= \lambda_n \eta_n^{ab} \end{aligned}$$

  with the correct periodicity conditions along the compactified directions.

- Decomposition for 10-dim γ-matrices:

$$\Gamma^\mu = \gamma_{(4)}^\mu \otimes \mathbb{I}_{(6)} \;\; , \;\; \Gamma^i = \gamma_{(4)}^5 \otimes \gamma_{(6)}^i$$

Solving the eigenvalue equations

- Let us start to consider the torus T^2 described by the coordinates (x^1, x^2):

$$x^1 \equiv x^1 + 2\pi R \qquad x^2 \equiv x^2 + 2\pi R$$

- or by the "flat" dimensionless ones:

$$z = \frac{x^1 + U x^2}{2\pi R} \qquad \bar{z} = \frac{x^1 + \bar{U} x^2}{2\pi R}$$

- The metric of the torus in the two coordinate systems is equal to:

$$G_{ij}^{(x^1,x^2)} = \frac{\mathcal{T}_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \;\; ; \;\; G_{ij}^{(z,\bar{z})} = \frac{\mathcal{T}_2}{2U_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Gauge covariant derivative:

$$\tilde{D}_z = \partial_z - iB_z \;\; ; \;\; \tilde{D}_{\bar{z}} = \partial_{\bar{z}} - iB_{\bar{z}}$$

- where

$$B = B_z dz + B_{\bar{z}} d\bar{z} = \frac{\pi m(\bar{z}dz - zd\bar{z})}{2iU_2}$$

- They imply ($F \equiv dB$):

$$\left[-i\tilde{D}_z, -i\tilde{D}_{\bar{z}}\right] = -\frac{\pi m}{U_2} \equiv iF_{z\bar{z}}$$

- The first Chern class must be an integer m:

$$\int \frac{F}{2\pi} = \int F_{z\bar{z}} dz \wedge d\bar{z} = m \Longrightarrow F_{z\bar{z}} = -\frac{\pi m}{iU_2}$$

- Compute Laplace operator on torus T^2:

$$\tilde{D}_k \tilde{D}^k = \tilde{D}_k G^{ki} \tilde{D}_i = \frac{2U_2}{\mathcal{T}_2} \left\{\tilde{D}_z, \tilde{D}_{\bar{z}}\right\}$$

- If $m > 0$ we can introduce the creation and annihilation operator:

$$-i\tilde{D}_z \equiv -i\left(\partial_z - \frac{\pi m\bar{z}}{2U_2}\right) = \sqrt{\frac{\pi m}{U_2}}a^\dagger$$

$$-i\tilde{D}_{\bar{z}} \equiv -i\left(\partial_{\bar{z}} + \frac{\pi m z}{2U_2}\right) = \sqrt{\frac{\pi m}{U_2}}a$$

- They satisfy the harmonic oscillator algebra:

$$[a, a^\dagger] = 1$$

- We get:

$$-\tilde{D}_k\tilde{D}^k = \frac{2\pi m}{\mathcal{T}_2}\left(aa^\dagger + a^\dagger a\right) = \frac{2\pi m}{\mathcal{T}_2}\left(2a^\dagger a + 1\right) \equiv \frac{2\pi m}{\mathcal{T}_2}(2N+1)$$

- The ground state for the torus T^2 is degenerate and there are m independent solutions given by:

$$\phi^{ab,n}_{T^2;+}(z) = e^{i\pi m z\frac{\mathrm{Im}z}{\mathrm{Im}U}}\,\Theta\begin{bmatrix}\frac{2n}{m}\\ 0\end{bmatrix}(mz|mU) \;\; ; \;\; n = 0\ldots m-1$$

- Definition of the Jacobi Θ-function

$$\Theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z|U) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi i\left[\frac{1}{2}(n+\frac{\alpha}{2})^2 U + (n+\frac{\alpha}{2})(z+\frac{\beta}{2})\right]}$$

- They are determined by solving the equation

$$a\,\phi^{ab}_{T^2}(z,\bar{z}) \equiv \tilde{D}_{\bar{z}}\phi^{ab}_{T^2}(z,\bar{z}) = 0$$

