HOLOGRAPHY AS AN INITIAL-VALUE FORMULATION OF GRAVITY

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WHAT IS HOLOGRAPHY?

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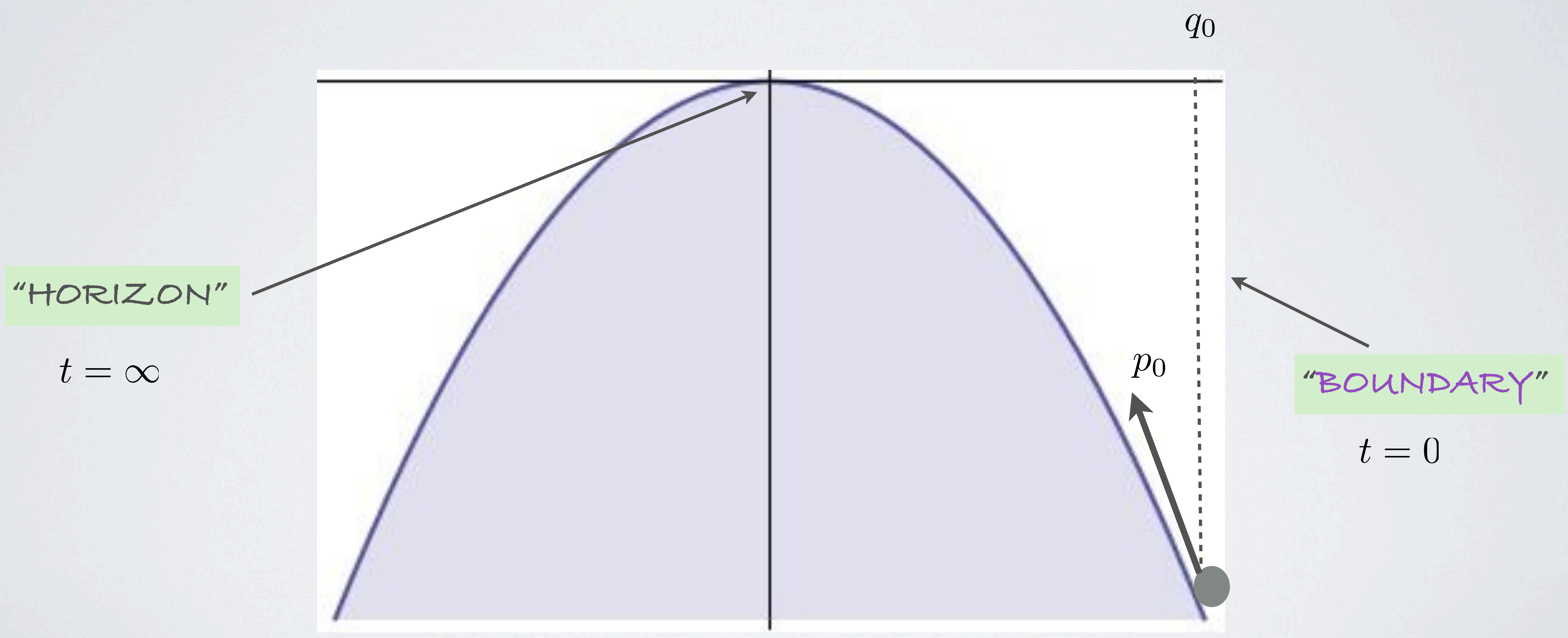
DISGUISED HAMILTONIAN DYNAMICS

SUMMARY

- HOLOGRAPHY VS HAMILTONIAN DYNAMICS - A TOY SYSTEM
- THE REFINED 3+1 SPLIT FORMALISM OF GRAVITY
- HOLOGRAPHY VS THE INITIAL VALUE FORMULATION OF GRAVITY
- EXAMPLES: HOLOGRAPHIC RENORMALIZATION, KOUNTERTERMS, BLACK-HOLE HOLOGRAPHY, SELF-DUAL CONFIGURATIONS.

A TOY-SYSTEM: THE INVERTED HARMONIC OSCILLATOR

$$I = \int_0^\infty dt \left[p\dot{q} - \frac{1}{2}\left(p^2 - \omega^2 q^2\right)\right]$$



THE GENERAL SOLUTION DESCRIBING THIS MOTION IS

$$q(t) = q_0 \cosh \omega t + \frac{1}{\omega} p_0 \sinh \omega t$$

IF ONE REQUIRES THAT THE MASS "JUST CLIMBS" THE POTENTIAL, ONE NEEDS TO IMPOSE THE "REGULARITY" CONDITION

$$q(\infty) = 0 \quad \Rightarrow \quad p_0 = -\omega q_0$$

"LINEAR RESPONSE"

THEN, THE VARIATION OF THE ON-SHELL ACTION IS

$$\delta I_{o.s.} = -p_0 \delta q_0 \quad \Rightarrow \quad \frac{\delta^2 I_{o.s.}}{\delta q_0^2} = \omega$$

"TWO-POINT FUNCTION"

FOR $\omega \mapsto i\omega$

THE "REGULARITY" CONDITION

BECOMES "IN FALLING B.C."

