Wilson loops and Amplitudes in N=4 SYM

Claude Duhr

In collaboration with V. Del Duca and V. A. Smirnov.

September 15, 2010
HP2.3, Florence

Why planar N=4 SYM..?

- Why are we interested in planar N=4 Super-Yang-Mills?

  In the end, the world is not N=4 SYM, so we should rather concentrate on QCD...

Why planar N=4 SYM..?

- Why are we interested in planar N=4 Super-Yang-Mills?
  In the end, the world is not N=4 SYM, so we should rather concentrate on QCD...
- But in QCD, life is (too) hard...
- Aim: Find a 'simpler' gauge theory, that can act as a toy model to explore the structure of gauge theory amplitudes to higher loop orders.

Why planar N=4 SYM..?

- N=4 planar SYM is such a simpler gauge theory!
  - ➡ It is conformal to all orders in perturbation theory.
  - ➡ AdS/CFT correspondence might even give some insight into the strongly coupled sector of the theory.
  - ➡ N=4 SYM amplitudes are part of QCD amplitudes, e.g., at one-loop level:

$$A_n^{\mathrm{YM}} = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1} + A_n^{\mathrm{scalar}}$$

- A lot of new developments were made in the last few years, and the field is developing very fast!

Outline

- Several intriguing conjectures/observations in N=4 SYM:
  - ➡ ABDK/BDS ansatz.
  - ➡ MHV amplitude - Wilson loop duality.
  - ➡ Computation of two-loop remainder functions.

The ABDK/BDS ansatz

- Anastasiou, Bern, Dixon and Kosower (ABDK) formulated a conjecture for a generic two-loop MHV amplitude in N=4 SYM:

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon),$$

- Bern, Dixon and Smirnov (BDS) extended this conjecture to all loop orders, by exponentiating the one-loop amplitude:

$$M_n(\epsilon) = 1 + \sum_{l=1}^{\infty} a^l\, M_n^{(l)}(\epsilon) = \exp \sum_{l=0}^{\infty} a^l \left[f^{(l)}(\epsilon)\, M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right],$$

The ABDK/BDS ansatz

	n=4	n=5	n=6
l=2			
l=3			

The ABDK/BDS ansatz

	n=4	n=5	n=6
l=2	✓		
l=3	✓		

[ABDK; BDS]

The ABDK/BDS ansatz

	n=4	n=5	n=6
l=2	✓	✓ (num.)	
l=3	✓		
	[ABDK; BDS]	[Bern, Czakon, Kosower, Roiban, Smirnov]	

The ABDK/BDS ansatz

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		
	[ABDK; BDS]	[Bern, Czakon, Kosower, Roiban, Smirnov]	[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]

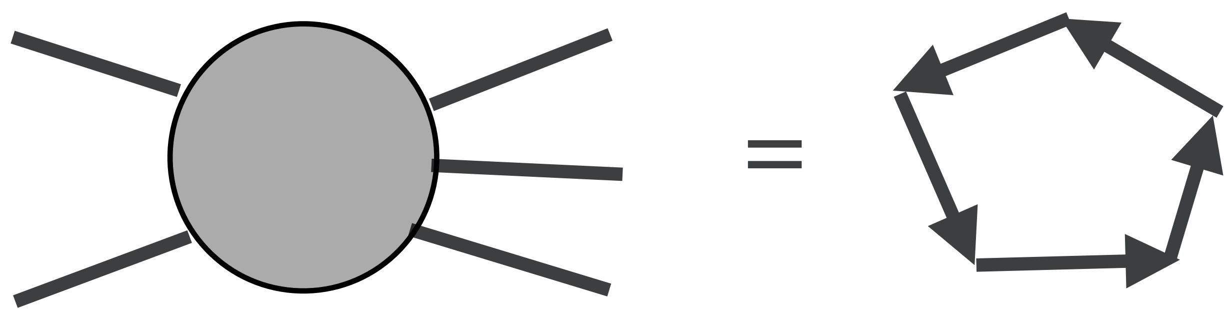
- What goes wrong for n = 6 ..?
- The answer comes from the Wilson loop!

Wilson loops in N=4 SYM

- Definition of a Wilson loop:

$$W[\mathcal{C}_n] = \mathrm{Tr}\,\mathcal{P}\exp\left[ig\oint \mathrm{d}\tau\,\dot{x}^\mu(\tau)A_\mu(x(\tau))\right]$$

- It is conjectured that Wilson loop along an n-edged polygon is equal to an n-point MHV scattering amplitude:



$$p_i = x_{i,i+1} = x_i - x_{i+1}$$

[Alday, Maldacena;
Drummond, Korchemsky, Sokatchev]

- Proven analytically at one-loop for arbitrary n, and at two-loops for n = 4, 5, 6. [Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, Spence]

Wilson loops in N=4 SYM

- Wilson loops possess a conformal symmetry, and it was shown that a solution to the corresponding Ward identities is the BDS ansatz, e.g., at two-loops,

[Drummond, Henn, Korchemsky, Sokatchev]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + \mathcal{O}(\epsilon)\,,$$

Wilson loops in N=4 SYM

- Wilson loops possess a conformal symmetry, and it was shown that a solution to the corresponding Ward identities is the BDS ansatz, e.g., at two-loops,

[Drummond, Henn, Korchemsky, Sokatchev]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} \; + R_n^{(2)}(u_{ij}) + \mathcal{O}(\epsilon)\,,$$

- ... but we can always add a arbitrary function of conformal invariants and we still obtain a solution to the Ward identities!

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

The breakdown of BDS

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		

The breakdown of BDS

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		

No non trivial conformal cross-ratios,

$$R_4^{(l)} = R_5^{(l)} = 0.$$

The breakdown of BDS

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		

No non trivial conformal cross-ratios,

$$R_4^{(l)} = R_5^{(l)} = 0.$$

There are three non trivial cross ratios:

$$u_1 = \frac{s_{12}\, s_{45}}{s_{123}\, s_{345}}, \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{123}\, s_{234}},$$

$$u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}},$$

Strong coupling

- At strong coupling, the AdS/CFT machinery was used to compute some special cases of the remainder function
  - ➡ for six edges, in 3+1 dimensions when all cross ratios are equal

$$R(u,u,u) = -\frac{\pi}{6} + \frac{1}{3\pi}\phi^2 + \frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right)$$

[Alday, Gaiotto, Maldacena]

  - ➡ for eight edges, in 1+1 dimensions

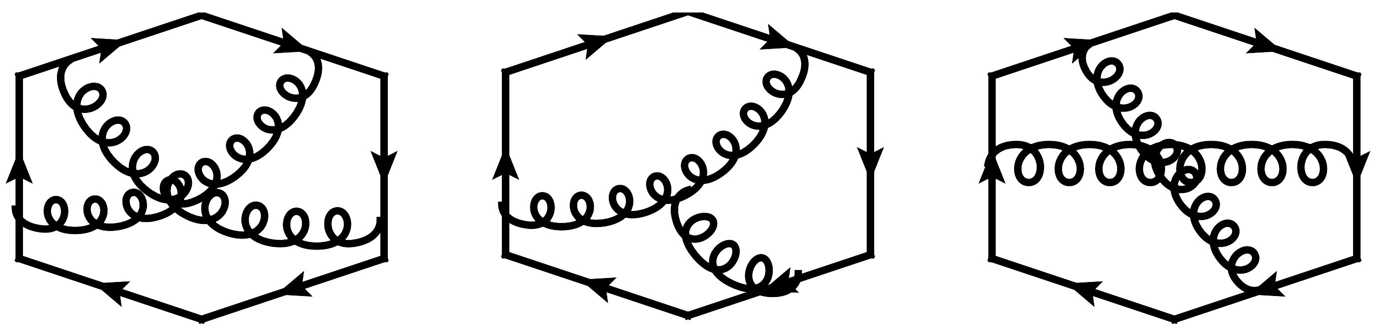
$$R^{\text{strong}}_{8,WL} = -\frac{1}{2}\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^+}\right) + \frac{7\pi}{6}$$

$$+\int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{|m|\,\sinh t}{\tanh(2t+2i\phi)}\ln\left(1+e^{-2\pi|m|\cosh t}\right)$$

[Alday, Maldacena]

Weak coupling

- Anastasiou, Brandhuber, Heslop, Khoze, Spence and Travaglini worked out the two-loop Wilson loop diagrams:



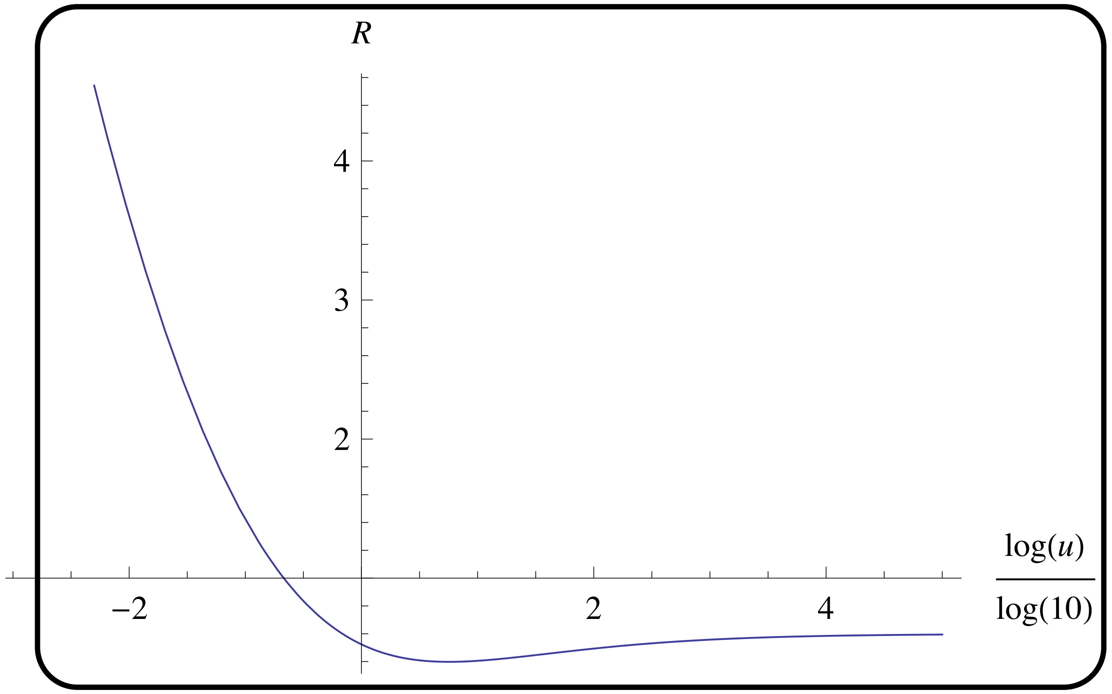
...

- Each of these diagrams is an integral, similar to a Feynman parameter integral.

- Numerical evaluations of the integrals allow to compare to the strong coupling answer.

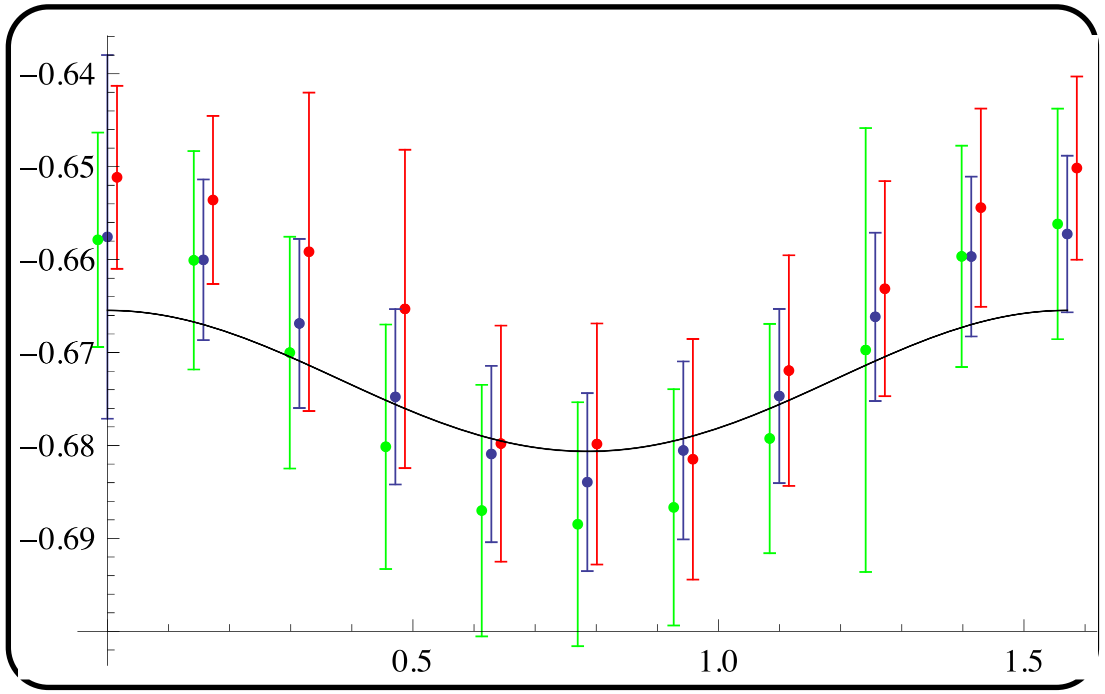
Weak coupling vs. strong coupling

- Hexgon



[Alday, Gaiotto, Maldacena]

- Octagon



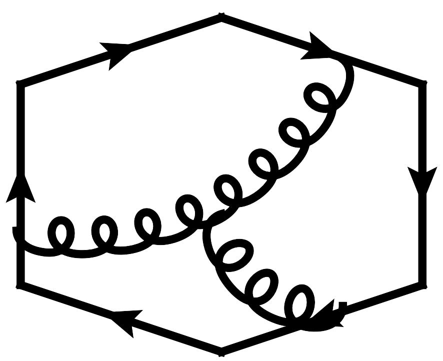
[Brandhuber, Heslop, Khoze Spence, Travaglini]

- Could it be that the strong coupling result is equal to the weak coupling result???

- Only analytic results at weak coupling can tell...

Weak coupling

- For n = 6, many of the integrals can be computed explicitly, but one is particularly 'hard':



$$f_H(p_1,p_2,p_3;Q_1,Q_2,Q_3)$$
$$:= \frac{\Gamma(2-2\epsilon_{\mathrm{UV}})}{\Gamma(1-\epsilon_{\mathrm{UV}})^2}\int_0^1\Big(\prod_{i=1}^3 d\tau_i\Big)\int_0^1\Big(\prod_{i=1}^3 d\alpha_i\Big)\delta(1-\sum_{i=1}^3\alpha_i)\;(\alpha_1\alpha_2\alpha_3)^{-\epsilon_{\mathrm{UV}}}\frac{\mathcal{N}}{\mathcal{D}^{2-2\epsilon_{\mathrm{UV}}}}\,,$$

$$\begin{aligned}\mathcal{N} &= 2(p_1p_2)(p_1p_3)\Big[\alpha_1\alpha_2(1-\tau_1)+\alpha_3\alpha_1\tau_1\Big] + 2(p_1p_3)(p_2p_3)\Big[\alpha_3\alpha_1(1-\tau_3)+\alpha_2\alpha_3\tau_3\Big]\\ &+ 2(p_1p_2)(p_2p_3)\Big[\alpha_2\alpha_3(1-\tau_2)+\alpha_1\alpha_2\tau_2\Big] + 2\alpha_1\alpha_2\Big[2(p_1p_2)(p_3Q_3)-(p_2p_3)(p_1Q_3)-(p_3p_1)(p_2Q_3)\Big]\end{aligned}$$

$$+\dots$$

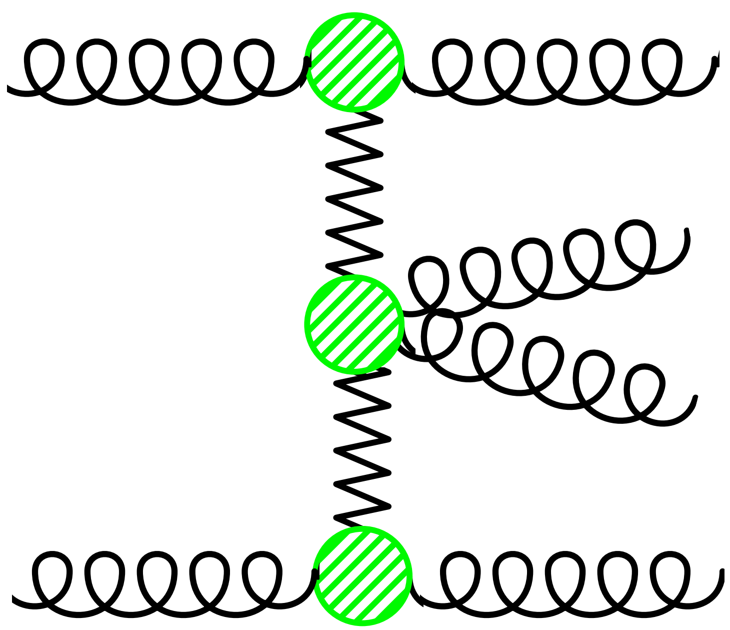
- The integrals do not explicitly depend on conformal ratios.
- But is all this complexity really needed..?
- Could we go to simplified kinematics?