- with the following periodicity conditions going around the two one-cycles of the torus:

$$\phi^{ab}(z+1,\bar{z}+1) = e^{i\,\chi_1(z,\bar{z})}\phi^{ab}(z,\,\bar{z})$$
$$\phi^{ab}(z+U,\bar{z}+\bar{U}) = e^{i\,\chi_2(z,\bar{z})}\phi^{ab}(z,\,\bar{z})$$

- where

$$\chi_1 = \frac{\pi m}{\mathrm{Im}U}\mathrm{Im}(z) \;\; ; \;\; \chi_2 = \frac{\pi m}{\mathrm{Im}U}\mathrm{Im}(\bar{U}\,z)$$

- χ_1 is determined by:

$$\begin{aligned} B_z(z+1,\bar{z}+1) &= B_z(z,\bar{z}) + \frac{\pi m}{2iU_2} = B_z(z,\bar{z}) + \partial_z\chi_1 \\ B_{\bar{z}}(z+1,\bar{z}+1) &= B_{\bar{z}}(z,\bar{z}) - \frac{\pi m}{2iU_2} = B_{\bar{z}}(z,\bar{z}) + \partial_{\bar{z}}\chi_1 \end{aligned}$$

- If $m < 0$ the identification of D_z and $D_{\bar{z}}$ with the creation and annihilation operators is exchanged; i.e.:

$$-i\tilde{D}_z = \sqrt{\frac{\pi|m|}{U_2}}a \;\; ; \;\; -i\tilde{D}_{\bar{z}} = \sqrt{\frac{\pi|m|}{U_2}}a^\dagger$$

- Then

$$-\tilde{D}_k \tilde{D}^k = \frac{2\pi|m|}{\mathcal{T}_2}\left(2a^\dagger a + 1\right) \equiv \frac{2\pi|m|}{\mathcal{T}_2}(2N+1)$$

- The wave functions of the (degenerate) ground state, are given by:

$$\phi^{ab,n}_{T^2;-} = e^{\pi i|m|\bar{z}\frac{\mathrm{Im}\bar{z}}{\mathrm{Im}U}}\,\Theta\left[\begin{array}{c}\frac{-2n}{m}\\ 0\end{array}\right](m\bar{z}|m\bar{U}) \;\; ; \;\; n = 0\ldots|m|-1$$

- On the torus $T^2 \times T^2 \times T^2$ one gets:

$$-\tilde{D}_k \tilde{D}^k \Longrightarrow \sum_{r=1}^{3} \frac{2U_2^{(r)}}{\mathcal{T}_2^{(r)}} \left\{ \tilde{D}_{z_r}, \tilde{D}_{\bar{z}_r} \right\} = \sum_{r=1}^{3} \frac{2\pi |m_r|}{\mathcal{T}_2^{(r)}} (2N_r + 1)$$

- Eigenvalue equation becomes:

$$-\tilde{D}_k \tilde{D}^k \phi_n^{ab} = m_n^2 \phi_n^{ab} \Longrightarrow \sum_{s=1}^{3} \frac{2\pi |m_s|}{\mathcal{T}_2^{(s)}} (2N_s + 1) \phi_n^{ab} = \hat{m}_n^2 \phi_n^{ab}$$

where

$$m_n^2 = \frac{\hat{m}^2}{(2\pi R)^2}$$

- Eigenvalue equation for fermions:

$$i\gamma^{i}_{(6)}\tilde{D}_i\eta^{ab}_n \quad = \quad \lambda_n\,\eta^{ab}_n$$

- Squaring the previous equation

$$\left(-\tilde{D}_i\tilde{D}^i\mathbb{I} - \frac{1}{2}[\gamma^i,\gamma^j]\tilde{D}_i\tilde{D}_j\right)\eta_n = \lambda_n^2\eta_n$$