THE IMAGINARY PART OF THE 2-PT

FUNCTION GIVES THE SUSCEPTIBILITY

E.G. "KUBO-FORMULA"

TRY TO HAVE SOME FUN WITH THE INVERTED H.O. - CHANGE TIME

$$t = Te^{\rho/T} \quad \Rightarrow \quad t\Big|_0^\infty \mapsto \rho\Big|_{-\infty}^\infty$$

$$I \mapsto I = \int_{-\infty}^\infty d\rho \left[pq' - \frac{1}{2}e^{\rho/T}\left(p^2 - \omega^2 q^2\right)\right]$$

GIVING THE (2ND-ORDER) E.O.M.

$$q'' - \frac{1}{T}q' - e^{2\rho/T}\omega^2 q = 0$$

CLEARLY, THE RESULTS ARE THE SAME AS BEFORE

SOLVE THIS BY FROBENIUS

$$\boxed{q(\rho) = \sum_{n=0}^\infty e^{n\rho/T} q_n}$$

THIS FORM IS DICTATED BY THE REQUIREMENT $\quad q(\rho = -\infty) = q_0$

RELATION TO HOLOGRAPHY - MASSIVE SCALAR ON FIXED E-ADS4

$$ds^2 = dt^2 + e^{2t/L} d\vec{x}^2$$

$t \to \infty$ BOUNDARY

$t \to -\infty$ "HORIZON"

THE 1ST-ORDER ACTION IS

$$I = \int dt\, d\vec{x} \left[\pi\dot{\phi} - \frac{1}{2\sqrt{g}} \left(\pi^2 - g e^{2t/L} \partial_i\phi\partial_i\phi - g m^2 \phi^2 \right) \right]$$

THE HAMILTONIAN E.O.M. ARE

$$\dot{\phi} = \frac{1}{\sqrt{g}}\pi \,, \quad \dot{\pi} = \sqrt{g} \left(m^2\phi - e^{2t/L} \vec{\partial}^2 \phi \right)$$

SOLVE THE RADIAL ("TIME") EVOLUTION BY FROBENIUS

$$\phi(t, \vec{x}) = \sum_{n=0}^{\infty} e^{(\Delta+n)t/L} \phi_n(\vec{x})$$

THE CANONICAL MOMENTUM IS ALSO EXPANDED AS

$$\pi(t,\vec{x}) = \sum_{n=0}^{\infty} \left(\frac{\Delta+n}{L}\right) e^{(\Delta-3+n)t/L}\phi_n(\vec{x})$$

THE FIRST TERMS IN THE EXPANSION READ

$$\left[\frac{\Delta(\Delta-3)}{L^2} - m^2\right]\phi_0(\vec{x})e^{(\Delta-3)t/L} + \left[\frac{(\Delta+1)(\Delta-2)t}{L^2} - m^2\right]\phi_1(\vec{x})e^{(\Delta-2)/L}$$

$$+\left[\left(\frac{(\Delta+2)(\Delta-1)}{L^2} - m^2\right)\phi_2(\vec{x}) + \vec{\partial}^2\phi_0(\vec{x})\right]e^{(\Delta-1)t/L}$$

$$+\left[\left(\frac{(\Delta+3)\Delta}{L^2} - m^2\right)\phi_3(\vec{x}) + \vec{\partial}^2\phi_1(\vec{x})\right]e^{\Delta t/L} + \cdots = 0$$

THE TERMS IN THE BRACKETS MUST VANISH. THIS YIELDS RELATIONSHIPS BETWEEN Δ, m^2 AND THE COEFFICIENTS OF THE SERIES.

SINCE THIS IS A HAMILTONIAN SYSTEM, ONE NEEDS TO IDENTIFY THE TWO INDEPENDENT COEFFICIENTS IN THE SERIES EXPANSION.

THESE WILL BE THE REQUIRED INITIAL DATA: "INITIAL POSITION" AND "INITIAL VELOCITY" FOR THE HAMILTONIAN EVOLUTION ALONG "TIME".