Regge limits

- Quasi-multi-Regge kinematics

$$y_3 \gg y_4 \simeq y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



- Conformal cross ratios are no longer trivial

[Del Duca, CD, Glover]

The six-point remainder function

- Conclusion: It is enough to compute the remainder function in this restricted area of phase space.
- In the limit, all integrals are
  - ➡ at most three-fold.
  - ➡ dependent on conformal cross ratios only.
- The resulting integrals are much simpler and can be solved in a closed form, and we can extract the two-loop six-point remainder function,

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

[Del Duca, CD, Smirnov]

The six-point remainder function

- The result is completely expressed in terms Goncharov's multiple polylogarithm,

$$G(\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a} G(\vec{w}';t) \qquad \Big| \qquad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{n-1}(t)$$

- Some of them depend on complicated arguments:

$$u_{jkl}^{(\pm)} = \frac{1-u_j-u_k+u_l \pm \sqrt{(u_j+u_k-u_l-1)^2 - 4\,(1-u_j)\,(1-u_k)\,u_l}}{2\,(1-u_j)\,u_l},$$

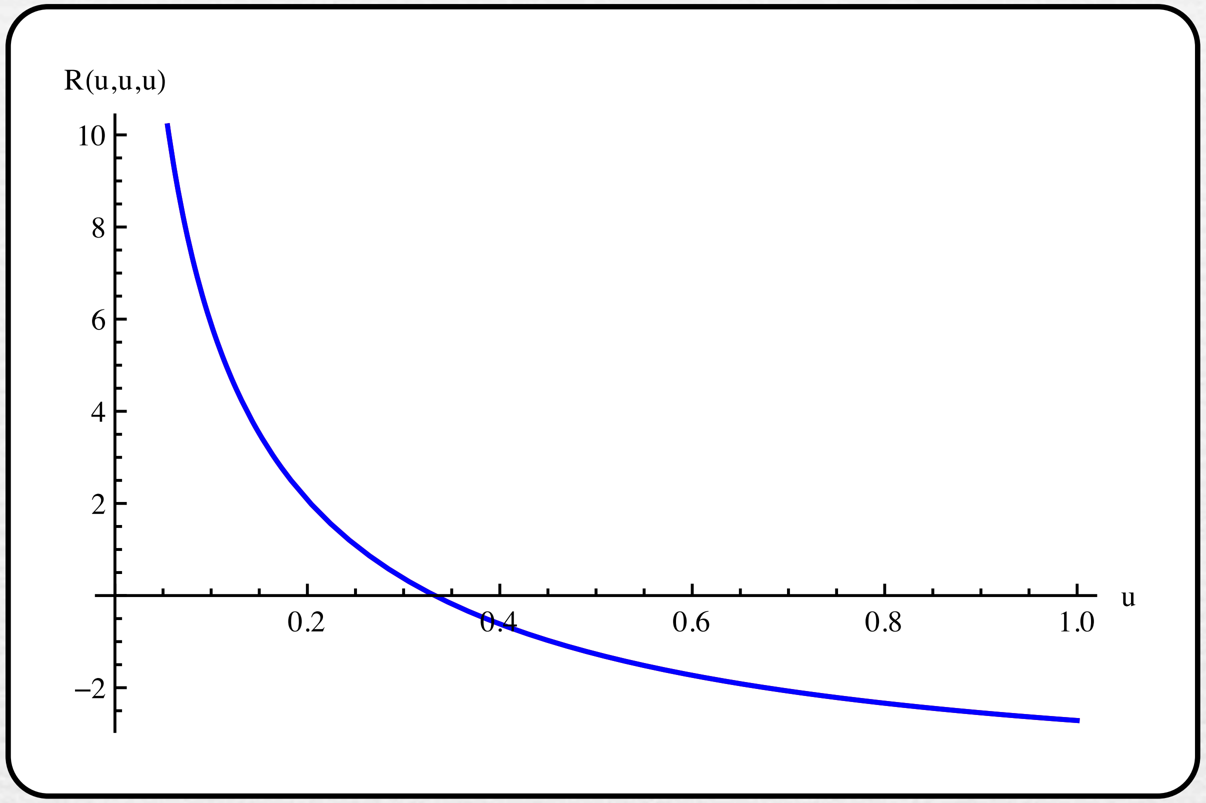
$$v_{jkl}^{(\pm)} = \frac{u_k-u_l \pm \sqrt{-4u_ju_ku_l+2u_ku_l+u_k^2+u_l^2}}{2\,(1-u_j)\,u_k}.$$

- The result is expressed as a very complicated combination of multiple polyogarithms.

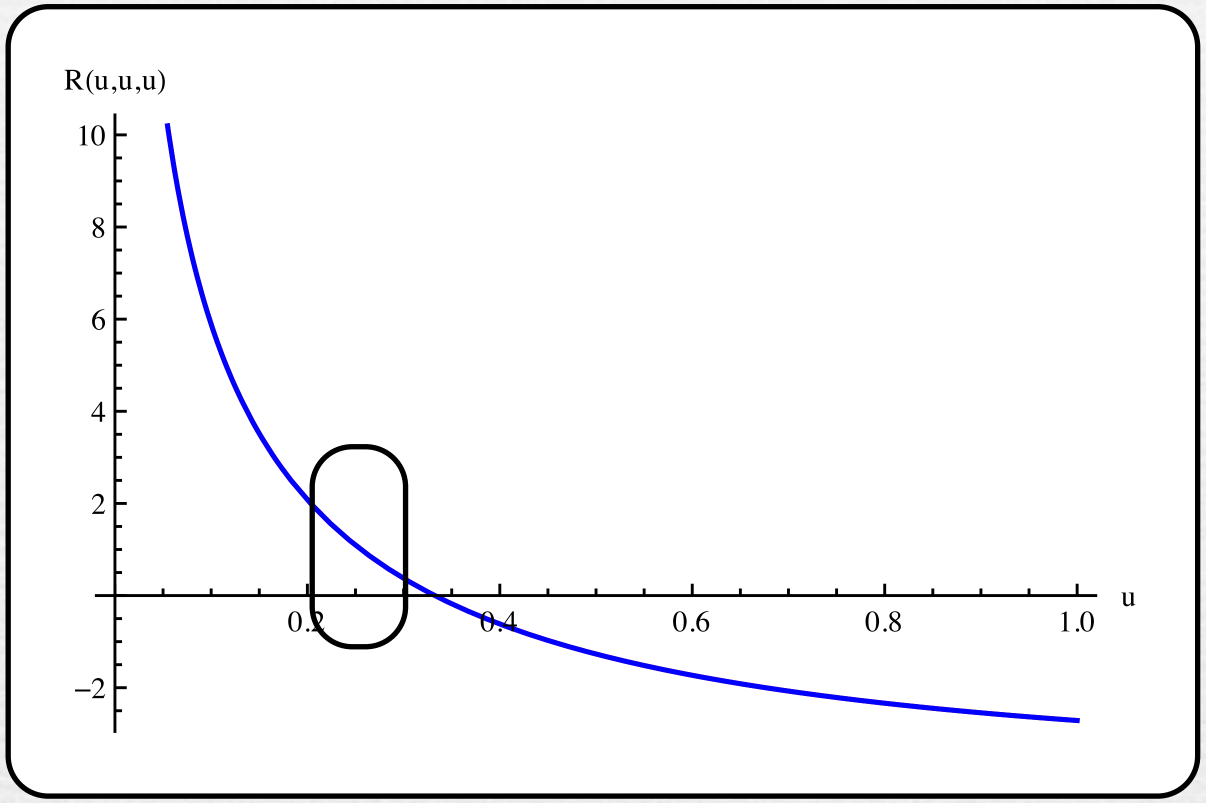
The six-point remainder function

$$
\begin{aligned}
&R^{(2)}_{6,WL}(u_1,u_2,u_3) = \qquad\qquad (\mathrm{H}.1)\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+\\
&G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+\\
&G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-
\end{aligned}
$$

