# On the spectrum of strings in $AdS_5 \times S^5$

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AdS/CFT:

progress largely using limited tools
of supergravity + probe actions
Need to understand quantum $AdS_5 \times S^5$ string theory

Problems for string theory:

- spectrum of states (energies/dimensions as functions of $\lambda$)
- construction of vertex operators: closed and open (?) string ones
- computation of their correlation functions
- expectation values of various Wilson loops
- gluon scattering amplitudes (?)
- generalizations to simplest less supersymmetric cases

............

## $AdS_5 \times S^5$

Recent remarkable progress in quantitative understanding
interpolation from weak to strong ‘t Hooft coupling
based on/checked by perturbative gauge theory (4-loop in $\lambda$)
and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$ ) “data”
and assumption of exact integrability
string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, C, m, ...) = \Delta(\lambda, C, m, ...)$$

$C$ - “charges” of $SO(2,4) \times SO(6)$: $S_1, S_2;\ J_1, J_2, J_3$
$m$ - windings, folds, cusps, oscillation numbers, ...
Operators: $\mathrm{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...)$

Solve susy 4-d CFT
= Solve superstring in R-R background (2-d CFT):
compute $E = \Delta$ for any $\lambda$ (and $C$,$m$)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

Last 7 years – remarkable progress:

"semiclassical" string states with large quantum numbers

dual to "long" gauge operators (BMN, GKP, ...)

$E = \Delta$ – same (in some cases !) dependence on $C, m, ...$

coefficients = interpolating functions of $\lambda$

Current status:

1. "Long" operators = strings with large quantum numbers:

asymptotic Bethe Ansatz (ABA) [Beisert, Eden, Staudacher 06]

firmly established (including non-trivial phase factor)

2. "Short" operators = general quantum string states

Partial progress based on impriving ABA by

"Luscher corrections" [Janik et al]

Attempts to generalize ABA to TBA [Arutyunov, Frolov 08]

Very recent (complete ?) proposal for underlying "Y-system"
[Gromov, Kazakov, Vieira 09]

To justify need first-principles understanding of quantum
$AdS_5 \times S^5$ superstring theory:
1. Solve string theory in $AdS_5 \times S^5$ on $R^{1,1}$
$\rightarrow$ relativistic 2d S-matrix (including dressing phase if needed);
asymptotic BA for the spectrum
2. Generalize to finite-energy closed strings – theory on $R \times S^1$
$\rightarrow$ TBA as for standard sigma models

Reformulation in terms of currents with Virasoro conditions solved
("Pohlmeyer reduction") seems promising approach
[Grigoriev, AT]

## String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [Metsaev, AT 98]

$$S = T \int d^2\sigma \Big[ G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x + \bar{\theta}\theta\bar{\theta}\theta\partial x\partial x + ... \Big]$$

tension $T = \frac{\mathrm{R}^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset $\sigma$-model (Luscher-Pohlmeyer 76)

also for $AdS_5 \times S^5$ superstring (Bena, Polchinski, Roiban 02)

Progress in understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04,...)

Computation of 1-loop quantum superstring corrections (Frolov, AT; Park, Tirziu, AT, 02-04, ...)

Quantum string results were used as input for 1-loop term
in strong-coupling expansion of the phase $\theta$ in BA
(Beisert, AT 05; Hernandez, Lopez 06)

Tree-level S-matrix of BMN states from $AdS_5 \times S^5$ GS string
agrees with limit of elementary magnon S-matrix
(Klose, McLoughlin, Roiban, Zarembo 06)

2-loop string corrections (Roiban, Tirziu, AT; Roiban, AT 07)
2-loop check of finiteness of the GS superstring;
agreement with BA
– implicit check of integrability of quantum string theory
– non-trivial confirmation of BES exact phase in BA
(Basso, Korchemsky, Kotansky 07)

Key example of weak-strong coupling interpolation:

Spinning string in $AdS_5$

Folded spinning string in flat space:

$X_1 = \epsilon \ \sin\sigma \ \cos\tau, \quad X_2 = \epsilon \ \sin\sigma \ \sin\tau$

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 = -dt^2 + dX_i dX_i$$

$$t = \epsilon\tau \ , \quad \rho = \epsilon \sin\sigma \ , \quad \phi = \tau$$

If tension $T = \frac{1}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}$

energy $E = \epsilon\sqrt{\lambda}$ and spin $S = \frac{\epsilon^2}{2}\sqrt{\lambda}$ satisfy Regge relation:

$$\mathcal{E} = \sqrt{2\sqrt{\lambda} S}$$

$AdS_5$:

(de Vega, Egusquiza 96; Gubser, Klebanov, Polyakov 02)

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\phi^2$$

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma)$$

$$\rho'^2 = \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho, \qquad 0 < \rho < \rho_{\rm max}$$

$$\coth \rho_{\rm max} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

$\epsilon$ measures length of the string

$$\sinh \rho = \epsilon \, {\rm sn}(\kappa \epsilon^{-1} \sigma, \ -\epsilon^2)$$

periodicity in $0 \leqslant \sigma < 2\pi$

$$\kappa = \epsilon \ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2)$$

classical energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ and spin $S = \sqrt{\lambda}\mathcal{S}$

$$\mathcal{E}_0 = \epsilon \ {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2), \qquad \mathcal{S} = \frac{\epsilon^2 \sqrt{1+\epsilon^2}}{2} \ {}_2F_1(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2)$$

solve for $\epsilon$ as in flat space – get analog of Regge relation

$$\mathcal{E}_0 = \mathcal{E}_0(\mathcal{S}) \ , \qquad E_0 = \sqrt{\lambda} \, \mathcal{E}_0(\frac{S}{\sqrt{\lambda}})$$

Flat space – AdS interpolation:

$\mathcal{E}_0 \sim \sqrt{\mathcal{S}}$ at $\mathcal{S} \ll 1$ , $\quad \mathcal{E}_0 \sim \mathcal{S}$ at $\mathcal{S} \gg 1$

Novel AdS "Long string" limit: $\epsilon \gg 1$, i.e. $\mathcal{S} \gg 1$

$$\mathcal{E}_0 = \mathcal{S} + \frac{1}{\pi} \ln \mathcal{S} + \ldots$$

$\mathcal{S} \to \infty$: ends of string reach the boundary ($\rho = \infty$)

solution drastically simplifies

$$t = \kappa\tau, \quad \phi \approx \kappa\tau, \quad \rho \approx \kappa\sigma \, , \quad \kappa \sim \epsilon \sim \ln \mathcal{S} \to \infty$$

string length is infinite, $R \times R$ effective world sheet

$E = S$ from massless end points at AdS boundary (null geodesic)

$E - S = \frac{\sqrt{\lambda}}{\pi} \ln S$ from tension/stretching of the string

$\rho = \kappa\sigma + \ldots$, $\mathcal{S} \sim e^{2\kappa}$,

$\kappa \sim \ln \mathcal{S}$=length of the string:

$\frac{1}{\mathcal{S}^n} \sim e^{n\kappa}$ – finite size corrections

For $\mathcal{S} \to \infty$ can compute quantum superstring corrections to $E$
remarkably, they respect the $S + \ln S$ structure:
string solution is homogeneous $\to$ const coeffs
$\kappa \sim \ln \mathcal{S} \to \infty$ is "volume factor"
Semiclassical string theory limit

$$1.\ \lambda \gg 1\ , \qquad \mathcal{S} = \frac{S}{\sqrt{\lambda}} = \text{fixed}, \qquad 2.\ \mathcal{S} \gg 1$$

$$E = S + f(\lambda) \ln S + \ldots\ ,$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi} \left[ 1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \ldots \right]$$

$a_n$–Feynmann graphs of 2d CFT – $AdS_5 \times S^5$ superstring
$a_1 = -3 \ln 2$: Frolov, AT 02
$a_2 = -K$: Roiban, AT 07
$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ = 0.915 (2-loop $\sigma$-model integrals)