- Restricting ourself to the case $T^2 \times T^2 \times T^2$ and decomposing the six-dimensional Dirac algebra in the product of three two dimensional representations according to the relation:

$$\begin{array}{ccc}
\gamma^4_{(6)} = \gamma^1_{(1)} \otimes \sigma^3 \otimes \sigma^3 & ; & \gamma^5_{(6)} = \gamma^2_{(1)} \otimes \sigma^3 \otimes \sigma^3 \\
\gamma^6_{(6)} = \mathbb{I} \otimes \gamma^1_{(2)} \otimes \sigma^3 & ; & \gamma^7_{(6)} = \mathbb{I} \otimes \gamma^2_{(2)} \otimes \sigma^3 \\
\gamma^8_{(6)} = \mathbb{I} \otimes \mathbb{I} \otimes \gamma^1_{(3)} & ; & \gamma^9_{(6)} = \mathbb{I} \otimes \mathbb{I} \otimes \gamma^2_{(3)}
\end{array}$$

with:

$$\gamma^2_{(r)} \equiv \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; : \; \gamma^1_{(r)} \equiv \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Eigenvalue equation becomes

$$\left(2\pi\sum_{r=1}^{3}(2N_r+1)\frac{|m_r|}{T_2^{(r)}}\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}-\frac{2\pi m_1}{T_2^{(1)}}\sigma_3\otimes\mathbb{I}\otimes\mathbb{I}-\frac{2\pi m_2}{T_2^{(2)}}\mathbb{I}\otimes\sigma_3\right.$$
$$\left.-\frac{2\pi m_3}{T_2^{(3)}}\mathbb{I}\otimes\mathbb{I}\otimes\sigma_3\right)\eta_n^1\otimes\eta_n^2\otimes\eta_n^3=(2\pi R)^2\lambda_n^2\eta_n^1\otimes\eta_n^2\otimes\eta_n^3$$

where we have decomposed

$$\eta_n=\eta_n^1\otimes\eta_n^2\otimes\eta_n^3$$

- Unique zero mode: 4-dim chiral fermion with chirality given by:

$$\chi_4=\chi_{10}\chi_1\chi_2\chi_3$$

χ_r $(r=1,2,3)$ is the chirality on the r-th torus.

- Since the zero mode eigenfunction on $T^2 \times T^2 \times T^2$ is the product of the zero mode eigenfunctions on each torus T^2 we will limit ourselves to the Dirac equation on the torus T^2:

$$\left(\gamma^{z}_{(r)}\tilde{D}_{z^r} + \gamma^{\bar{z}}_{(r)}\tilde{D}_{\bar{z}^r}\right)\eta^{ab}_{r}(z^r,\,\bar{z}^r) = 0$$

- If $m_r > 0$ solution is

$$\eta_{r,+} = \begin{pmatrix} \eta^{ab}_{r,+} \\ 0 \end{pmatrix} \quad ; \quad \eta^{ab,n_r}_{r,+} = \phi^{ab,n_r}_{r,+}$$

- If $m_r < 0$ solution is

$$\eta_{r,-} = \begin{pmatrix} 0 \\ \eta^{ab}_{r,-} \end{pmatrix} \quad ; \quad \eta^{ab,n_r}_{r,-} = \phi^{ab,n_r}_{r,-}$$

Kähler metrics

- We can now compute the kinetic term for the twisted fields:

$$S_2^{(\Phi)} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \sum_n \prod_{r=1}^{3} \left[(2\pi R)^2 \int d^2 z_r \sqrt{G^r} \right] \phi_n^{ba} \phi_n^{ab}$$
$$\times \left\{ \sum_{r=1}^{3} \left[\varphi_{nr}^{ba,z}(x) \left[D_\mu D^\mu - m_n^2 + \frac{4\pi m_r}{(2\pi R)^2 \mathcal{T}_2^{(r)}} \right] \varphi_{nrz}^{ab}(x) \right] \right.$$
$$\left. + \sum_{r=1}^{3} \left[\varphi_{nr}^{ba,\bar{z}}(x) \left[D_\mu D^\mu - m_n^2 - \frac{4\pi m_r}{(2\pi R)^2 \mathcal{T}_2^{(r)}} \right] \varphi_{nr\bar{z}}^{ab}(x) \right] \right\}$$