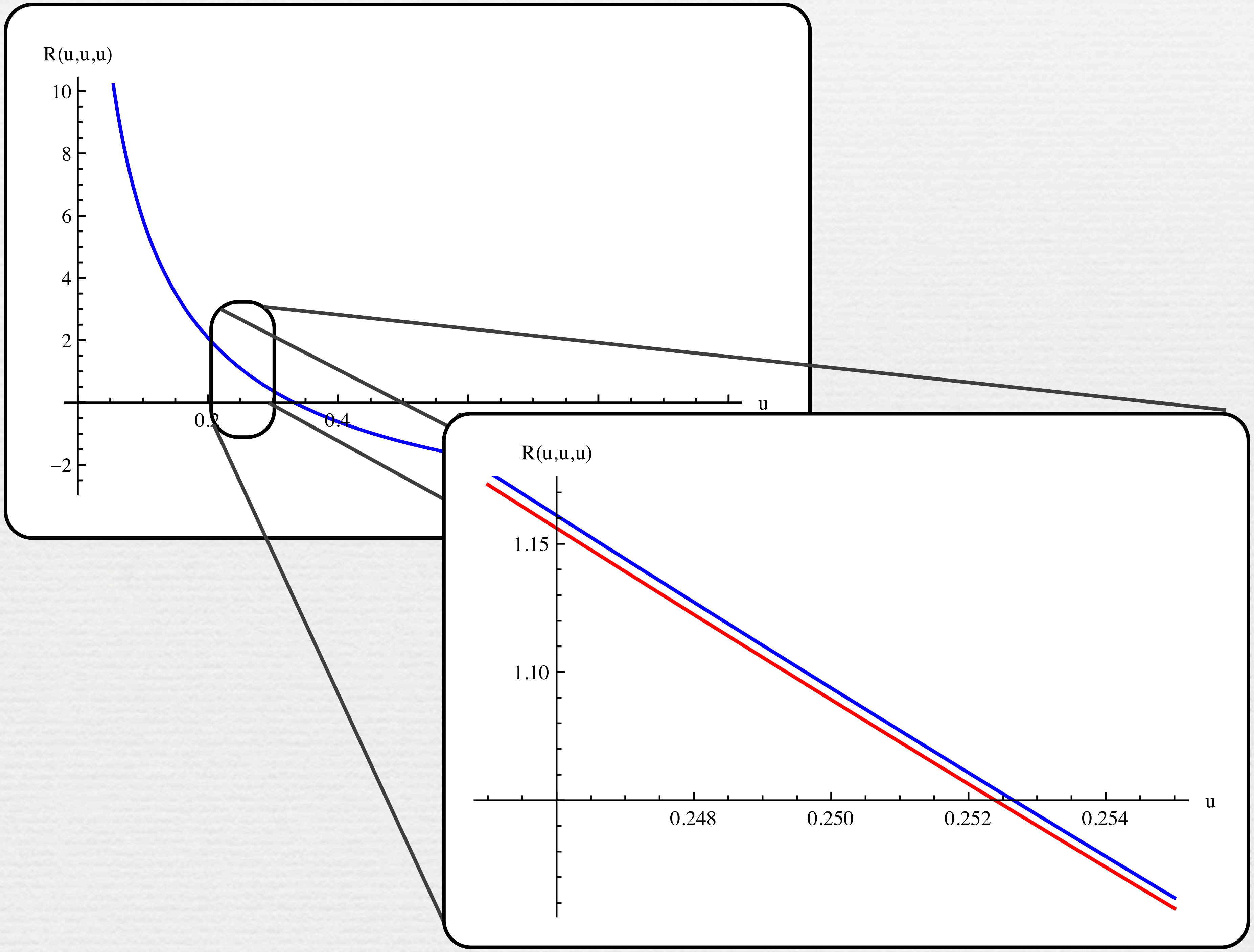
Weak coupling vs. strong coupling



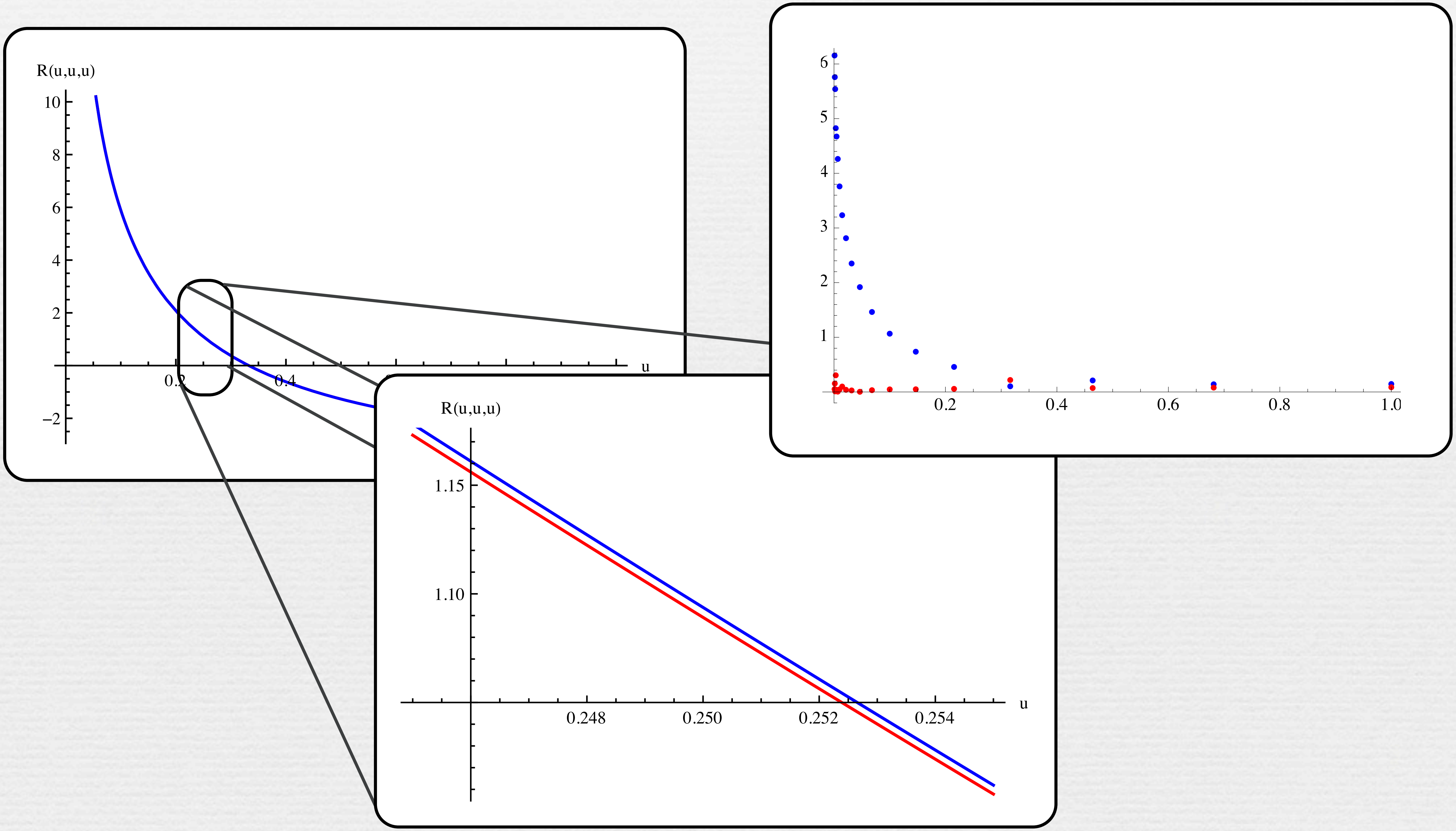
Weak coupling vs. strong coupling



Weak coupling vs. strong coupling



Weak coupling vs. strong coupling



The six-point remainder function

- If we want to compare directly the analytic expressions, we need identities among multiple polylogarithms...
  - ➡ Needs the intervention of a mathematician!