NEVERTHELESS, THE ISSUE IS WHETHER THE "BULK" CANONICAL VARIABLES ARE WELL-DEFINED:

NAMELY, WHETHER THEIR LEADING BOUNDARY VALUES YIELD THE TWO INDEPENDENT COEFFICIENTS RESPECTIVELY. THIS IS WHAT HAPPENS IN FLAT SPACE.

E.G. FOR $\Delta = 0\,, \Rightarrow m^2 = 0\,, \phi_1 = 0\,,\ \ \phi_0\ \phi_3$ ARE INDEPENDENT COEFFICIENTS

HENCE $\pi(t, \vec{x})$ IS NOT WELL-DEFINED

CONSIDER THE MINIMAL CASE OF A CONFORMALLY COUPLED SCALAR

$$\boxed{m^2 L^2 = -2\,, \quad \Delta = 1}$$

THE GENERAL SOLUTIONS OF THE E.OM. ARE

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{\mathrm{i}\vec{p}\vec{x}} \phi(t,\vec{p})$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{\mathrm{i}\vec{p}\vec{x}} e^{t/L} \left[\phi_0(\vec{p}) \cosh\left(e^{t/L} L|\vec{p}|\right) + \frac{\phi_1(\vec{p})}{L|\vec{p}|} \sinh\left(e^{t/L} L|\vec{p}|\right)\right]$$

$$\pi(t,\vec{x}) = \sqrt{g}\dot{\phi}(t,\vec{x})$$

$$\pi(t,\vec{x}) = e^{-3t/L}\frac{1}{L}\phi(t,\vec{x}) + \int \frac{d^3\vec{p}}{(2\pi)^3} e^{\mathrm{i}\vec{p}\vec{x}} e^{-t/L} \left[\frac{\phi_1(\vec{p})}{L} \cosh\left(e^{t/L} L|\vec{p}|\right) + |\vec{p}|\phi_0(\vec{p}) \sinh\left(e^{t/L} L|\vec{p}|\right)\right]$$

CLEARLY, THE BULK CANONICAL MOMENTUM IS NOT WELL-DEFINED.

IN A HAMILTONIAN SYSTEM, THE ONLY THINGS WE CAN DO ARE CANONICAL TRANSFORMATIONS - THE RELEVANT ONE HERE IS:

$$\boxed{\hat{\pi}(t,\vec{x}) \equiv \pi(t,\vec{x}) - \sqrt{g}\frac{1}{L}\phi(t,\vec{x})}$$

AS USUAL, C.T. CORRESPOND TO BOUNDARY TERMS IN THE ACTION. HERE:

$$I \mapsto \hat{I} = I - \frac{1}{2L}\int_{\partial\mathcal{M}} d^3\vec{x}\,\phi^2(t,\vec{x})$$

A SHORT CALCULATION THEN YIELDS

$$\delta\hat{I}_{on-shell} = -\frac{1}{L}\int_{\partial\mathcal{M}} d^3\vec{x}\phi_1(x)\delta\phi_0(\vec{x}) + \cdots$$

BUT THIS IS NOT HOLOGRAPHY JUST YET! WE NEED TO IMPOSE THE REGULARITY CONDITION I.E. REQUIRE THAT THE FIELD VANISH AT THE "HORIZON". THIS GIVES

$$\boxed{\phi_0(\vec{p}) + \frac{1}{L|\vec{p}|}\phi_1(\vec{p}) = 0}$$

$$\boxed{\delta\hat{I}_{on-shell} = \int \frac{d^3\vec{p}}{(2\pi)^3}|\vec{p}|\phi_0(\vec{p})\delta\phi_0(\vec{p})}$$

THE E.O.M. FOR THE SCALAR ON ADS4 IS

$$\ddot{\phi}(t,\vec{p}) + \frac{3}{L}\dot{\phi}(t,\vec{p}) - e^{2\rho/L}\vec{p}^2\phi(t,\vec{p}) - m^2\phi(t,\vec{p}) = 0$$

FOR $m^2L^2 = -2$ THE ABOVE CANONICAL TRANSFORMATION IS EQUIVALENT TO

$$\boxed{\phi(t,\vec{p}) = e^{\rho/L}f(t,\vec{p}) \Rightarrow \ddot{f}(t,\vec{p}) - \frac{1}{L}\dot{f}(t,\vec{p}) - e^{2\rho/L}\vec{p}^2 f(t,\vec{p}) = 0}$$

HENCE, IT IS EQUIVALENT TO THE H.O. WITH THE IDENTIFICATION $\vec{p}^2 \leftrightarrow \omega^2$