- Introduce the fields φ^I with "flat" indices :

$$\varphi_{\bar{z}}^{ab} = G_{\bar{z}z} e^{z}{}_{I} (\varphi^I)^{ab} \equiv \sqrt{\frac{\mathcal{T}_2}{2U_2}} \varphi_{+}^{ab}$$

$$(\varphi^{\bar{z}})^{ba} = e^{\bar{z}}{}_{I} (\varphi^I)^{ba} \equiv \sqrt{\frac{2U_2}{\mathcal{T}_2}} (\varphi_{+}^{ab})^{\dagger}$$

$$\varphi_{z}^{ab} = G_{z\bar{z}} e^{\bar{z}}{}_{I} (\varphi^I)^{ab} \equiv \sqrt{\frac{\mathcal{T}_2}{2U_2}} \varphi_{-}^{ab}$$

$$(\varphi^{z})^{ba} = e^{z}{}_{I} (\varphi^I)^{ba} \equiv \sqrt{\frac{2U_2}{\mathcal{T}_2}} (\varphi_{-}^{ab})^{\dagger}$$

where

$$(\varphi_+)^{ab} = \left(\frac{\varphi^1 + i\,\varphi^2}{\sqrt{2}} \right)^{ab} \;;\; (\varphi_-)^{ab} = \left(\frac{\varphi^1 - i\,\varphi^2}{\sqrt{2}} \right)^{ab} \;;\; \varphi_{+}^{ba} = (\varphi_{-}^{ab})^{\dagger}$$

- Keeping only the lowest modes of the two towers of Kaluza-Klein states:

$$
S_2^{(\phi_0)} = -\frac{1}{2g^2} \prod_{s=1}^{3} \left[(2\pi R)^2 \int d^2 z_s \sqrt{G^{(z_s, \bar{z}_s)}} \right] (\phi_0^{ab})^\dagger (\phi_0^{ab}) \times
$$

$$
\times \int d^4 x \sqrt{G_4} \sum_{r=1}^{3} N_{\varphi_r}^2
$$

$$
\times \Big[(D_\mu (\varphi_{r,+}^{ab})^\dagger (x))(D^\mu \varphi_{r,+}^{ab}(x)) + (M_{0,r}^{+})^2 (\varphi_{r,+}^{ab})^\dagger (x) \varphi_{r,+}^{ab}(x)
$$

$$
+ (D_\mu (\varphi_{r,-}^{ab})^\dagger (x))(D^\mu \varphi_{r,-}^{ab}(x)) + (M_{0,r}^{-})^2 (\varphi_{r,-}^{ab})^\dagger (x) \varphi_{r,-}^{ab}(x) \Big]
$$

where

$$
(M_{n,r}^{\pm})^2 = m_n^2 \pm \frac{4\pi m_r}{(2\pi R)^2 \mathcal{T}_2^{(r)}} = \frac{1}{(2\pi R)^2} \left[\sum_{s=1}^{3} \frac{2\pi |m_s|}{\mathcal{T}_2^{(s)}} (2N_s + 1) \pm \frac{4\pi m_r}{\mathcal{T}_2^{(r)}}
$$

- A normalization factor, in general moduli dependent, has been added.

- CIM determine it by requiring that the quadratic terms are canonically normalized.
- We will fix it by requiring the holomorphicity of the Yukawa couplings.

- Massless scalar only if the following condition is satisfied for $m_r > 0$ or $m_r < 0$:

$$\sum_{s=1}^{3} \frac{2\pi|m_s|}{\mathcal{T}_2^{(s)}} - \frac{4\pi|m_r|}{\mathcal{T}_2^{(r)}} = 0 \Longrightarrow \frac{1}{2}\sum_{s=1}^{3} \frac{|m_s|}{\mathcal{T}_2^{(s)}} - \frac{|m_r|}{\mathcal{T}_2^{(r)}} = 0$$

restauring $\mathcal{N} = 1$ supersymmetry

- The integral over the extra dimensions can be explicitly done:

$$Z \equiv \frac{1}{2g^2} \prod_{r=1}^{3} \left[(2\pi R)^2 \int d^2 z_r \sqrt{G^r} \right] \phi_0^{ba} \phi_0^{ab} = \frac{1}{2g^2} \times$$

$$\times \prod_{r=1}^{3} \left[\frac{(2\pi R)^2 \mathcal{T}_2^{(r)}}{(2|m_r|U_2^{(r)})^{1/2}} \right] = \frac{\mathrm{e}^{-\phi_{10}}}{8\pi} \prod_{r=1}^{3} \left[\left(\frac{T_2^{(r)}}{2U_2^{(r)}} \right)^{1/2} \left(\frac{T_2^{(r)}}{|m_r|} \right)^{1/2} \right]$$