- The theory of motives provides a way to handle such expression
  - ➡ Hand-waving idea: Associate a ‘tensor calculus’ to polylogarithms that incorporates the functional identities.

[Goncharov, Spradlin, Vergu, Volovich]

The six-point remainder function

$$R(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right)$$

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},$$

$$\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3.$$

[Goncharov, Spradlin,
Volovich, Vergu]

Towards remainder functions with more legs

- The techniques we developed for the computation of the six-point remainder function can also be applied to Wilson loops with more edges.
- We focus on the 1+1 dimensional setup studied at strong coupling.

Towards remainder functions with more legs

- The techniques we developed for the computation of the six-point remainder function can also be applied to Wilson loops with more edges.
- We focus on the 1+1 dimensional setup studied at strong coupling.
- The final answer involves 25.000 terms...

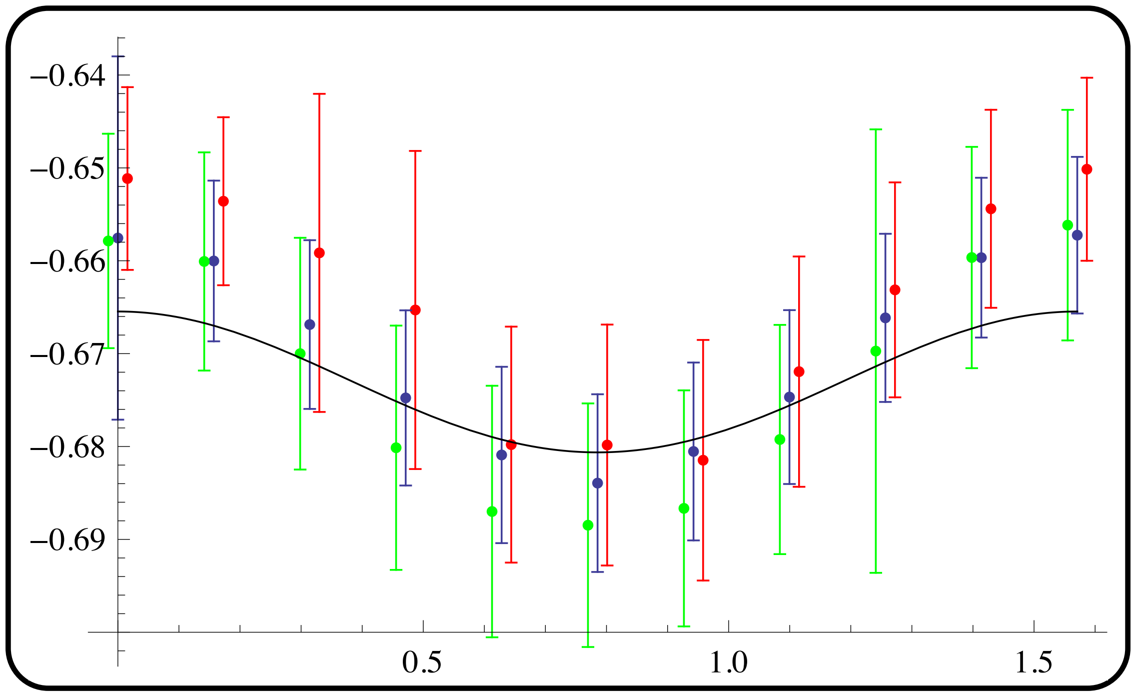
Towards remainder functions with more legs

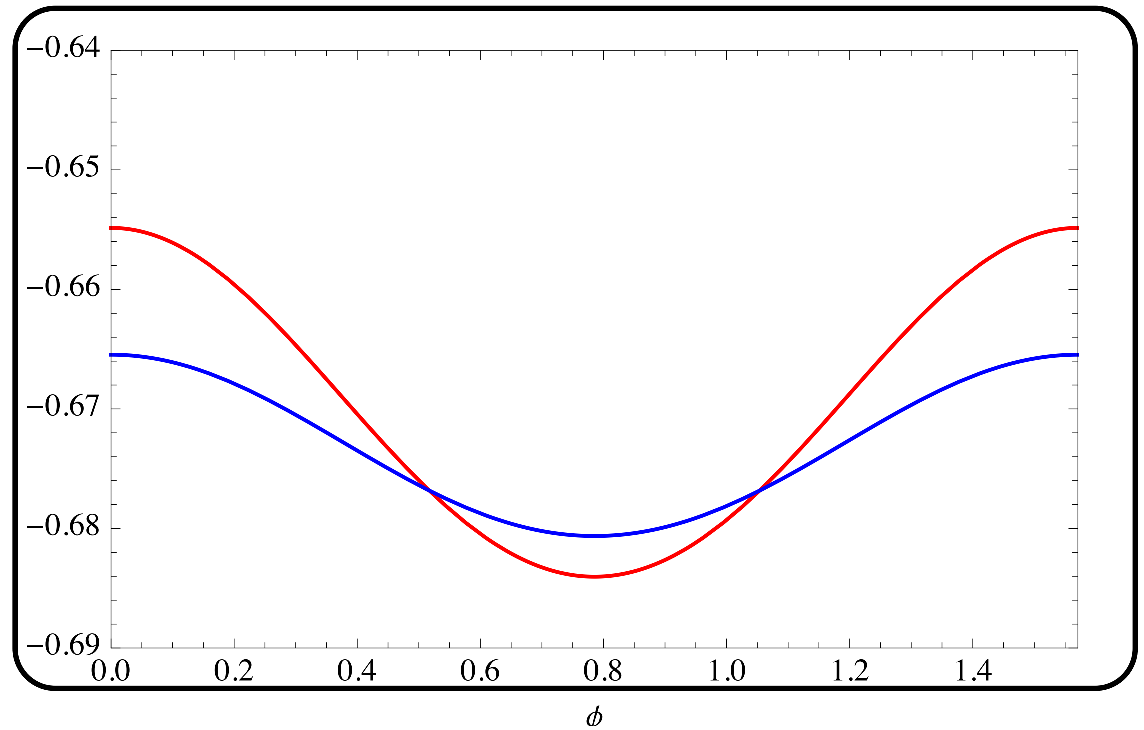
- The techniques we developed for the computation of the six-point remainder function can also be applied to Wilson loops with more edges.
- We focus on the 1+1 dimensional setup studied at strong coupling.
- The final answer involves 25.000 terms...

  ... but they all collapse to

$$R^{(2)}_{8,WL}(\chi^+,\chi^-) = -\frac{\pi^4}{18} - \frac{1}{2}\ln\left(1+\chi^+\right)\ln\left(1+\frac{1}{\chi^+}\right)\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^-}\right)$$

Octagon in 1+1 dimensions





- Same pattern as for the hexagon:

  Even though the two ansers are very close everywhere, they are not identical...

Conclusion

- In the last ten months, a lot of progress was made to compute two-loop multi-leg amplitudes/Wilson loops:
  - ➡ Hexgon in 3+1 dimensions
  - ➡ Octagon in special kinematics (1+1 dimensions)
  - ➡ All even-sided polygons in 1+1 dimensions. [Heslop, Khoze]
- Intriguing connection between strong and weak coupling to be understood.
- Along the way, we can start to fill up our tool box for multi-leg multi-loop computations:
  - ➡ Multiple polylogarithms
  - ➡ New insights from the theory of motives
- Interesting times are ahead in the N=4 SYM world!