Gauge theory: dual operators – minimal twist ones

$\mathrm{Tr}(\Phi D_+^S \Phi), \quad \Delta - S - 2 = O(\lambda)$

Remarkably, same $\ln S$ asymptotics of anomalous dimensions on gauge theory side [symmetry argument: Alday, Maldacena]

Perturbative gauge theory limit:

$$1.\ \lambda \ll 1\ , \quad S = \text{fixed}; \qquad 2.\ S \gg 1$$

$$\Delta - S - 2 = f(\lambda) \ln S + ...$$

$$f(\lambda \ll 1) = c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + c_4\lambda^4 + ...$$

$$= \frac{1}{2\pi^2}\left[\lambda - \frac{\lambda^2}{48} + \frac{11\lambda^3}{2^8 \times 45} - \left(\frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6}\right)\frac{\lambda^4}{2^7} + ...\right]$$

$c_n$ are given by Feynmann graphs of 4d CFT – N=4 SYM

$c_3$: Kotikov, Lipatov, et al 03;

$c_4$: Bern, Czakon, Dixon, Kosower, Smirnov 06;

The two limits are formally different
but for leading $\ln S$ term that does not appear to matter $\rightarrow$
single $f(\lambda)$ provides smooth interpolation
from weak to strong coupling

remarkably, both expansions are reproduced from one
Beisert-Eden-Staudacher integral equation for $f(\lambda)$
[strong coupling expansion:
numerical – Benna, Benvenuti, Klebanov, Scardicchio 07;
analytic – Basso, Korchemsky, Kotansky 07;
Kostov, Serban, Volin 08]

exact expression for $f(\lambda)$ from BES equation?
true meaning of non-perturbative $e^{-\frac{1}{2}\sqrt{\lambda}}$ terms
in strong-coupling expansion?

One direction: study in detail semiclassical string states
for various values of parameters including $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections
Principles of comparison: gauge states vs string states
1. look at states with same global $SO(2,4) \times SO(6)$ charges
e.g., $(S, J)$ – "SL(2) sector" – $\mathrm{Tr}(D_+^S \Phi^J)$
$J$=twist=spin-chain length
2. assume no "level crosing" while changing $\lambda$
min/max energy $(S, J)$ states should be in correspondence

Gauge theory:
$\Delta \equiv E = S + J + \gamma(S, J, m, \lambda)$ , $\quad \gamma = \sum_{k=1}^{\infty} \lambda^k \gamma_k(S, J, m)$
$m$ stands for other conserved charges labelling states
(e.g., winding in $S^1 \subset S^5$ or number of spikes in $AdS_5$)
fix $S$, $J$, ... and expand in $\lambda$; may then expand in large/small $S$, $J$, ...

String theory:
$E = S + J + \gamma(\mathcal{S}, \mathcal{J}, m, \sqrt{\lambda})$ ,

$\gamma = \sum_{k=-1}^{\infty} \frac{1}{(\sqrt{\lambda})^k} \widetilde{\gamma}_k(\mathcal{S}, \mathcal{J}, m)$

$\mathcal{S} = \frac{S}{\sqrt{\lambda}}\ , \quad \mathcal{J} = \frac{J}{\sqrt{\lambda}}\ , \quad m$

- semiclassical parameters fixed in the $\frac{1}{\sqrt{\lambda}}$ expansion

Various possible limits:

(i) BMN-like "fast-string" limit – "locally-BPS" long oprators

GT: $J \gg 1, \quad \frac{S}{J}$=fixed, $m$=fixed

ST: $\mathcal{J} \gg 1, \quad \frac{\mathcal{S}}{\mathcal{J}}$=fixed, $m$=fixed

direct agreement of first few orders in $\frac{1}{\mathcal{J}}$

(including 1- and 2-loop string corrections)

to 1- and 2-loop gauge theory spin chain results

including $1/J$ and $1/J^2$ finite size corrections

(Frolov, AT 03; Beisert, Minahan, Staudacher, Zarembo 03; ...)

"non-renormalization" due to susy (and structure)

no interpolation functions of $\lambda$, no need to resum $J$ dependence

$$E = S + J + \frac{\lambda}{J}\Big[h_1(\tfrac{S}{J}, m) + \tfrac{1}{J} h_2(\tfrac{S}{J}, m) + ...\Big] + ...$$

captured by effective Landau-Lifshitz model

on both string and spin chain side

need interpolation functions at higher orders (dressing phase)

(ii) “Slow Long strings” – long non-BPS operators like $\mathrm{Tr}(\Phi D_+^S \Phi)$

GT: $\ln S \gg J \gg 1$

ST: $\ln \mathcal{S} \gg \mathcal{J},\ \mathcal{J} = 0$ or $\mathcal{J}$=fixed

$E = S + f(\lambda) \ln S + ...$

$S$ dependence is same but need an interpolatig function

$f(\lambda \gg 1) = a_1 \sqrt{\lambda} + ... , \qquad f(\lambda \ll 1) = c_1 \lambda + ...$

(iii) “Fast Long strings”

GT: $S \gg J \gg 1,\, j \equiv \frac{J}{\ln S}$=fixed

ST: $\mathcal{S} \gg \mathcal{J} \gg 1,\, \ell \equiv \frac{\mathcal{J}}{\ln \mathcal{S}}$=fixed = $\frac{j}{\sqrt{\lambda}}$

GT: $E = S + f(j, \lambda) \ln S + ...$

$f = a_1(\lambda) j + a_2(\lambda) j^3 + ...$

ST: $E = S + f(\ell, \sqrt{\lambda}) \ln \mathcal{S} + ...$

$f = \sqrt{\lambda}\sqrt{1+\ell^2} + (c_1 + c_2 \ell^2 \ln \ell + ...) + \frac{1}{\sqrt{\lambda}}(c_3 \ell^2 \ln^2 \ell + ...) + ...$

[Belitsky, Gorsky, Korchemsky 06; Frolov, Tirziu, AT 06;

Alday, Maldacena 07, Freyhult, Rej, Staudacher 07;

Roiban, AT 07; Kostov, Serban, Volin 08; Basso, Korchemsky 08;

Gromov 08, Fioravanti et al 08, ...]

need a resummation in both $\lambda$ and $\ell$ (or $j$) to match

general situation – G and S limits do not commute

## Large $S$ expansion for spinning string:

1. subleading terms in large-spin expansion?
compare to gauge theory – also partially controlled by
functional relation and reciprocity?
2. dependence on spin parameter is same (i.e. coefficients are
interpolating functions as in cusp anomaly case) or we
do need to resum also the spin dependence to compare?
3. formal small-spin limit – may shed light on dimensions of short
operators at strong coupling if (?) limits commute
[Beccaria, Forini, Tirziu, AT 08; Tirziu, AT 08]

## Subleading terms in large $S$ expansion

string has large but finite length: does not reach boundary
$E_0 = \sqrt{\lambda}\,\mathcal{E}(\mathcal{S})$: expand in large $\mathcal{S}$

$$E_0(\mathcal{S} \gg 1) = S + a_0 \ln \mathcal{S} + a_1 + \frac{1}{\mathcal{S}}(a_2 \ln \mathcal{S} + a_3)$$

$$+\frac{1}{\mathcal{S}^2}(a_4 \ln^2 \mathcal{S} + a_5 \ln \mathcal{S} + a_6) + O(\frac{\ln^3 \mathcal{S}}{\mathcal{S}^3})$$

$a_0 = \frac{\sqrt{\lambda}}{\pi}, \;\; a_1 = \frac{\sqrt{\lambda}}{\pi} \ln(8\pi) - 1, \; ....$