THE REFINED 3+1 SPLIT FORMALISM FOR GRAVITY

[R. G. Leigh and T. P. (07)]

THE FIRST-ORDER FORMULATION OF 4-D GRAVITY STARTS FROM THE ACTION

$$I_{EH} \equiv -16\pi G_4 S_{EH} = \int \left(R^{ab} \wedge e^c \wedge e^d - \frac{\Lambda}{6} e^a \wedge \cdots \wedge e^d \right) \epsilon_{abcd}$$

WITH THE USUAL DEFINITIONS FOR THE VIELBEIN AND THE SPIN-CONNECTION

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\,, \;\; T^a = de^a + \omega^a{}_b \wedge e^b \qquad (a, b = \underbrace{0}_{\sigma_\perp}, \underbrace{1, 2, 3}_{\sigma_3})$$

TO MAKE CONTACT WITH THE METRIC FORMALISM WE NOTE:

$$S_{EH} \to G_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,, \quad \Lambda = -\frac{3}{L^2}\sigma_\perp\,, \;\; \sigma_3\sigma_\perp = \sigma = \pm 1$$

THE 3+1 SPLIT IS A REFINED ADM FORMULATION FOR 4-D GRAVITY:

WE ASSUME A LOCAL 3-D SLICING AND SPLIT EVERYTHING ACCORDINGLY

$$
\begin{aligned}
e^0 &= N dt\,, \qquad e^\alpha = N^\alpha dt + \tilde{e}^\alpha\,, \\
\omega^{0\alpha} &= q^{0\alpha} dt + \sigma_\perp K^\alpha\,, \qquad \omega^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma}\left(Q_t + B_\gamma\right).
\end{aligned}
\qquad \alpha, \beta, \gamma = 1, 2, 3
$$

NOVELTY IS THE INTRODUCTION OF THE "ELECTRIC" AND "MAGNETIC" FIELDS

$K^\alpha\,, \quad B^\alpha$ SO(3)-SO(2,1) VECTOR-VALUED 1-FORMS

THE MAIN ISSUE IS TO IDENTIFY THE CORRECT DYNAMICAL VARIABLES

AFTER SOME WORK THE GRAVITATIONAL ACTION TAKES THE FORM

$$
\begin{aligned}
S_{\mathrm{EH}} \;=\; & -\frac{\sigma_\perp}{8\pi G}\int dt \wedge \Big\{-K_\alpha \wedge \dot{\Sigma}^\alpha + N\tilde{W}_\alpha \wedge \tilde{e}^\alpha + \sigma_\perp \hat{Q} \wedge K_\beta \wedge \tilde{e}^\beta \\
& \qquad + \sigma_\perp q^{0\alpha}\tilde{\mathcal{D}}\Sigma_\alpha - N^\alpha \epsilon_{\alpha\beta\gamma}\tilde{\mathcal{D}}K^\beta \wedge \tilde{e}^\gamma\Big\} \\
& -\frac{1}{8\pi G}\int_\partial \left(q^{0\alpha} dt + \sigma_\perp K^\alpha\right) \wedge \Sigma_\alpha\,,
\end{aligned}
$$

$$
\hat{Q} \equiv Q_\alpha \tilde{e}^\alpha \qquad \Sigma^\alpha = {}^*\tilde{e}^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma}\tilde{e}^\beta \wedge \tilde{e}^\gamma \qquad \tilde{W}_\alpha \equiv \rho_\alpha - \frac{1}{2}\epsilon_{\alpha\beta\gamma}K^\beta \wedge K^\gamma + \frac{1}{\ell^2}\Sigma_\alpha\,.
$$

$$
\rho_\alpha = \tilde{d}B_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}B^\beta \wedge B^\gamma \qquad \tilde{\mathcal{D}}V^\alpha = \tilde{d}V^\alpha + \epsilon^\alpha{}_{\beta\gamma}B^\beta \wedge V^\gamma
$$