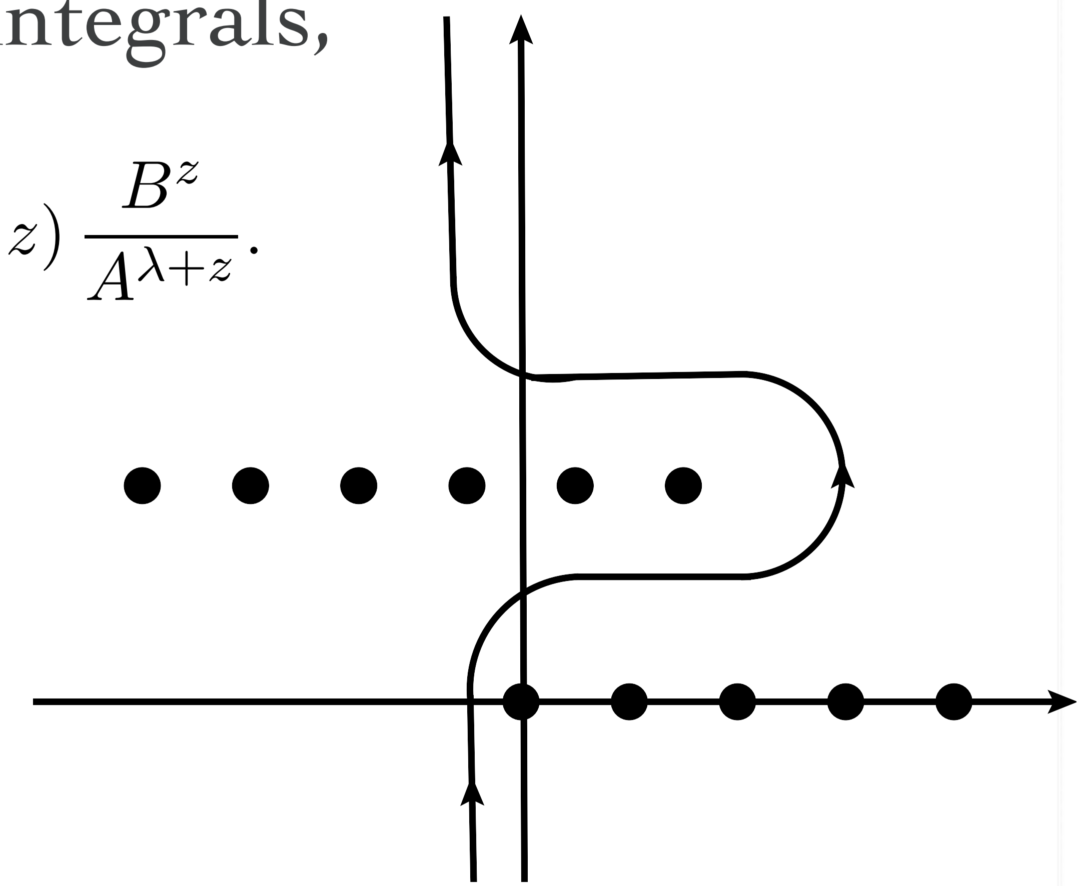
Back ups

A Recipe to compute Wilson loops

- Step 1:

  We write down a Mellin-Barnes representation for each diagram, i.e., we replace denominators in the Feynman parameter integrals by contour integrals,

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z\, \Gamma(-z)\, \Gamma(\lambda+z)\, \frac{B^z}{A^{\lambda+z}}.$$



- This turns the Feynman parameter integral into residue calculus:

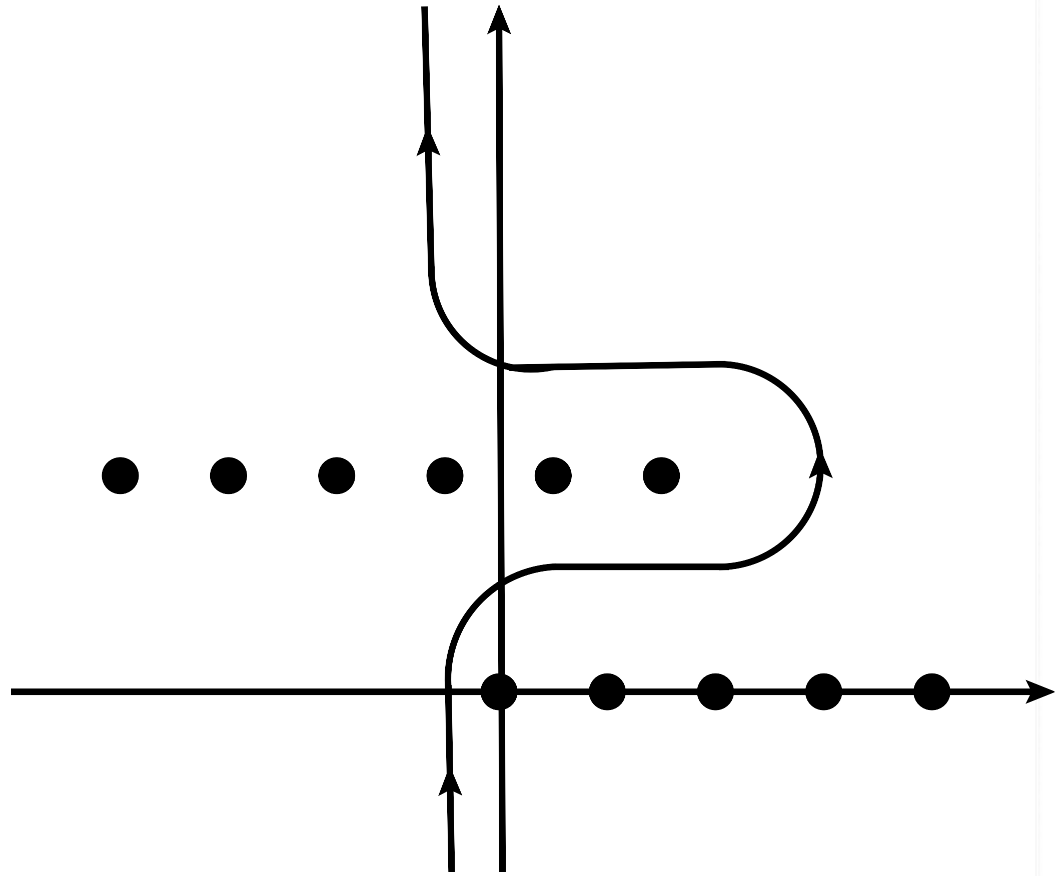
$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$$

A Recipe to compute Wilson loops

- Step 2:

  We exploit Regge exactness and we only compute the leading behavior of each integral in the quasi-multi-Regge limit

- The Mellin-Barnes approach is very suitable for this!

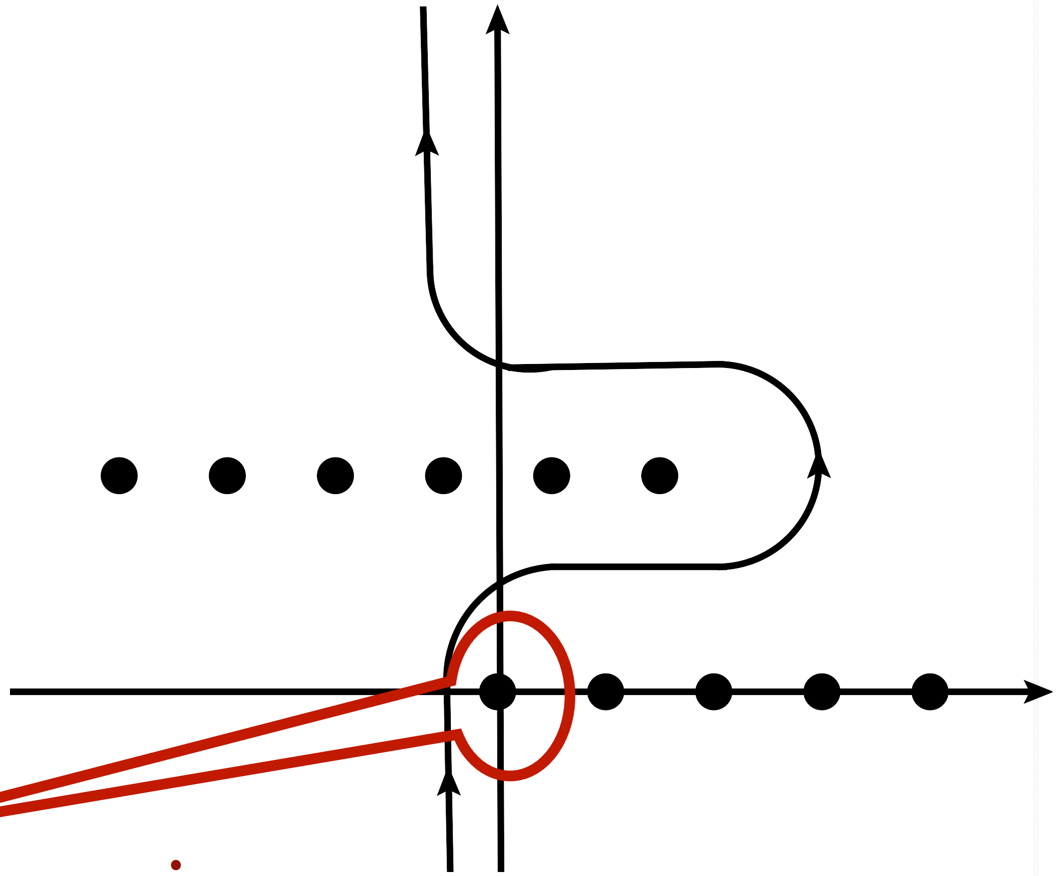


A Recipe to compute Wilson loops

- Step 2:

  We exploit Regge exactness and we only compute the leading behavior of each integral in the quasi-multi-Regge limit

- The Mellin-Barnes approach is very suitable for this!