Coefficients of $\frac{\ln^k \mathcal{S}}{\mathcal{S}^k}$ terms happen to be related to coefficient of $\ln \mathcal{S}$ as suggested by "functional relation" (Basso, Korchemsky 06)

$$E - S = f(E + S) = a_0 \ln(S + \frac{1}{2} a_0 \ln S + ...) + ...$$

$$a_2 = \frac{1}{2} a_0^2, \qquad a_4 = -\frac{1}{8} a_0^3, ...$$

Simple explanation:

look at near boundary limit where for large $\mathcal{S}$

string end moves moves along nearly null line at the boundary:

pp-wave limit: cusp anomaly as "pp-wave" anomaly

(Kruczenski, AT 08)

pp-wave limit effectively establishes contact with
collinear conformal group in the boundary theory
(Ishizeki, Kruczenski, Titziu, AT 08)

Some of coefficients in large $S$ expansion are related
due to reciprocity property in gauge theory
true also at strong coupling
(Basso, Korchemski 06; Beccaria, Forini, Tirziu, AT 08)

Dimensions of short operators = quantum string states:

progress in understanding spectrum of conformal dimensions
of planar $N = 4$ SYM or spectrum of strings in $AdS_5 \times S^5$
based on (partly proved/checked) assumption of
quantum integrability
Spectrum of states with large quantum numbers –
solution of ABA equations
key example: cusp anomaly function
Recent proposal of how to extend this to "short" states
with any quantum numbers – TBA or "Y-system" approach
so far not checked/compared to direct quantum string results

Aim: compute leading $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ correction to dimension of
"lightest" massive string state
dual to Konishi operator in SYM theory
– data for checking future (numerical) prediction of "Y-system"

Konishi operator:

a family of operators related by susy – same anomalous dimension

lowest canonical dimension examples:

$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1, 2, 3, \quad \Delta = 2 + \gamma\,, \quad \gamma = O(\lambda)$

$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector $\Delta = 4 + \gamma$

$\mathrm{Tr}\Phi_1 D_+^2 \Phi_1$ in $sl(2)$ sector $\Delta = 4 + \gamma$

special case:

does not mix with others, eigen-state of anom. dim. matrix

with lowest eigenvalue

Weak coupling expansion: $g^2 = \frac{\lambda}{(4\pi^2)}, \quad \lambda = g_{\rm YM}^2 N_c$

$$\gamma = 12g^2 - 48g^4 + 336g^6 + [-2496 + 576\zeta(3) - 1440\zeta(5)]g^8 + ....$$

[4-loop results with wrapping:

Fiamberti, Santambrogio, Sieg, Zanon 08;

Bajnok, Janik; Velizhanin 08]

Finite radius of convergence ($N_c = \infty$) – if we could sum up and then re-expand at large $\lambda$ – what to expect?

As discussed below:

$$\lambda \gg 1 : \qquad \Delta(\Delta - 4) = 4\sqrt{\lambda} + a + O(\tfrac{1}{\sqrt{\lambda}})$$

$$\Delta = 2 + 2\sqrt{\sqrt{\lambda}}\Big[1 + \frac{a+4}{8\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

$$a = \text{first correction to mass of string state}$$

weak coupling = perturbative gauge theory:
operators built out of free fields, canonical dimension (and susy) – constraints on mixing
operators of different canonical dimension do not mix in gauge perturbation theory;
strong coupling = perturbative string theory:
string states built out of "flat-string" oscillators

large degeneracy of mass spectrum

how one interpolates from small to large $\lambda$ ?

states from different flat-space levels do not mix in string pert.theory

AdS/CFT duality suggests that dual string state is

"lightest" massive type IIB string state

at large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$

– small string at the center of $AdS_5$ – nearly flat space

flat case: $\alpha' m^2 = 4(n-1), \ \ n = \frac{1}{2}(N + \bar{N}) = 1, 2, ...$

$n = 1$ massless (supergravity = BPS) level

$n = 2$ is the first massive level (many states, highly degenerate)

l.c. vacuum $|0>$: $(8+8)^2 = 256$ states

first excited level $[(a^i_{-1} + S^a_{-1})|0>]^2 = [(8+8) \times (8+8)]^2$

in $SO(9)$ reps:

$([2,0,0,0] + [0,0,1,0] + [1,0,0,1])^2 = (44 + 84 + 128)^2$

curved background will lift degeneracy in mass

state with lightest mass at 1-st excited level
should correspond to Konishi op. (and its susy descendents)

Strategy: collect information about mass shifts of
different states at first massive string level

dimension of such state in $AdS_5$ (with fixed $n$)

$[-\nabla^2 + m^2]\Phi + ... = 0$

$\Delta(\Delta - 4) = (m\mathrm{R})^2 + O(\alpha') = 4(n-1)\frac{\mathrm{R}^2}{\alpha'} + O(\alpha')$

$\Delta = 2 + \sqrt{(m\mathrm{R})^2 + 4 + O(\alpha')}$

$$\Delta(\lambda \gg 1) = \sqrt{4(n-1)\sqrt{\lambda}} + ...$$

[Gubser, Klebanov, Polyakov 98]

first massive level:

$n = 2: \qquad \Delta = 2\sqrt{\sqrt{\lambda}} + ...$

How to compute strong-coupling corrections for short strings?

strong-coupling expansion for *massive* string states:
can use near-flat-space expansion
label states as in flat space: discrete set of oscillator states
"non-intersection principle" (Polyakov 01):
no level crossing for states with same quantum numbers
as $\lambda$ changes from strong to weak coupling

Possible approaches:

(i) semiclassical approach:
identify short string state as a small-spin limit of semiclassical string state
– reproduce the structure of strong-coupling corrections to short operators
[Gubser,Klebanov,Polyakov 02; Frolov,AT 02,03;Tirziu,AT 08]

(ii) vertex operator approach:
use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in anomalous dimension of corresponding vertex operator
[Polyakov 01; AT 03]

(iii) space-time effective action approach:
use near flat space expansion and NSR vertex operators
to reconstruct $\alpha'$ corrections to corresponding
massive string state equation of motion
[Burrington, Liu 05]

(iv) "light-cone" quantization approach:
start with light-cone gauge $AdS_5 \times S^5$ string action
and compute corrections to energy of
corresponding flat-space oscillator string state
[Metsaev, Thorn, AT 00; Roiban, AT ]

## Semiclassical expansion: spinning strings

classical string solution with energy $E$ and charge (spin) $J$

expand $E$ in $\alpha' \to 0$ or large $\sqrt{\lambda}$ with $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ kept fixed

$$E = E(\frac{J}{\sqrt{\lambda}}, \sqrt{\lambda}) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + ...$$

in "short" string limit $\mathcal{J} \ll 1$

$$\mathcal{E}_n = \sqrt{c_0 \mathcal{J}}\ (a_{0n} + a_{1n}\mathcal{J} + a_{2n}\mathcal{J}^2 + ...)$$

expansion valid for $\sqrt{\lambda} \gg 1$ and $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ fixed, i.e. $J \sim \sqrt{\lambda} \gg 1$

imagine we knew all terms in this expansion – could express $\mathcal{J}$ in terms of $J$, fix $J$ to finite value and re-expand in $\sqrt{\lambda}$

$$E = \sqrt{c_0\sqrt{\lambda}J}\Big[a_{00} + \frac{a_{10}J + a_{01}}{\sqrt{\lambda}} + \frac{a_{20}J^2 + a_{11}J + a_{02}}{(\sqrt{\lambda})^2} + ...\Big]$$

$a_{kn}$ coefficients of $n$-loop string corrections

If set $J$ to finite value to trust the coefficient of $\frac{1}{(\sqrt{\lambda})^n}$

need to know the coefficients of up to $n$-loop terms

knowledge of classical $a_{10}$ and 1-loop $a_{01}$ coefficient

is sufficient to fix $\frac{1}{\sqrt{\lambda}}$ term $E$

but to fix the $\frac{1}{(\sqrt{\lambda})^2}$ term need also 2-loop coefficient $a_{02}$