VARYING WRT THE LAGRANGE MULTIPLIERS WE GET THE CONSTRAINTS

$$
\begin{aligned}
-8\pi G\sigma_\perp \frac{\delta S}{\delta N} &= \tilde{W}_\alpha \wedge \tilde{e}^\alpha = 0\,, \\
-8\pi G\sigma_\perp \frac{\delta S}{\delta N^\alpha} &= -\epsilon_{\alpha\beta\gamma}\tilde{\mathcal{D}}K^\beta \wedge \tilde{e}^\gamma = 0\,, \\
-8\pi G\sigma_\perp \frac{\delta S}{\delta q^{0\alpha}} &= \sigma_\perp \tilde{\mathcal{D}}\Sigma_\alpha = \boxed{\sigma_\perp \epsilon_{\alpha\beta\gamma}\tilde{T}^\beta \wedge \tilde{e}^\gamma = 0\,,} \\
-8\pi G\sigma_\perp \frac{\delta S}{\delta \hat{Q}} &= \sigma_\perp K_\alpha \wedge \tilde{e}^\alpha = 0\,, \\
-8\pi G\sigma_\perp \frac{\delta S}{\delta B^\alpha} &= \boxed{N\tilde{T}^\alpha + \left(\tilde{d}N + \sigma_\perp K_\beta N^\beta - \hat{q}\right) \wedge \tilde{e}^\alpha = 0}
\end{aligned}
$$

$$\hat{q} \equiv q^0{}_\alpha \tilde{e}^\alpha$$

$$\boxed{\tilde{d}N + \sigma_\perp K_\beta N^\beta - \hat{q} = 0 \qquad \tilde{T}^\alpha = \mathcal{D}\tilde{e}^\alpha = 0}$$

THE MAGNETIC FIELD IS THE LAGRANGE MULTIPLIER FOR THE ZERO TORSION CONSTRAINT

A NATURAL GAUGE-FIXING IS THEN $N = N(t), N^\alpha = 0 \Rightarrow \hat{q} = 0$

THIS ACCOUNTS FOR ALMOST ALL HOLOGRAPHIC BACKGROUNDS

TO PROCEED WITH THE REMAINING GAUGE FIXING NOTE THAT UNDER SO(3,1)

$$\begin{aligned} e &\mapsto e' = ge\,, \\ \omega &\mapsto \omega' = g\omega g^{-1} + gdg^{-1} \end{aligned}$$

A TRANSFORMATION THAT PRESERVES THE FORM OF THE VIERBEIN IS

$$e^0 = Ndt \Rightarrow g^0{}_\alpha = 0$$

HENCE, WE ARE HAVE AT OUR DISPOSAL TRANSFORMATIONS IN

$$g^\alpha{}_\beta \qquad \mathrm{L} = \left\{ \begin{array}{ll} \mathrm{SO}(3) & \text{if } \sigma_\perp = -1 \\ \mathrm{SO}(2,1) & \text{if } \sigma_\perp = +1 \end{array} \right\} \subset \mathrm{SO}(3,1)$$

WITH THEIR HELP ONE CAN GAUGE-FIX TO ZERO Q:

$$\omega^{\alpha\beta} = \epsilon^{\alpha\beta\gamma}(Q_\gamma dt + B_\gamma) \Rightarrow -\epsilon^\alpha{}_{\beta\gamma}Q^\gamma = (g^{-1})^\alpha{}_\gamma \dot{g}^\gamma{}_\beta$$

T-INDEPENDENT L-ROTATIONS REMAIN:

CAN SET TO ZERO THE ANTISYMMETRIC PART OF THE MAGNETIC FIELD

VARIATION WRT TO THE DYNAMICAL VARIABLES GIVES THE DYNAMICAL E.O.M.

$$-8\pi G\sigma_\perp \frac{\delta S}{\delta K^\alpha} = -\epsilon_{\alpha\beta\gamma}\left(\dot{\tilde{e}}^\beta + K^\beta\right)\wedge \tilde{e}^\gamma = 0\,,$$
$$-8\pi G\sigma_\perp \frac{\delta S}{\delta \tilde{e}^\alpha} = \tilde{W}_\alpha + \frac{2}{\ell^2}\Sigma_\alpha + \epsilon_{\alpha\beta\gamma}\tilde{e}^\beta \wedge \dot{K}^\gamma = 0$$

GATHER EVERYTHING - ALL CLASSICAL BACKGROUNDS IN 4D SATISFY:

$$\boxed{K_\alpha \wedge \tilde{e}^\alpha = 0\,, \qquad \tilde{\mathcal{D}}\tilde{e}^\alpha = 0\,, \qquad \dot{\tilde{e}}^\alpha + K^\alpha = 0}$$

$$\boxed{\tilde{W}_\alpha \wedge \tilde{e}^\alpha = 0\,, \qquad \epsilon_{\alpha\beta\gamma}\tilde{\mathcal{D}}K^\beta \wedge \tilde{e}^\gamma = 0\,, \tilde{W}_\alpha + \epsilon_{\alpha\beta\gamma}\left(\dot{K}^\beta + \frac{1}{\ell^2}\tilde{e}^\beta\right)\wedge \tilde{e}^\gamma = 0}$$