- The action for the twisted fields become:

$$S_2^{\phi_0} = \int d^4x \sqrt{G_4} \sum_{r=1}^{3} Z_r$$
$$\times \Big[(D_\mu(\varphi_{r,+}^{ab})^\dagger(x))(D^\mu \varphi_{r,+}^{ab}(x)) + (M_{0,r}^{+})^2 (\varphi_{r,+}^{ab})^\dagger(x)\varphi_{r,+}^{ab}(x)$$
$$+ (D_\mu(\varphi_{r,-}^{ab})^\dagger(x))(D^\mu \varphi_{r,-}^{ab}(x)) + (M_{0,r}^{-})^2 (\varphi_{r,-}^{ab})^\dagger(x)\varphi_{r,-}^{ab}(x) \Big]$$

- But the holomorphic variables to be used in string theory and in supergravity are not the same.
- Those to be used in supergravity are:

$$s_2 = \mathrm{e}^{-\phi_{10}} \prod_{r=1}^{3} \mathcal{T}_2^{(r)} \quad ; \quad t_2^{(r)} = \mathrm{e}^{-\phi_{10}} \mathcal{T}_2^{(r)} \quad ; \quad u_2^{(r)} \equiv U_2^{(r)}$$

- We have to write Z in terms of the supergravity variables.

- After going to Einstein frame one gets the following Kähler metrics:

$$Z = \frac{N_\varphi^2}{2 s_2^{1/4}} \prod_{r=1}^{3} \left[\frac{1}{(2u_2^{(r)})^{1/2} (t_2^{(r)})^{1/4}} \left(\frac{T_2^{(r)}}{|m_r|} \right)^{1/2} \right]$$

- N_φ is determined by computing the Yukawa couplings and requiring that they are holomorphic.
- One obtains:

$$N_{\varphi_1} = \left(\frac{|m_1|}{T_2^{(1)}} \right)^{1/2}$$

- One gets:

$$Z_{ab}^{chiral} = \frac{1}{2 s_2^{1/4}} \prod_{r=1}^{3} \left[\frac{1}{(2u_2^{(r)})^{1/2} (t_2^{(r)})^{1/4}} \right] \left(\frac{\nu_1^{ab}}{\pi \nu_2^{ab} \nu_3^{ab}} \right)^{1/2} \; ; \; \pi \nu_r \equiv \frac{|m_r|}{T_2^{(r)}}$$

- The dependence on the magnetizations is consistent with previous stringy calculations for small ν

$$\left[\frac{\Gamma(1-\nu_1^{ab})}{\Gamma(\nu_1^{ab})}\frac{\Gamma(\nu_2^{ab})}{\Gamma(1-\nu_2^{ab})}\frac{\Gamma(\nu_3^{ab})}{\Gamma(1-\nu_3^{ab})}\right]^{1/2} \Longrightarrow \left(\frac{\nu_1^{ab}}{\nu_2^{ab}\nu_3^{ab}}\right)^{1/2}$$

- In the field theory limit ($\alpha' \to 0$)

$$\tan\pi\nu_r = \frac{|m_r|}{T_2^{(r)}} \Longrightarrow \pi\nu_r = \frac{|m_r|}{T_2^{(r)}} \quad ; \quad T_2^{(r)} = \frac{V_{T_2}}{(2\pi\sqrt{\alpha'})^2}$$

- But the rest of the dependence on the moduli, obtaiined indirectly with instanton calculations, has not yet been obtained in a complete stringy calculation.