[cf. "long/fast string" expansion $\mathcal{J} \gg 1$ [Frolov, AT 03]:

for fixed $J$ the tension $\sqrt{\lambda}$ appeared in positive powers –

strong coup. expansion at fixed $J$ – need to resum the series]

Example 1: short folded string in $AdS_5$

($J \to S$ =spin in $AdS_5$) [Tirziu, AT 08]

$$c_0 = 2, \quad a_{00} = 1, \quad a_{10} = \frac{3}{8}, \quad a_{01} = 0.227(?)$$

$$a_{20} = -\frac{21}{128}, \quad a_{11} = -\frac{1219}{576} + \frac{3}{2}\ln 2 - \frac{3}{4}\zeta(3), \ ...$$

[ $a_{01} = -0.25...$ (numerical result of Gromov 08), =- 1/4 ]

Example 2: small circular string in $S^5$ with $J_1 = J_2 = J$:
[Frolov, AT 03]
remarkable feature: classical energy same as in flat space:
$a_{01} = a_{02} = ... = a_{0k} = 0$

$$c_0 = 4, \quad a_{00} = 1, \quad a_{10} = 0, \quad a_{01} = -\frac{1}{2}$$

$$a_{20} = 0\ , \quad a_{02} = 0, \quad a_{11} = -\frac{3}{4} - \frac{3}{2}\zeta(3), \ ...$$

knowledge of 1-loop semiclassical string correction –
allows to predict leading strong-coupling correction
to energy for finite $J$

$$E = 2\sqrt{\sqrt{\lambda}J}\,\Big[1 - \frac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

not quite right at any $J$ – misses possible finite integer shift
of $J$ (and of $E$) due to exact zero-mode quantization
– will need to compare this with vertex operator approach

Some details:

Konishi state: $J_1 = J_2 = 2$

try represent it by "short" classical string with same charges

flat space $R_t \times R^4$: circular string solution ($\sigma \in (0, 2\pi)$)

$$x_1 + ix_2 = \ a\, e^{in(\tau+\sigma)} \ , \qquad x_3 + ix_4 = \ a\, e^{in(\tau-\sigma)}$$

$$E = \sqrt{\tfrac{4}{\alpha'} nJ}, \qquad J = \tfrac{na^2}{\alpha'}$$

this solution can be directly embedded into

$R_t \times S^5$ in $AdS_5 \times S^5$: [Frolov, AT 03]

string is on *small* sphere inside $S^5$ $X_1^2 + ... + X_6^2 = 1$ (e.g. $n = 1$)

$$t = \kappa\tau \ , \qquad X_1 + iX_2 = \tfrac{\sin\gamma_0}{\sqrt{2}}\ e^{i(\tau+\sigma)},$$
$$X_3 + iX_4 = \tfrac{\sin\gamma_0}{\sqrt{2}}\ e^{i(\tau-\sigma)} \ , \qquad X_5 + iX_6 = \cos\gamma_0$$
$$\mathcal{J} = \mathcal{J}_1 = \mathcal{J}_2 = \tfrac{1}{2}\sin^2\gamma_0 \ , \ \ \mathcal{E}^2 = \kappa^2 = 2\sin^2\gamma_0 = 4\mathcal{J}$$

Remarkably, as in flat space

$$E = \sqrt{\lambda}\mathcal{E} = \sqrt{4\sqrt{\lambda}J}\ , \qquad J = \sqrt{\lambda}\mathcal{J}$$

[cf. another (unstable) branch of $J_1 = J_2$ solution with $\mathcal{J} > \frac{1}{2}$:
$E_0 = \sqrt{J^2 + \lambda} = \sqrt{\lambda}(1 + \frac{J^2}{2\sqrt{\lambda}} + ...)$ ]

1-loop quantum string correction to the energy:
sum of bosonic and fermionic fluctuation frequencies ($n = 0, 1, 2, ...$)
Bosons (2 massless + massive):

$$AdS_5 : \quad 4\times \qquad \omega_n^2 = n^2 + 4\mathcal{J}$$
$$S^5 : \qquad 2\times \qquad \omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}) \pm 2\sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}^2}$$

Fermions:

$$4\times \qquad \omega_{n\pm}^{2\,f} = n^2 + 1 + \mathcal{J} \pm \sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}}$$

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Big[4\omega_n + 2(\omega_{n+} + \omega_{n-}) - 4(\omega_{n}{}_{+}^{f} + \omega_{n}{}_{-}^{f})\Big]$$

expand in small $\mathcal{J}$ and do sums –

compute finite coefficients (UV divergences cancel)

normalize to flat space result in the $\mathcal{J} \to 0$ limit:

in flat space theory is gaussian – trivial 1-loop correction

$$E_1 = \frac{1}{\sqrt{\mathcal{J}}}\Big[ -\mathcal{J} - \frac{3}{2}(1+2\zeta(3))\mathcal{J}^2 - \frac{1}{4}\big(5+6\zeta(3)+30\zeta(5)\big)\mathcal{J}^3 + \ldots\Big]$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}J}\Big[1 - \frac{1}{2\sqrt{\lambda}} - \frac{3J}{4\lambda}(1+2\zeta(3)) + ...\Big]$$

If we could interpolate to $J_1 = J_2 = 2$ that would suggest for Konishi state ($2J = J_1 + J_2 \to J_1 + J_2 - 2 = 2$)

$$E = 2\sqrt{\sqrt{\lambda}}\,\Big[1 - \frac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

Similar expressions found for short folded string in $AdS_5$ [Tirziu,AT 08; Gromov 08]

$$E = \sqrt{2\sqrt{\lambda}S}\,\Big[1 + \frac{a_0 S + a_1}{\sqrt{\lambda}} + ...\Big]\,, \quad a_0 = \frac{3}{8}\,, \quad a_1 = -\frac{1}{4}$$

and for folded string with $J_1 = J_2$ in $S^5$:
[Beccaria, Tirziu, AT 08]

$$E = \sqrt{4\sqrt{\lambda} J} \left[ 1 + \frac{a_0 J + a_1}{\sqrt{\lambda}} + ... \right] , \quad a_0 = \frac{3}{8} , ...$$

Aim:
compare this with dimensions of the corresponding quantum states
(eigen-states, not coherent states)

## Dimensions of quantum string states from target space anomalous dimension operator

Flat space: $k^2 = m^2 = \frac{4(n-1)}{\alpha'}$
e.g. leading Regge trajectory
$(a_1^\dagger \bar{a}_1^\dagger)^{S/2}|0>$ or $(\partial x \bar{\partial} x)^{S/2} e^{ikx}, \ \ n = S/2$

Mass spectrum in (weakly) curved background?
solve marginality (1,1) conditions on vertex operators
deformed by curved background: determine anomalous
dimension operator $(L_0 + \bar{L}_0)$ and diagonalize it

Example of scalar anomalous dimension operator $\widehat{\gamma}(G, B)$:
acts on $T(x) = \sum c_{n...m} x^n ... x^m$ or on coefficients $c_{n...m}$
differential operator in target space
found from $\beta$-function for the corresponding perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2z[(G_{mn} + B_{mn})(x)\partial x^m \bar{\partial} x^n + T(x)]$$
$$\beta_T = -2T - \tfrac{\alpha'}{2}\, \widehat{\gamma}\, T + O(T^2)$$
$$\widehat{\gamma} = \Omega^{mn} D_m D_n + ... + \Omega^{m...k} D_m ... D_k + ...$$
$$\Omega^{mn} = G^{mn} + p_1 \alpha' R^{mn} + p_2 \alpha' H^m_{\ kl} H^{nkl} + O(\alpha'^3)$$