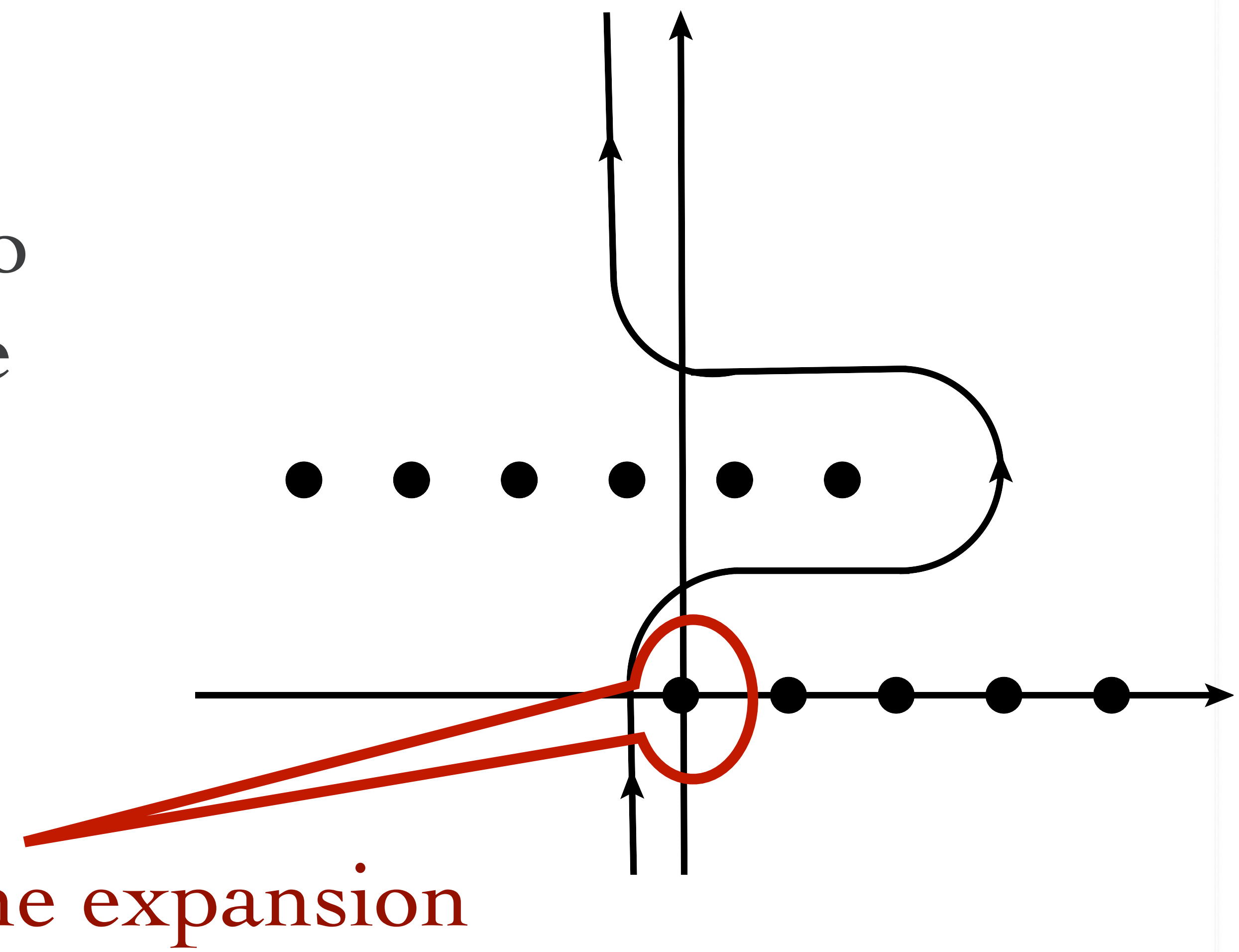
Leading term in the expansion

A Recipe to compute Wilson loops

- Step 3:

  Iterate the limits: There are six different ways to take the limits, corresponding to the six cyclic permutations of the external legs.

- Regge-exactness allows us to take all six limits at the same time!

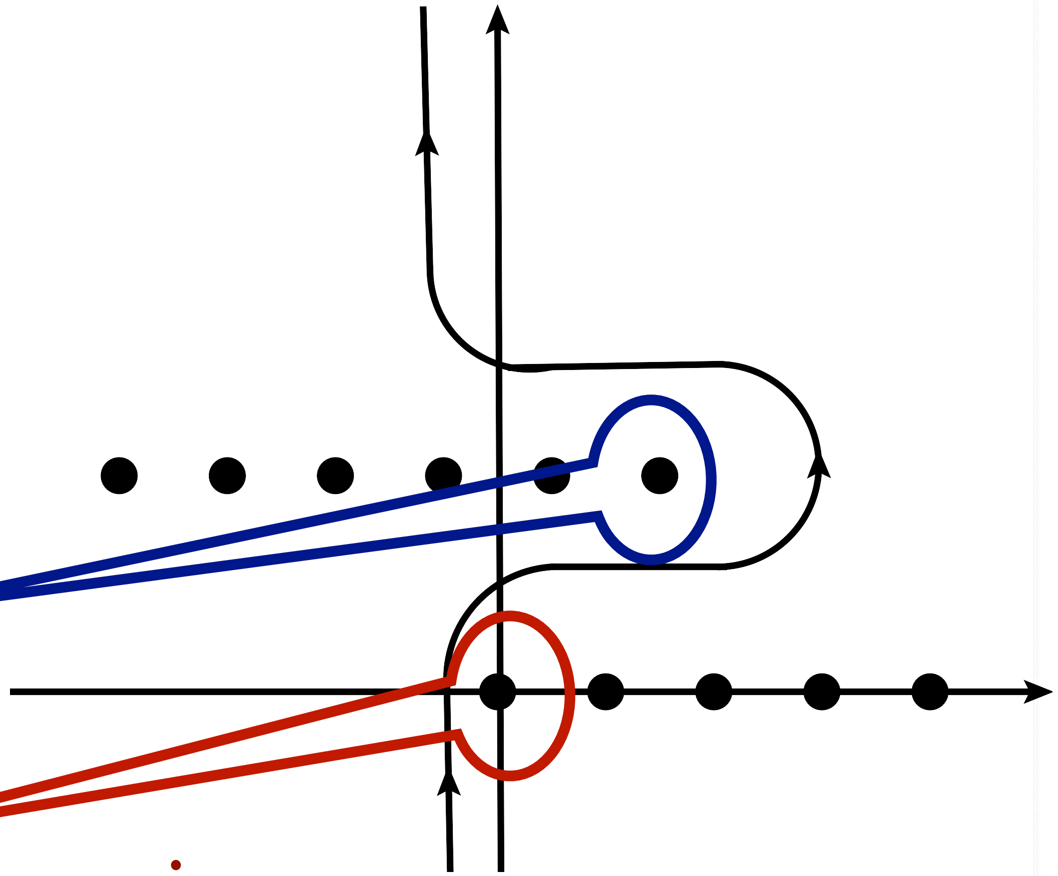


A Recipe to compute Wilson loops

- Step 3:

  Iterate the limits: There are six different ways to take the limits, corresponding to the six cyclic permutations of the external legs.

- Regge-exactness allows us to take all six limits at the same time!