- If one of the m_r's is vanishing and the other two are equal

$$\frac{|m_1|}{\mathcal{T}_2^{(1)}} = \frac{|m_2|}{\mathcal{T}_2^{(2)}} \quad ; \quad m_3 = 0 \ .$$

then we have two massless excitations corresponding to the two complex scalars of the hypermultiplet of $\mathcal{N} = 2$ supersymmetry.
- One gets for them the following effective action:

$$-\frac{1}{2g^2} \int d^4x \sqrt{G_4} \Big[N^2_{\varphi_1} (D_\mu \varphi^{ba}_{1,-}(x))(D^\mu \varphi^{ab}_{1,+}(x)) +$$
$$\times N^2_{\varphi_2} (D_\mu \varphi^{ba}_{2,-}(x))(D^\mu \varphi^{ab}_{2,+}(x)) \Big] \prod_{r=1}^{3} \left[(2\pi R)^2 \int d^2 z_r \sqrt{G^r} \right] \phi_0^{ba} \phi_0^{ab}$$

- But now the wave function contains only the Θ-functions corresponding to the first two tori, while the wave function along the third torus is just a constant.

- After going to Einstein frame one gets the following Kähler metrics:

$$
\begin{aligned}
Z_i^{hyper} &= \frac{e^{2\phi_4}}{2} e^{-\phi_{10}} N_i^2 T_2^{(3)} \prod_{r=1}^{2} \left[\left(\frac{T_2^{(r)}}{2\,U_2^{(r)}} \right)^{1/2} \left(\frac{T_2^{(r)}}{|m_r|} \right)^{1/2} \right] = \\
&= \frac{N_i^2}{2\left(4 u_2^{(1)} u_2^{(2)} t_2^{(1)} t_2^{(2)}\right)^{1/2}} \prod_{r=1}^{2} \left[\left(\frac{T_2^{(r)}}{|m_r|} \right)^{1/2} \right]
\end{aligned}
$$

- N_i is determined from the requirement that the Yukawa couplings are holomorphic:

$$
N_1 = N_2 = \left(\frac{|m_1|}{T_2^{(1)}} \right)^{1/2} = \left(\frac{|m_2|}{T_2^{(2)}} \right)^{1/2}
$$

- For the hypermultiplet one gets

$$Z_i^{hyper} = \frac{1}{2\left(4u_2^{(1)}u_2^{(2)}t_2^{(1)}t_2^{(2)}\right)^{1/2}}$$

- The dependence on the magnetization cancels as obtained with other methods!
- The previous approach can be trivially extended to the case of the adjoint scalars, where the wave-function is a constant and one gets:

$$-\frac{1}{2}\,\mathrm{e}^{-\phi_{10}}\,T_2^{(1)}T_2^{(2)}T_2^{(3)}\mathrm{e}^{2\phi_4}\int d^4x\sqrt{G_4}\left[\sum_{r=1}^{3}\frac{1}{T_2^{(r)}U_2^{(r)}}\partial^\mu\bar{\varphi}_r^a(x)\partial_\mu\varphi_r^a(x)\right]$$

- where (in terms of the ten-dimensional fields)

$$\varphi_r^a \equiv i\,\frac{\bar{U}C^a_{2r+2} - C^a_{2r+3}}{\sqrt{4\pi}}$$

- One can read the following Kähler metric: $Z_r = \frac{1}{t_2^{(r)}u_2^{(r)}}$

Field theory limit

- The field theory limit is obtained by sending $\alpha' \to 0$ keeping the physical volume V_{T^2} of the torus fixed.
- In this limit

$$\tan \pi\nu_r^{a,b} = \frac{m_r^{(a,b)}}{n_r^{(a,b)}\, T_2^{(r)}} = \frac{m_r^{(a,b)}}{n_r^{(a,b)}\, V_{T^2}^{(r)}}(2\pi\sqrt{\alpha'})^2 \sim \pi\nu_r^{a,b}$$

- The field theory limit corresponds to small values of $\nu_r^{(a,b)}$ and of $\nu_r \equiv \nu_r^{(a)} - \nu_r^{(b)}$!!
- What are the states that survive in this limit?
- The mass spectrum of the NS sector is given by:

$$M^2 = \frac{1}{\alpha'}\left[N_4^X + N_4^\psi + N_{comp.}^X + N_{comp}^\psi + \frac{1}{2}\sum_{r=1}^{3}\nu_r - \frac{1}{2}\right]$$

- Only those states for which $[\ldots] \sim [C + \nu] \sim \alpha' \to C = 0$.