$p_1 = 0,\ \ p_2 = -\frac{1}{4}$ in DR with minimal subtraction

$\Omega^{\cdots} \sim \alpha'^n R^p_{....} H^q_{...}$

for $H_{mnk} = 0$: to 3-loop order $\widehat{\gamma} = D^2 + ...$

Solve

$$-\,\widehat{\gamma}\, T + m^2 T = 0, \qquad\qquad m^2 = -\frac{4}{\alpha'}$$

i.e. diagonalize $\widehat{\gamma}$ – find anomalous dimension spectrum:

generalization of $\alpha' k^2 = -4$ in flat space

similar approach for massless (graviton, ...) and massive states e.g. $\beta^G_{mn} = \alpha' R_{mn} + ...$

gives Lichnerowitz operator as anomalous dimension operator

$$R_{mn}(G+h) = R_{mn} + \frac{1}{2}\widehat{\gamma}^{kl}_{mn} h_{kl} + O(h^2)$$
$$(\widehat{\gamma}h)_{mn} = -D^2 h_{mn} + 2R_{mknl}h^{kl} - 2R_{k(m}h^k_{n)}$$

Equivalent approach to find $\widehat{\gamma}$:

reconstruct quadratic in $T$ effective action in curved background

from tachyon-graviton amplitudes in flat space

$$\int d^D x [T(m^2 - \partial^2)T + hT\partial\partial T + ...] \to \int d^D x [T(m^2 - D^2)T + ...]$$

Effective superstring action for graviton

$$S = \int d^D x \sqrt{g}[R + \alpha'^3 RRRR + ...]$$

$$(\widehat{\gamma} h)_{mn} = -D^2 h_{mn} + 2R_{mknl} h^{kl} + O(\alpha'^3)$$

Massive string states in curved background:

$$\int d^D x \sqrt{g} [\Phi_{...}(m^2 - D^2 + X)\Phi_{...} + ...]$$

$$m^2 = \frac{4}{\alpha'}(n-1) \, , \qquad X = R_{....} + O(\alpha')$$

strategy: reconstruct from string scattering amplitudes
using known vertex operators in flat space

Apply this to the case of $AdS_5 \times S^5$ background

$$R_{mn} - \frac{1}{96}(F_5 F_5)_{mn} = 0, \quad R = 0 \, , \quad F_{mnklp} F^{mnklp} = 0$$

leading-order term in $X$ should vanish for scalar state
prediction – leading $\alpha'$ correction to scalar string mass =0 (?!)
i.e. for a scalar (singlet) state should have

$$[-D^2 + m^2 + O(\tfrac{1}{\sqrt{\lambda}})]\Phi \ = 0 \, ,$$

$$\Delta_{(n)} = 2 + \sqrt{4(n-1) + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$

$$\Delta_{(n=2)} = 2 + 2\sqrt{\sqrt{\lambda}}\Big[1 + \frac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

natural guess for the leading terms in strong-coupling expansion

of singlet Konishi state dimension

What about non-singlet Konishi states ?

– they should have the same dimension

$\mathrm{Tr}[\Phi_1, \Phi_2]^2$ corresponds to $SO(6)$ (2,2,0) state $J_1 = J_2 = 2$

tensor wave function $\Phi_{mn;kl}$

or vertex operator like (see below)

$\sim N_+^{-\Delta} \partial n_x \bar{\partial} n_x \partial n_y \bar{\partial} n_y$

$S^5$: $\quad n_a n_a = 1, n_x = n_1 + i n_2, \quad n_y = n_3 + i n_4$

$AdS_5$: $\quad N_+ = N_0 + i N_5, \quad N_+ N_- - N_k N_k = 1$

$\mathrm{Tr}(\Phi_1 D_+^2 \Phi_1)$ should correspond to state with spins $S = J = 2$

In more detail:

# Effective action approach

derive equation of motion for a massive string field
in a background from quadratic effective action $S$
reconstructed from flat-space S-matrix
Example: totally symmetric NS-NS 10-d tensor state
corresponding to leading Regge trajectory in flat space

generic weakly curved background with 5-form flux
find quadratic terms in $S$
from correlators of flat-space NSR vertex operators
[Burrington, Liu 05]

symmetric massive string field $\Phi_{\mu_1\ldots\mu_{2n}}$ in metric+RR background

$$L = [R - \tfrac{1}{2\cdot 5!}F_5^2 + O(\alpha'^3)]$$
$$-\tfrac{1}{2}(D_\mu\Phi D^\mu\Phi + m^2\Phi^2) + \sum_{k\geq 1}(\alpha')^{k-1}\Phi X_k(R, F_{(5)}^2, D^2)\Phi + \ldots$$

assumption: $\alpha' n R \ll 1, \quad i.e. \quad n \ll \sqrt{\lambda}$
small massive string in the middle of $AdS_5$:
near-flat-space expansion should be applicable

$X_k$ in general is mixing matrix
assume that totatlly symmetric tracells transverse $\Phi$
does not mix with other states at same level
(justified at least for $AdS_5 \times S^5$ background)
minimal $S$ reproducing on-shell eqs. for $\Phi$ to leading $\alpha'$ order

$$[-D^2 + m^2 + X_1 + O(\alpha')]\Phi_{\mu_1 \cdots \mu_{2n}} = 0$$

ignore terms vanishing on-shell :
$R_{\mu\nu} \sim (F_5 F_5)_{\mu\nu}, \quad F_5 F_5 = 0, \quad R = 0$
Then:

$$\begin{aligned}\Phi X_1 \Phi = c_1 \Phi_{\mu_1 \mu_2 \cdots \mu_{2n}} R^{\mu_1 \nu_1 \mu_2 \nu_2} \Phi_{\nu_1 \nu_2}{}^{\mu_3 \cdots \mu_{2n}} \\ + c_2 \Phi_{\mu_1 \cdots \mu_{2n}} F^{\mu_1 \nu_1 \alpha_3 \cdots \alpha_5} F^{\mu_2 \nu_2}{}_{\alpha_3 \cdots \alpha_5} \Phi_{\nu_1 \nu_2}{}^{\mu_3 \cdots \mu_{2n}} \\ + c_3 \Phi_{\mu_1 \mu_2 \cdots \mu_{2n}} F^{\mu_1 \alpha_2 \cdots \alpha_5} F^{\nu_1}{}_{\alpha_2 \cdots \alpha_5} \Phi_{\nu_1}{}^{\mu_2 \cdots \mu_{2n}}\end{aligned}$$

$$c_i = c_i(n) = ?$$

to fix $X_1$ compute interactions of $\Phi$ with graviton and RR field:

3-point NS-NS scattering amplitude to fix $\Phi R_{....}\Phi$

4-point NS-RR scattering amplitude to fix $\Phi F_5 F_5 \Phi$

in flat space:

states on leading Regge trajectory in type IIB NS-NS sector

$\alpha' m^2 = 4(n-1)$ and spin $S = 2n$

$$V = \zeta_{\mu_1\cdots\mu_{2n}}(\partial X^{\mu_1}\cdots\partial X^{\mu_n}\bar{\partial}X^{\mu_{n+1}}\cdots\bar{\partial}X^{\mu_{2n}} + \text{fermions})\, e^{ik\cdot X}$$

$\Phi$-$h_{\mu\nu}$-$\Phi$ function: [Giannakis, Liu, Porrati, 98]

closed string vertex= (left) x (right) parts (in -1 picture)

$-k^2 = m^2 = \frac{4(n-1)}{\alpha'}$

$$V_{-1} = \zeta_{\mu_1\cdots\mu_n} e^{-\phi}\psi^{\mu_1}\partial X^{\mu_2}\cdots\partial X^{\mu_n} e^{ik\cdot X}, \quad V_0 = \xi_\mu\big(\partial X^{\mu_1} + i\tfrac{\alpha'}{2}\psi^{\mu_1} k\cdot\psi\big)e^{ik\cdot X}$$

$$\zeta_{\mu_1\cdots\cdots\mu_n} = \text{tot.symm.}, \quad k^{\mu_i}\zeta_{\mu_1\cdots\mu_i\cdots\mu_n} = 0, \quad \eta^{\mu_i\mu_j}\zeta_{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_n} = 0$$

result:

$$c_1 = n^2$$

$n = 1$: agrees with Lichnerowitz operator

$\Phi$-$F_5$-$F_5$-$\Phi$ function [Burrinton, Liu 05]