AN IMPORTANT QUANTITY IN THE FOLLOWING IS THE ON-SHELL WEYL TENSOR

$$\boxed{W^{ab} = R^{ab} + \Lambda e^a \wedge e^b}$$

WHOSE COMPONENTS ARE

$$\boxed{\sigma_\perp W^{0\alpha} = dt \wedge \left(\dot{K}^\alpha + \frac{1}{\ell^2}\tilde{e}^\alpha\right) + \tilde{\mathcal{D}}K^\alpha\,, \quad W^\alpha = \frac{\sigma_\perp}{2}\epsilon^\alpha{}_{\beta\gamma}W^{\beta\gamma} = dt \wedge \dot{B}^\alpha + \tilde{W}^\alpha}$$

HOLOGRAPHY VS THE INITIAL VALUE FORMULATION OF GRAVITY

CONSIDER A 3-DIM MANIFOLD Σ

WITH A DREIBEIN, SPIN-CONNECTION AND 1-FORM $(\epsilon^\alpha, b^\alpha, \kappa^\alpha)$

SATISFYING

$$\tilde{\mathcal{D}}_b \epsilon^\alpha = 0\,, \qquad \tilde{w}_\alpha \wedge \varepsilon^\alpha = 0\,, \qquad \epsilon_{\alpha\beta\gamma} \tilde{\mathcal{D}}_b \kappa^\beta \wedge \varepsilon^\gamma = 0$$

$$\tilde{w}_\alpha = \tilde{d} b_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma} b^\beta \wedge b^\gamma - \frac{1}{2}\epsilon_{\alpha\beta\gamma}\kappa^\beta \wedge \kappa^\gamma + \frac{1}{2\ell^2}\epsilon_{\alpha\beta\gamma}\varepsilon^\beta \wedge \varepsilon^\gamma$$

THERE EXISTS A UNIQUE (IN THE ABSENCE OF A COSM. CONST.) SPACETIME

$$(\mathcal{M}, g) \qquad\qquad g = \sigma_\perp dt \otimes dt + \tilde{e}^\alpha \otimes \tilde{e}_\alpha$$

WITH DREIBEIN, SPIN-CONNECTION AND EXTRINSIC CURVATURE OF THE SLICES

$$\lim_{t\to t_0} \tilde{e}^\alpha = \varepsilon^\alpha \qquad \lim_{t\to t_0} B^\alpha = b^\alpha\,, \qquad \lim_{t\to t_0} K^\alpha = k^\alpha$$

Σ_{t_0} CAUCHY SURFACE

TRY TO IMPLEMENT A SIMILAR ARGUMENT IN THE PRESENCE OF A C.C.

THE FEFFERMAN-GRAHAM EXPANSION

[C. Fefferman, C. R. Graham "The ambient metric" 0710.0919]

$$\tilde{e}^\alpha = e^{t/\ell} E^\alpha(x) + e^{-t/\ell} \sum_{k=0} F^\alpha_{[k+2]}(x) e^{-kt/\ell}$$

CONFORMAL CLASS

IN THE BOUNDARY

$t \to \infty$ BOUNDARY

$$K^\alpha = -\frac{1}{\ell} e^{t/\ell} E^\alpha + \frac{1}{\ell} e^{-t/\ell} \sum_{k=0} (k+1) F^\alpha_{[k+2]} e^{-kt/\ell}$$

$$B^\alpha = \sum_{k=0} B^\alpha_{[k]} e^{-kt/\ell}$$

THE FIRST FEW TERMS IN THE EXPANSION YIELD

$$F^\alpha_{[2]} \wedge E_\alpha = F^\alpha_{[3]} \wedge E_\alpha = F^\alpha_{[4]} \wedge E_\alpha = 0$$