Leading term in the expansion in limit 2

Leading term in the expansion in limit1

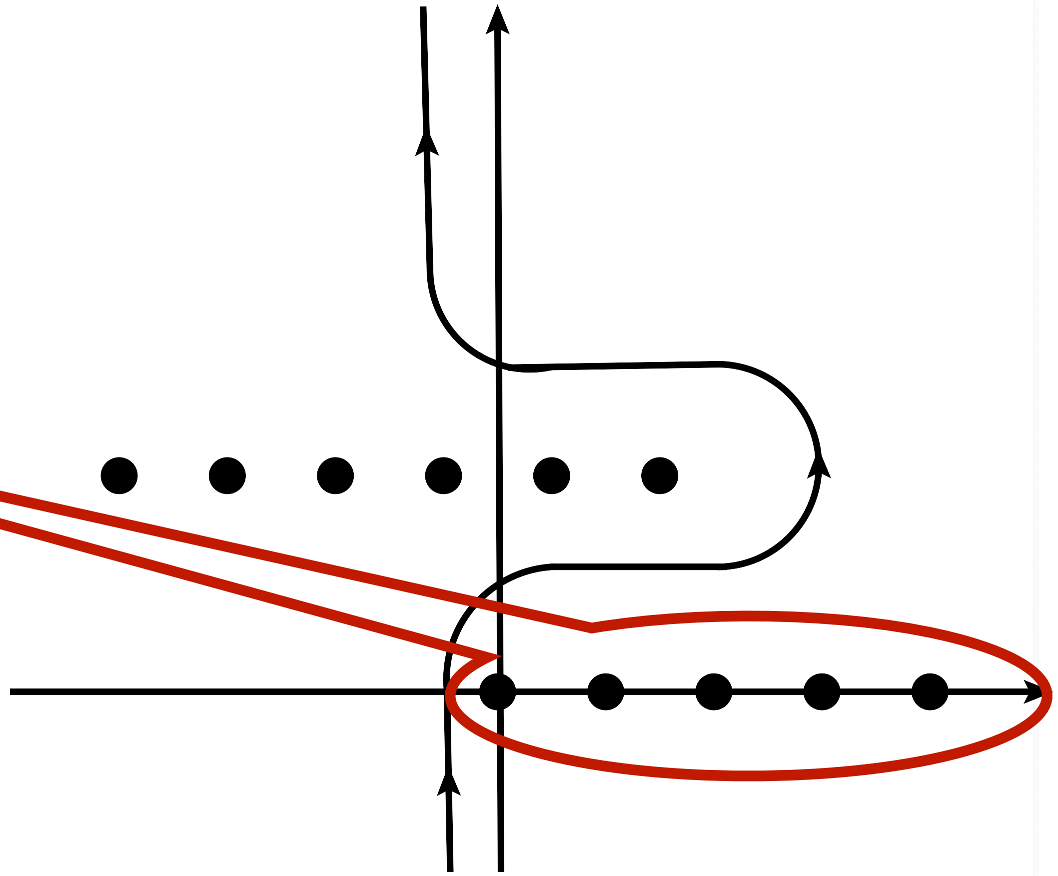
A Recipe to compute Wilson loops

- Step 4:

  Sum up the remaining towers of residues:

$$\sum_{n=1}^{\infty} \frac{u_i^n}{n} = -\ln(1-u_i)$$

$$\sum_{n=1}^{\infty} \frac{u_i^n}{n^k} = \mathrm{Li}_k(u_i)$$

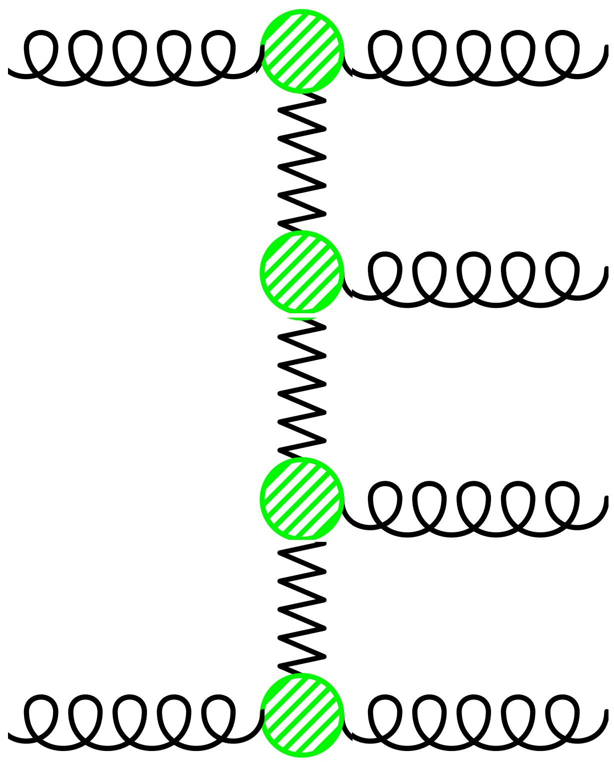


Regge limits

- Multi-Regge kinematics

$$y_3 \gg y_4 \gg y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



- s-type invariants are large.

  t-type invariants are small.

  Conformal cross ratios become trivial

[Del Duca, CD, Glover]

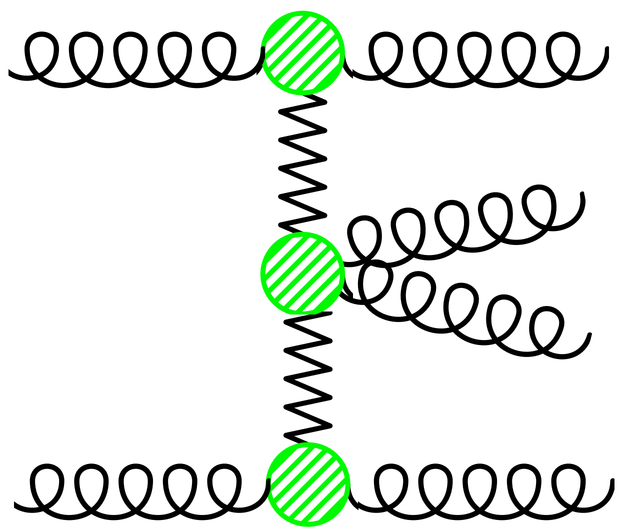
Regge-exactness of Wilson loops

- The result is in fact even stronger:

  The Wilson-loop is **Regge-exact** in this limit, i.e., it is the same in this special kinematics and in arbitrary kinematics

$$y_3 \gg y_4 \simeq y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



- This result is in fact true for Wilson loops with an arbitrary number of edges and loops! [Del Duca, CD, Smirnov]
- Bottomline: it is enough to perform the computation in these **simplified** kinematics to obtain the two-loop six-point Wilson loop in **arbitrary** kinematics!

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

conformal ratios are invariant.

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \mathcal{M}_n^{(1)}$$

[Brandhuber, Heslop, Travaglini]

conformal ratios are invariant.

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \mathcal{M}_n^{(1)}$$

[Brandhuber, Heslop, Travaglini]

conformal ratios are invariant.

Structure of the one-loop amplitude:

$$\ln s_{ij} \quad + \quad \mathrm{Li}_2(1-u_{ij})$$

[Bern, Dixon, Dunbar, Kosower]

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \mathcal{M}_n^{(1)}$$

[Brandhuber, Heslop, Travaglini]

conformal ratios are invariant.

Structure of the one-loop amplitude:

$$\ln s_{ij} \qquad + \qquad \mathrm{Li}_2(1-u_{ij})$$

[Bern, Dixon, Dunbar, Kosower]

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \mathcal{M}_n^{(1)}$$

[Brandhuber, Heslop, Travaglini]

conformal ratios are invariant.

Structure of the one-loop amplitude:

$$\ln s_{ij} \qquad + \qquad \mathrm{Li}_2(1-u_{ij})$$

[Bern, Dixon, Dunbar, Kosower]

Log's are not power suppressed.

Symbols

- Simple example:

$$\mathrm{Li}_2(x) + \ln(1-x)\ln x = -\mathrm{Li}_2(1-x) - \frac{\pi^2}{6}$$

$$\mathrm{Symbol}(\mathrm{Li}_2(x)) = -(1-x)\otimes x$$

$$\mathrm{Symbol}(\ln(1-x)\ln x) = (1-x)\otimes x + x\otimes(1-x)$$

$$\mathrm{Symbol}(const) = 0$$

$$\mathrm{Symbol}(\mathrm{Li}_2(x) + \ln(1-x)\ln x) = x\otimes(1-x)$$

$$= -\mathrm{Symbol}(\mathrm{Li}_2(1-x))$$