Ramond-Ramond vertex: in - 1/2 picture (left half)

$$V_{-1/2} = u_{\dot\alpha} S^{\dot\alpha}_{-1/2} e^{ik\cdot X},$$
$$V_{1/2} = [\partial X^\mu + i\tfrac{\alpha'}{2} k\cdot\psi\psi^\mu] u_{\dot\alpha} \Gamma_\mu{}^{\dot\alpha}{}_\beta S^\beta_{1/2} e^{ik\cdot X}$$

extract leading-order part in $\alpha'$: 0-momentum part in $F_5$

subtract massless exchanges, extract contact terms

assume $\Phi$ does not mix with massive RR fields

result:

$$c_2 = -\frac{1}{4!}\,, \qquad c_3 = -\frac{1}{4\times 4!}$$

check: reproduces eq for graviton perturbation around

$R_{\mu\nu} - \frac{1}{4\times 4!}(F_5 F_5)_{\mu\nu} = 0$

$c_3 F_5 F_5$ term appears from $R_{\mu\nu}$ term in Lichnerowitz operator

$$R_{mn}(g+h) = -\frac{1}{2}(\Delta_L h)_{mn} + O(h^2)$$
$$(\Delta_L h)_{mn} = -D^2 h_{mn} + 2R_{manb}h^{ab} - 2R_{a(m}h^a_{n)}$$

$c_3$ term actually cancels against $c_2$ term

$AdS_5 \times S^5$ background

let $M, N, \ldots = 0, 1, ...9, \quad \mu, \nu \ldots$ in $AdS_5$ and $m, n, \ldots$ in $S^5$

$$R_{\mu\nu} = -\frac{4}{\mathrm{R}^2} g_{\mu\nu}, \qquad R_{mn} = \frac{4}{\mathrm{R}^2} g_{mn},$$
$$F_{\mu\nu\rho\lambda\sigma} = \frac{4}{\mathrm{R}} \epsilon_{\mu\nu\rho\lambda\sigma}, \qquad F_{mnpqr} = \frac{4}{\mathrm{R}} \epsilon_{mnpqr},$$
$$R_{\mu\nu\rho\sigma} = -\frac{1}{\mathrm{R}^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \qquad R_{mnpq} = \frac{1}{\mathrm{R}^2}(g_{mp}g_{nq} - g_{mq}g_{np})$$

the two $F_5 F_5$ terms cancels against each other

only the full Riemann tensor term survives

i.e. $\Phi$-$F_5$-$F_5$-$\Phi$ contact terms do not contribute
to the leading mass shift in a maximally symmetric background:

$$F^{MN\cdots}F^{PQ}{}_{\dots} + \tfrac{1}{4}F^{M\cdots}F^{P}{}_{\dots} \equiv \mathcal{T}^{MNPQ} + \tfrac{1}{4}g^{PQ}\mathcal{T}^{MLN}{}_{L}$$

vanishes for $\mathcal{T}_{MNPQ} \sim g_{MP}g_{NQ} - g_{MQ}g_{NP}$
contracted between symmetric tracefree fields $\Phi$:
i.e. Ramond-Ramond background can be essentially ignored...
suggests that to this order fermions are not relevant
apart from making $AdS_5 \times S^5$ background consistent solution
(e.g., satisfaction of BPS conditions)

$$\begin{aligned} L &= \tfrac{1}{2}\Phi_{M_1\cdots M_{2n}}(-D^2 + m^2)\Phi^{M_1\cdots M_{2n}} \\ &\quad + \frac{n^2}{\mathrm{R}^2}\left(\Phi_{\mu_1\mu_2 M_3\cdots M_{2n}}\Phi^{\mu_1\mu_2 M_3\cdots M_{2n}} - \Phi_{m_1 m_2 M_3\cdots M_{2n}}\Phi^{m_1 m_2 M_3\cdots M_{2n}}\right) + \dots \end{aligned}$$

background is direct product – can consider a particular tensor
with $S$ indices in $AdS_5$ and $K$ indices in $S^5$:

end up with anomalous dimension operator

$$[-D^2 + (m^2 + \frac{K^2 - S^2}{2\mathrm{R}^2})]\Phi = 0\ , \qquad D^2 = D^2_{AdS_5} + D^2_{S_5}$$
$$m^2 = \tfrac{4}{\alpha'}(n-1) = \tfrac{2}{\alpha'}(S+K-2), \quad 2n = S+K$$

symmetric transverse traceless tensors – highest-weight state –
in terms of Young labels $(\Delta, S, 0; J, K, 0)$, $\quad J \geqslant K$
extract $AdS_5$ radius and set $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$

$$[-D^2_{AdS_5} + M^2]\Phi = 0$$
$$M^2 = 2\sqrt{\lambda}(S+K-2) + \frac{1}{2}(K^2 - S^2) + J(J+4) - K$$

For symmetric traceless rank $S$ tensor in $AdS_5$:
same by analytic continuation from $SO(6)$ [Metsaev 98]

$$-D^2_{AdS_5} + M^2 \quad \to \quad -\Delta(\Delta - 4) + M^2 + S$$
$$\Delta = 2 + \sqrt{M^2 + S + 4}$$
$$= 2 + \sqrt{2\sqrt{\lambda}(S+K-2) + \frac{1}{2}(S+K-2)(K-S) + J(J+4) + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$

BPS cases: $J = K + J', \quad J' = 0, 1, 2, ...$

$S = 2, K = 0, \Delta = 4 + J'$;

$K = 2, S = 0,\ J = 2 + J', \Delta = 6 + J'$

$S = K = 1,\ \Delta = 5 + J'$

[generalizations: Bianchi, Morales, Samtleben 03]

$S = 0,\ J = K$ case: $(J, J, 0)$ state

$$\Delta = 2 + \sqrt{2\sqrt{\lambda}(J-2) + \frac{3}{2}J^2 + 3J + 4}$$

large $J$ limit:

$$\Delta_{J\gg 1} = \sqrt{2\sqrt{\lambda}J}\ (1 + \frac{3}{8}\frac{J}{\sqrt{\lambda}} + ...)$$

agrees with expansion of energy of
classical folded string on $S^5$ with $J_1 = J_2 = K \gg 1$

$K = 0,\ S \neq 0$ case:

$$\Delta = 2 + \sqrt{2\sqrt{\lambda}(S-2) - \frac{1}{2}S(S-2) + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$

for large $S$

$$\Delta = 2 + \sqrt{2\sqrt{\lambda}}(1 - \frac{S}{8\sqrt{\lambda}} + ...)$$

[does not match folded string expression
$E = \sqrt{2\sqrt{\lambda}}(1 + \frac{3}{8}\frac{S}{\sqrt{\lambda}} + ...)$
folded string in $AdS_5$ is represented by a different state ?]