- The only states are the following:

$$\psi^{(r)\dagger}_{1/2-\nu_r}\prod_{s=1}^{3}(a^{(s)\dagger}_{\nu_s})^{N_s}|0> \;\; ; \;\; \psi^{(r)\dagger}_{1/2+\nu_r}\prod_{s=1}^{3}(a^{(s)\dagger}_{\nu_s})^{N_s}|0>$$

- For them we get:

$$(M^{(r)}_{\pm})^2 = \frac{1}{\alpha'}\left[\frac{1}{2}\sum_{s=1}^{3}\nu_s(2N_s+1)\pm\nu_r\right]$$

- Since $(\nu_r \equiv \nu_r^{(a)}; \nu_r^{(b)} = 0)$

$$\nu_r = \frac{m_r}{\pi n_r T_2^{(r)}} = \frac{4\pi}{(2\pi R)^2}\frac{m_r}{n_r \mathcal{T}_2^{(r)}}\cdot\alpha' \quad ; \quad \frac{1}{T_2^{(r)}} = \frac{1}{\mathcal{T}_2^{(r)}}\cdot\frac{\alpha'}{R^2}$$

  R is an arbitrary dimensional quantity.

- For those states the dependence on α' cancels and we get

$$(M^{(r)}_{\pm})^2 = \frac{4\pi}{(2\pi R)^2}\left[\frac{1}{2}\sum_{s=1}^{3}\frac{m_s}{n_s\mathcal{T}_2^{(s)}}(2N_s+1)\pm\frac{m_r}{n_r\mathcal{T}_2^{(r)}}\right] \; ; \; \mathcal{T}_2 = \frac{V_{T^2}}{(2\pi R)^2}$$

Yukawa couplings

- The Yukawa couplings can be computed from the following tri-linear terms of the original Lagrangian:

$$S_3^{\Phi} = \frac{1}{2g^2} \int d^4x \sqrt{G_4} \int d^6y \sqrt{G_6} \sum_{n,m,l} \bar{\psi}_n^{ca} \gamma_{(4)}^5$$

$$\times \left[\varphi_{i,\,m}^{ab} \psi_l^{bc} \otimes (\eta_n^{ac})^\dagger \gamma_{(6)}^i \phi_m^{ab} \eta_l^{bc} - \varphi_{i,\,m}^{bc} \psi_l^{ab} \otimes (\eta_n^{ac})^\dagger \gamma_{(6)}^i \phi_m^{bc} \eta_l^{ab} \right]$$

- with the constraints:

$$I_r^{ab} + I_r^{bc} + I_r^{ca} = 0 \;\; ; \;\; r = 1, 2, 3$$

- Let us focus on the massless scalar relative to the first torus:

$$\frac{|I_1^{ab}|}{T_2^{(1)}} = \frac{|I_2^{ab}|}{T_2^{(2)}} + \frac{|I_3^{ab}|}{T_2^{(3)}}$$

and assume that

$$I_1^{ca} < 0 \;\; ; \;\; I_1^{ab} > 0 \;\; ; \;\; I_1^{bc} < 0$$

- One gets:

$$(S_3^{\Phi}) = \int d^4x \sqrt{G_4}\,\bar{\psi}^{ca}\,\gamma^5_{(4)}\varphi_1^{ab}\,\psi^{bc}\,Y^s$$

where

$$Y^s = \frac{\mathrm{e}^{-\phi_{10}}}{\sqrt{8\pi}}\sigma N_\varphi N_\psi N_\psi \prod_{r=1}^{3}\left[\frac{T_2^{(r)}}{\left(2U_2^{(r)}|I_r^{ab}|\chi_r^{ab}|I_r^{bc}|\chi_r^{bc}|I_r^{ca}|\chi_r^{ca}\right)^{1/2}}\right.$$
$$\left.\times\Theta\left[\begin{array}{c} 2\left(\frac{n'}{I_r^{ca}}+\frac{m'}{I_r^{bc}}+\frac{l'}{I_r^{ab}}\right) \\ 0 \end{array}\right](0|-I_r^{ab}I_r^{bc}I_r^{ca}U_f^{(r)})\right]$$

where

$$\begin{aligned}\chi_r^{ab} &= (1+sign(I_r^{bc}I_r^{ca}))/2\\ \chi_r^{bc} &= (1+sign(I_r^{ab}I_r^{ca}))/2\\ \chi_r^{ca} &= (1+sign(I_r^{bc}I_r^{ab}))/2\end{aligned}$$