$$\mathcal{D}_{[0]} E^\alpha = B^\alpha_{[1]} = 0$$

THE COEFFICIENTS ARE SYMMETRIC

B_0 IS BOUNDARY SPIN-CONNECTION

NEXT WE COMPUTE THE ON-SHELL WEYL TENSOR

$$
\begin{aligned}
\dot{K}^\alpha + \frac{1}{\ell^2}\tilde{e}^\alpha &= -\frac{1}{\ell^2}\sum_{k=0}^{\infty}[(k+2)^2-1]F^\alpha_{[k+3]}e^{-(k+2)t/\ell}\,, \\
\tilde{\mathcal{D}}K^\alpha &= -e^{-t/\ell}\frac{2}{\ell}\epsilon^\alpha{}_{\beta\gamma}B^\beta_{[2]}\wedge E^\gamma \\
&\quad -e^{-2t/\ell}\frac{3}{\ell}\epsilon^\alpha{}_{\beta\gamma}B^\beta_{[3]}\wedge E^\gamma + \mathcal{O}\left(e^{-3t/\ell}\right)\,, \\
\dot{B}^\alpha &= -\frac{1}{\ell}\sum_{k=0}^{\infty}(k+2)B^\alpha_{[k+2]}e^{-(k+2)t/\ell}\,, \\
\tilde{W}^\alpha &= \rho^\alpha_{[0]} + \frac{2}{\ell^2}\epsilon^\alpha{}_{\beta\gamma}F^\beta_{[2]}\wedge E^\gamma + e^{-t/\ell}\frac{3}{\ell^2}\epsilon^\alpha{}_{\beta\gamma}F^\beta_{[3]}\wedge E^\gamma \\
&\quad + e^{-2t/\ell}\left[\mathcal{D}_{[0]}B^\alpha_{[2]} + \frac{4}{\ell^2}\epsilon^\alpha{}_{\beta\gamma}F^\beta_{[4]}\wedge E^\gamma\right] + \mathcal{O}\left(e^{-3t/\ell}\right)
\end{aligned}
$$

THESE GIVE

$$\epsilon_{\alpha\beta\gamma}\tilde{\mathcal{D}}K^\beta \wedge \tilde{e}^\gamma = 0 \quad \Rightarrow \quad B^\alpha_{[2]} \wedge E_\alpha = 0\,, B^\alpha_{[3]} \wedge E_\alpha = 0$$

GAUSS LAW

SYMMETRIC TENSORS

$$\boxed{\tilde{W}_\alpha + \epsilon_{\alpha\beta\gamma}\left(\dot{K}^\beta + \frac{1}{\ell^2}\tilde{e}^\beta\right) \wedge \tilde{e}^\gamma = 0}$$

$$\boxed{\rho^\alpha_{[0]} + \frac{2}{\ell^2}\epsilon^\alpha{}_{\beta\gamma}F^\beta_{[2]} \wedge E^\gamma + e^{-2t/\ell}\left[\mathcal{D}_{[0]}B^\alpha_{[2]} - \frac{4}{\ell^2}\epsilon^\alpha{}_{\beta\gamma}F^\beta_{[4]} \wedge E^\gamma\right] + \mathcal{O}\left(e^{-3t/\ell}\right) = 0}$$

THIS ASSOCIATES THE F-COEFFICIENTS TO BOUNDARY GEOMETRIC QUANTITIES

$$\rho^\alpha_{[0]} + \frac{2}{\ell^2}\epsilon^\alpha{}_{\beta\gamma}F^\beta_{[2]} \wedge E^\gamma = 0$$

$$-\frac{2\sigma_\perp}{\ell^2}F^\alpha_{[2]} = {}^{(3)}S^\alpha = \mathrm{Ric}^\alpha - \frac{R}{4}E^\alpha\,,\ \mathrm{Ric}^\alpha = E_\beta \rfloor \rho^{\beta\alpha}\,,\ R = E_\alpha \rfloor \mathrm{Ric}^\alpha$$

SCHOUTEN TENSOR

THE MAGNETIC FIELD HAS AN INTRIGUING GEOMETRICAL MEANING

$$B^\alpha_{[2]} = -\sigma_\perp {}^{\tilde{*}}\mathcal{D}_{[0]} F^\alpha_{[2]} = \frac{\ell^2}{2} {}^{\tilde{*}}C^\alpha$$

$$C^\alpha = \mathcal{D}_{[0]} {}^{(3)}S^\alpha$$

COTTON TENSOR:

ZERO IF 3D METRIC CONFORMALLY FLAT

FURTHER COEFFICIENTS CAN BE FOUND I.E. $F^\alpha_{[4]} = \sigma_\perp \frac{\ell^4}{8} {}^{\tilde{*}}\mathcal{D}_{[0]} {}^{\tilde{*}}C^\alpha$

HOWEVER:

$F^\alpha_{[3]}$ IS AN INDEPENDENT DATA, SYMMETRIC, TRACELESS AND CONSERVED

$$F^\alpha_{[3]} \wedge E_\alpha = 0\,,\ ,\ \epsilon_{\alpha\beta\gamma} F^\alpha_{[3]} \wedge E^\beta \wedge E^\gamma = 0\,,\ \epsilon_{\alpha\beta\gamma} \mathcal{D}_{[0]} F^\beta_{[3]} \wedge E^\gamma = 0$$