To summarize: string states in $\mathrm{AdS}_5 \times S^5$ labeled by
$SU(2,2|4) \supset SO(2,4) \times SO(6)$ quantum numbers $(E, S_1, S_2; J_1, J_2, J_3)$
condition of marginality of corresponding (1,1) operator

$$
\begin{aligned}
0 = & -\sqrt{\lambda}(S + K - 2) \\
& + \frac{1}{2}[\Delta(\Delta - 4) + \frac{1}{2}S(S-2) - \frac{1}{2}K(K-2) - J(J+4)] + O(\tfrac{1}{\sqrt{\lambda}})
\end{aligned}
$$

symmetry: analytic continuation between $AdS_5$ and $S^5$
$\Delta \leftrightarrow -J, \ K \leftrightarrow S$

Implications for Konishi state dimension ?
states from same first massive level
$S = 0, \ K = 4$:

$$
\Delta = 2 + 2\sqrt{\sqrt{\lambda} + 10 + O(\tfrac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt{\sqrt{\lambda}}(1 + \frac{5}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2}))
$$

$S = 1,\ K = 3$:

$$\Delta = 2 + 2\sqrt{\sqrt{\lambda} + \tfrac{27}{4} + O(\tfrac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt{\sqrt{\lambda}}(1 + \frac{27}{8\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2}))$$

$S = 2,\ K = J = 2$:

$$\Delta = 2 + 2\sqrt{\sqrt{\lambda} + 4 + O(\tfrac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt{\sqrt{\lambda}}(1 + \frac{2}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2}))$$

Konishi operator should have lowest dimension...

$S = 4,\ K = J = 0$:

$$\Delta = 2 + 2\sqrt{\sqrt{\lambda} + O(\tfrac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt{\sqrt{\lambda}}(1 + O(\tfrac{1}{(\sqrt{\lambda})^2}))$$

cf. a scalar state at level 2 that gets no leading correction to mass

$$\Delta = 2 + \sqrt{4\sqrt{\lambda} + 4 + O(\tfrac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt{\sqrt{\lambda}}(1 + \frac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2}))$$

how to reproduce same dim. for other states in Konishi multiplet?

## Vertex operator approach

[Polyakov 01; AT 03]

superstring theory in $AdS_5 \times S^5$ :

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma [\partial N_a \bar{\partial} N^a + \partial n_k \bar{\partial} n_k + \text{fermions} ]$$

$$N_+ N_- - N_u N_u^* - N_v N_v^* = 1 \ , \qquad n_x n_x^* + n_y n_y^* + n_z n_z^* = 1$$

$$N_\pm = N_0 \pm i N_5, \quad N_u = N_1 + i N_2, \ ..., \ n_x = n_1 + i n_2, ...$$

construct marginal (1,1) operatots in terms of $N_a$ and $n_k$

Scalar vertex operators in Poincare patch:

$(-D^2 + m^2) T = 0, \qquad \widehat{V} = \int d^2\xi \ T(\widehat{x}(\xi))$

$T(\widehat{x}) = \int d^4x' \ K(\widehat{x}, x') T_0(x') \ , \ \ \widehat{x} = (z, x), \ \ ds^2 = \frac{dz^2 + dx_\mu dx_\mu}{z^2}$

$T_0(x)$ – “source” function at the boundary of AdS

$K$ = Dirichlet bulk-to-boundary propagator,

$$K = c(\Delta) [\frac{z}{z^2 + (x - x')^2}]^\Delta \ , \qquad K_{z \to 0} \to \delta^{(4)}(x - x')$$

$\Delta$ is determined from $0 = \gamma = -\frac{1}{2}\alpha' m^2 + \frac{1}{2\sqrt{\lambda}}\Delta(\Delta - 4) + ....$

vertex operator that enters correlation functions –

integrated over world sheet and depending on bndry point

$$\widehat{V}(x) = \int d^2\xi \, \mathrm{V}(\xi), \ \ \mathrm{V} = K(\widehat{x}(\xi), x), \ \ \widehat{V}(T_0) = \int d^4x \, \widehat{V}(x) \, T_0(x)$$

AdS/CFT correspondence:

string generating functional $Z[T_0(x)]$

= gauge-theory generating functional $< e^{\int \, d^4x \ T_0(x)\mathcal{O}(x)} >$,

$\mathcal{O}(x)$ = gauge operator with same quant. numbers and dim.

vertex operator for dilaton-type sugra mode (chiral primary)

$$\mathrm{V}_J(\xi) = \ (N_+)^{-\Delta} \ (n_x)^J \ (-\partial N_M \bar{\partial} N_M + \partial n_k \bar{\partial} n_k + \text{fermions})$$

$N_+ \equiv N_0 + iN_5 = \frac{1}{z}(z^2 + x_m x_m) \sim e^{it}$

$n_x \equiv n_1 + in_2 \sim e^{i\varphi}$

rotation along the big circle of $S^5$

localize at the boundary – form linear superposition:

$$\widehat{V}_J(x) = \int d^2\xi \, \mathrm{V}_J(x(\xi) - x, z(\xi), \varphi(\xi))$$

arbitrary $x$ or 4-momentum

$$< \widehat{V}_J(x)\widehat{V}_{-J}(x') > \sim \ |x - x'|^{-\Delta}$$

determine $\Delta = \Delta(J)$ in expansion in inverse string tension

$$0 = \gamma = 2 - 2 + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - J(J + 4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

$$\Delta = 4 + J + O(\tfrac{1}{\sqrt{\lambda}})$$

should be no corrections to all orders – BPS state

cf. vertex operator for bosonic string state
on leading Regge trajectory in flat space

$$\mathrm{V}_S(\xi) = e^{-iEt}\big(\partial X \bar{\partial} X\big)^{S/2}$$

$X = x_1 + i x_2, \qquad \bar{X} = x_1 - i x_2$

marginality condition

$$\gamma = 0 = 2 - S - \frac{1}{2}\alpha' E^2 = 0 \,, \qquad \text{i.e.} \qquad \alpha' E^2 = 2(S-2)$$

candidate operators for states on leading Regge trajectory:

$$\mathrm{V}_J(\xi) = (N_+)^{-\Delta}\big(\partial n_x \bar{\partial} n_x\big)^{J/2} \,, \qquad n_x \equiv n_1 + i n_2$$

$$\mathrm{V}_S(\xi) = (N_+)^{-\Delta}\big(\partial N_u \bar{\partial} N_u\big)^{S/2} \,, \qquad N_u \equiv N_1 + i N_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

How these mix with operators with same

quantum numbers and canonical dimension?

in general $\big(\partial n_x \bar{\partial} n_x\big)^{J/2}$ mixes with

$$(n_x)^{2p+2q}(\partial n_x)^{J/2-2p}(\bar{\partial} n_x)^{J/2-2q}(\partial n_m \partial n_m)^p(\bar{\partial} n_k \partial n_k)^q$$

$p, q = 0, ..., J/4\ , \quad m, k = 1, ..., 6$

$(N_+)^{-\Delta}\big(\partial N_u \bar\partial N_u\big)^{S/2}$ mixes with

$$N_+^{-\Delta-p-q} N_x^{p+q} (\partial N_+)^p (\partial N_x)^{S/2-p} (\bar\partial N_+)^q (\bar\partial N_x)^{S/2-q} + O(\partial N_a \partial N_a \bar\partial N_b \bar\partial N_b)$$

$p, q = 0, ..., S/4\ , \quad a, b = 0, 1, ...5$

true vertex operators = eigenstates of anomalous dimension matrix

are particular linear combinations

Recall:

in general $S = \frac{1}{\pi\alpha'} \int d^2\xi\ G_{mn}(x) \partial x^m \bar\partial x^n$ perturbed by

$V(f) = f_{m_1 ... m_J}(x) \partial^{k_1} x^{m_1} .... \bar\partial^{k_h} x^{m_J}$

compute the renormalization of $f_{m_1 ... m_j}$ and set $\beta_f = \widehat{\gamma} f + ...$=0

$\widehat{\gamma} f = [2 - J + \frac{1}{2}\alpha' D^2 + \sum c_k \alpha'^k (R....)^n ... D^p] f = 0$

diagonalize "anomalous dimension" operator

Solving for $f$ = finding eigenvalues and eigen-vectors

of anomalous dimension operator

but form of $\widehat{\gamma}$ for generic $f$ and $G$ is not known
even to leading (1-loop) order in $\alpha'$
(with exceptions of WZW models or plane-wave models)
not able to use universal expression for $\widehat{\gamma}$ –
need to calculate anomalous dimensions from "first principles".