- and

$$U_f^{(r)} = \begin{cases} U^{(r)} & \text{for} \quad sign(I^{ca} I^{bc} I^{ab}) < 0 \\ \bar{U}^{(r)} & \text{for} \quad sign(I^{ca} I^{bc} I^{ab}) > 0 \end{cases}$$

- The previous result is in fact completely general.
- Going to the Einstein frame and remembering the choice for the first torus one gets:

$$Y^E = \frac{\mathrm{e}^{K/2}}{\sqrt{8\pi}} \sigma N_{\varphi_1}^{ab} N_{\psi}^{ca} N_{\psi}^{bc} \left(\frac{T_2^{(1)}}{2 I_1^{ab}} \right)^{1/2} \left(\frac{T_2^{(2)}}{2 I_2^{ca}} \right)^{1/2} \left(\frac{T_2^{(3)}}{2 I_3^{bc}} \right)^{1/2} \times$$

$$\times \prod_{r=1}^{3} \left[\Theta \left[\begin{array}{c} 2 \left(\frac{n'}{I_r^{ca}} + \frac{m'}{I_r^{bc}} + \frac{l'}{I_r^{ab}} \right) \\ 0 \end{array} \right] (0| - I_r^{ab} I_r^{bc} I_r^{ca} U_f^{(r)}) \right]$$

- where

$$K = -\log s_2 - \sum_{r=1}^{3} \left[\log t_2^{(r)} + u_2^{(r)} \right]$$

- The requirement of supersymmetry in all three channels (ab, bc, ca) imposes for instance that

$$\chi_2^{bc} = \chi_3^{ca} = 0 \;\; ; \;\; \chi_3^{bc} = \chi_2^{ca} = 1$$

- Using these values we see that, if we choose the normalization factors as follows:

$$N_{\varphi_1}^{ab} = \left(\frac{|I_1^{ab}|}{T_2^{(1)}} \right)^{1/2} \;\; ; \;\; N_{\psi}^{ca} = \left(\frac{|I_2^{ca}|}{T_2^{(2)}} \right)^{1/2} \;\; ; \;\; N_{\psi}^{bc} = \left(\frac{|I_3^{bc}|}{T_2^{(3)}} \right)^{1/2}$$

  the Yukawa coupling becomes a holomorphic function of the moduli!

- Looking at the term with the scalar in the adjoint we get the following normalization factors for the fermions in the hypermultiplet:

$$N_{\psi_\uparrow} = N_{\psi_\downarrow} = \left(\frac{|I_1^{ab}|}{T_2^{(1)}} \right)^{1/2} = \left(\frac{|I_2^{ab}|}{T_2^{(2)}} \right)^{1/2}$$

Conclusions

- We have given a procedure for computing the Kähler metric of the various scalar fields: twisted, hypermultiplet and adjoint.
- But we have made two assumptions.
- The normalization factor contains only the minimal amount of terms that make the Yukawa couplings holomorphic.
- Our reasoning is based on a specific form of the scalar field.
- On the other hand, the presence of the normalization factor allows us to actually rescale the field with a quantity and at the same time rescale the normalization factor with the inverse quantity without changing the Kähler metrics and the Yukawa couplings.
- Therefore the presence of the normalization factor does not allow us to determine the absolute normalization of the scalar field.

- Let us find the relation between the field that we have used and the original ten-dimensional fields.
- One gets:

$$\varphi_{r\,-} = \sqrt{\frac{2\,U_2^{(r)}}{\mathcal{T}_2^{(r)}}}\varphi_{r\,z} = \frac{i}{\sqrt{2U_2^{(r)}\mathcal{T}_2^{(r)}}}(\bar{U}^{(r)}\varphi_{2r+2} - \varphi_{2r+3})$$

- If we compare the previous expression with the correspondent scalars in the adjoint, we see a factor $\sqrt{2U_2^{(r)}\mathcal{T}_2^{(r)}}$ not present in the adjoint.
- If we want to have a holomorphic relation we can eliminate this factor by including it in the normalization factor.
- There is still a bit of arbitrariness, but may be it is not relevant.