use global coordinates with linearly realized symmetry:
e.g. for $S^5 = SO(6)/SO(5)$

$$S = \frac{\sqrt{\lambda}}{\pi} \int d^2\xi \, \partial n_m \bar{\partial} n_m \ , \qquad n_m n_m = 1$$

$$\dot{\rm g} = -\epsilon {\rm g} + 4{\rm g}^2 + 4{\rm g}^3 + ... \ , \qquad {\rm g} \equiv \frac{1}{\sqrt{\lambda}} = \frac{\alpha'}{{\rm R}^2} \ , \qquad \epsilon = d - 2$$

running is cancelled if embedded into $AdS_5 \times S^5$ string theory
for states on leading Regge trajectory (no $\partial^k n$, $k > 1$)

$$O_{\ell,s} = f_{k_1...k_\ell m_1...m_{2s}} n_{k_1} ... n_{k_\ell} \partial n_{m_1} \bar{\partial} n_{m_2} ... \partial n_{m_{2s-1}} \bar{\partial} n_{m_{2s}}$$

their renormalization studied before [Wegner 90]

renormalization of composite operators to leading order in $\frac{1}{\sqrt{\lambda}}$

use "pairing rules" (and ignore "on-shell" operators):

$< AB >=< A > B + A < B > + < A, B >$

$< A, B >= \int d^2\xi d^2\xi' < n_k(\xi), n_m(\xi') > \frac{\delta A}{\delta n_k(\xi)} \frac{\delta B}{\delta n_m(\xi')}$

$< A(n) >= \frac{1}{2} \int d^2\xi d^2\xi' < n_k(\xi), n_m(\xi') > \frac{\delta^2 A}{\delta n_k(\xi)\delta n_m(\xi')}$, etc.

$$< n_k >= -\frac{5}{2} I n_k \ , \ \ < n_k, n_l >= -I(n_k n_l - \delta_{kl}) \ , \qquad I = -\frac{1}{2\pi\epsilon} \to \infty$$

$$< n_k, \partial n_l >= -I \partial n_k n_l \ , \ \ < n_k, \bar{\partial} n_l >= -I \bar{\partial} n_k n_l \ ,$$

$$< \partial n_k, \partial n_l >= I n_k n_l \partial n_m \partial n_m \ , \qquad < \bar{\partial} n_k, \bar{\partial} n_l >= I n_k n_l \bar{\partial} n_m \bar{\partial} n_m \ ,$$

$$< \partial n_k, \bar{\partial} n_l >= -I(\bar{\partial} n_k \partial n_l - \delta_{kl} \partial n_m \bar{\partial} n_m)$$

$$< (\partial n_k \bar{\partial} n_k) >= 0, \ \ < (\partial n_k \partial n_k) >= -4I \partial n_k \partial n_k \ , \qquad < (\bar{\partial} n_k \bar{\partial} n_k) >= -4I \bar{\partial} n_k \bar{\partial} n_k$$

simplest case:

$f_{k_1...k_\ell} n_{k_1}...n_{k_\ell}$ with traceless $f_{k_1...k_\ell}$ – mapped into itself has same anom. dim. $\gamma$ as its highest-weight representative

$$V_J = (n_x)^J$$

$$\gamma = 2 - \frac{1}{2\sqrt{\lambda}}[5J + J(J-1)] + O(\tfrac{1}{(\sqrt{\lambda})^2}) = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

scalar spherical harmonic that solves Laplace eq. on $S^5$

similar for $AdS_5$ or $SO(2,4)$ model:
replacing $n_x^J$ and $\partial n_m \bar{\partial} n_m$ with $N_+^{-\Delta}$ and $\partial N^a \bar{\partial} N_a$, with $J = -\Delta$ and g $= \frac{1}{\sqrt{\lambda}} \to -\frac{1}{\sqrt{\lambda}}$

e.g. dimension of $n_x^J \partial n_m \bar{\partial} n_m$: $\gamma = -\frac{1}{2\sqrt{\lambda}} J(J+4) + O(\frac{1}{(\sqrt{\lambda})^2})$

dimension of $N_+^{-\Delta} \partial N^a \bar{\partial} N_a$: $\gamma = \frac{1}{2\sqrt{\lambda}} \Delta(\Delta - 4) + O(\frac{1}{(\sqrt{\lambda})^2})$.

the number of $\partial n_k \bar{\partial} n_k$ factors never increases

can be used as quantum number

to characterise leading term in eigen-operator

example of scalar higher-level operator:

$$N_+^{-\Delta}[(\partial n_k \bar{\partial} n_k)^r + ...]$$

[Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$\gamma = -2(r-1) + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) + 2r(r-1)]$$
$$+ \frac{1}{(\sqrt{\lambda})^2}[\tfrac{2}{3}r(r-1)(r-\tfrac{7}{2}) + 4r] + ...$$

$r = 1$ is BPS:

fermionic contributions should make $r = 1$ exact zero of $\gamma$

$r = 2$: 1-st massive level – candidate for Konishi state

$$\Delta(\Delta - 4) = 4\sqrt{\lambda} - 4 + O(\tfrac{1}{\sqrt{\lambda}}) \, , \quad \Delta = 2 + 2\sqrt{\sqrt{\lambda}}\,[1 + O(\tfrac{1}{(\sqrt{\lambda})^2})]$$

same as $S=4, K=0$ state above (!)

still for a scalar operator expect no
leading correction to $\widehat{\gamma}=-\frac{1}{2}D^2$
fermionic contribution should cancel 1-loop mass shift $r(r-1)$?!
if that happens

$$\Delta(\Delta-4)=4\sqrt{\lambda}+O(\tfrac{1}{\sqrt{\lambda}})\,, \quad \Delta=2+2\sqrt{\sqrt{\lambda}}\,[1+\frac{1}{2\sqrt{\lambda}}+O(\tfrac{1}{(\sqrt{\lambda})^2})]$$

states of higher dimension:

$$(\partial n_k \partial n_k \bar{\partial} n_m \bar{\partial} n_m)^{r/2} \quad : \qquad \gamma=2-2r-\frac{4r}{\sqrt{\lambda}}+O(\tfrac{1}{(\sqrt{\lambda})^2})$$

$r=2$ – first massive level – gives positive shift
of string mass (above candidate Konishi value)

Examples of operators with spin in $S^5$:

$$N_+^{-\Delta}[(\partial n_x \bar{\partial} n_x)^{J/2} + ...]$$

$$\gamma(\Delta, J) = 2 - J + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}J(J + 10)] + O(\frac{1}{(\sqrt{\lambda})^2})$$

inclusion of fermions should shift $J(J+10) \to J(J-2)$

two spins $J, K$ in $S^5$:

$$O_{K,J} = N_+^{-\Delta} \sum_{u,v=0}^{K/2} c_{uv} M_{uv}$$

$$M_{uv} \equiv n_y^{J-u-v} n_x^{u+v} (\partial n_y)^u (\partial n_x)^{K/2-u} (\bar{\partial} n_y)^v (\bar{\partial} n_x)^{K/2-v}$$

highest and lowest eigen-values of 1-loop anom. dim. matrix

$$\gamma_{min} = 2 - K + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}K(K + 10) - J(J + 4) - 2JK] + O(\frac{1}{(\sqrt{\lambda})^2})$$

$$\gamma_{max} = 2 - K + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}K(K + 6) - J(J + 4)] + O(\frac{1}{(\sqrt{\lambda})^2})$$

fermions may again alter terms linear in $K$
to make $K = 2$ the zero of $\gamma$ (BPS)

$K = 4$: same level as Konishi state
– identify operators with right representations
[R.Roiban, AT, in progress]

**Light-cone quantization approach**

may be next time...