THE “INITIAL POSITION” IS E^α

THE “INITIAL VELOCITY” IS $F^\alpha_{[3]}$

THE PROBLEM, HOWEVER, IS THAT BOTH THE DREIBEIN AND THE ELECTRIC FIELDS GIVE THE SAME BOUNDARY DATA - RECALL THE SCALAR FIELD CASE

$$K^{\alpha} = -\frac{1}{\ell}\tilde{e}^{\alpha} + \mathcal{O}\left(e^{-t/\ell}\right)$$

THE QUANTITY THAT APPEARS TO YIELD THE INDEPENDENT BOUNDARY DATA IS RELATED TO THE WEYL TENSOR

$$\tilde{W}^{\alpha} = e^{-t/\ell}\frac{3}{\ell^2}\epsilon^{\alpha}{}_{\beta\gamma}F^{\beta}_{[3]} \wedge E^{\gamma} + \mathcal{O}\left(e^{-2t/\ell}\right)$$

$$\tilde{W}^{\alpha} = \sigma_{\perp}\epsilon^{\alpha}{}_{\beta\gamma}\mathcal{P}^{\beta} \wedge \tilde{e}^{\gamma}$$

$$\mathcal{P}^{\alpha} = \sigma_{\perp}\frac{3}{\ell^2}e^{-2t/\ell}F^{\alpha}_{[3]} + O\left(e^{-3t/\ell}\right)$$

IF ONE WANTS TO PROPERLY DEFINE THE HAMILTONIAN EVOLUTION IT APPEARS THAT ONE HAS A CHOICE OF TWO SETS OF VARIABLES

$$\{\tilde{e}^{\alpha}, \mathcal{P}^{\alpha}\}, \qquad \{K^{\alpha}, \mathcal{P}^{\alpha}\}$$

THE CHOICE IS TO EITHER

FIX IN THE BOUNDARY $\tilde{e}^\alpha$

DO A CANONICAL TRANSFORMATION $K^\alpha \mapsto \mathcal{P}^\alpha$

FIX IN THE BOUNDARY K^α

DO A CANONICAL TRANSFORMATION $\tilde{e}^\alpha \mapsto \mathcal{P}^\alpha \equiv \Sigma_\alpha \mapsto \tilde{W}_\alpha$

IN THE FIRST CASE WE ADD THE GIBBONS-HAWKING TERM AND OBTAIN

$$\delta S\Big|_{\mathrm{os}} = \frac{\sigma_\perp}{8\pi G}\int_{\partial\mathcal{M}} \epsilon_{\alpha\beta\gamma} K^\alpha \wedge \tilde{e}^\beta \wedge \delta\tilde{e}^\gamma$$

IN THE SECOND CASE WE DO NOT ADD THE GIBBONS-HAWKING TERM AND OBTAIN

$$\delta S_{\mathrm{EH}}\Big|_{\mathrm{os}} = -\frac{\sigma_\perp}{8\pi G}\int_{\partial\mathcal{M}} \Sigma_\alpha \wedge \delta K^\alpha$$

WE EXPECT THAT IN BOTH CASES WE OBTAIN THE SAME PHYSICAL INFO

EXAMPLES: HOLOGRAPHIC RENORMALIZATION, KOUNTERTERMS, BLACK-HOLE HOLOGRPHY, SELF-DUALITY

HOLOGRAPHIC RENORMALIZATION

WE WISH TO CANONICALLY TRANSFORM K TO THE QUANTITY W' SUCH THAT

$$\epsilon_{\alpha\beta\gamma}K^\beta \wedge \tilde{e}^\gamma \equiv \ell \tilde{W}'_\alpha - \ell\rho_\alpha - \frac{2}{\ell}\Sigma_\alpha$$

$$\tilde{W}'_\alpha = e^{-t/\ell}\frac{3}{\ell^2}\epsilon_{\alpha\beta\gamma}F^\beta_{[3]} \wedge E^\gamma + O\left(e^{-2t/\ell}\right)$$

USING W' INSTEAD OF K, THE ACTION BECOMES

VANISHES

$$S = \frac{\sigma_\perp \ell}{8\pi G}\int_{\mathcal{M}} \tilde{W}'_\alpha \wedge d\tilde{e}^\alpha - \frac{\sigma_\perp \ell}{8\pi G}\int_\partial \left[\dot{B}_\alpha \wedge \tilde{e}^\alpha \wedge dt + \rho_\alpha \wedge \tilde{e}^\alpha + \frac{1}{3\ell^2}\epsilon_{\alpha\beta\gamma}\tilde{e}^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma\right]$$

MINUS THE HOLOGRAPHIC RENORM. TERMS

SUBTRACTING THOSE, WE OBTAIN:

$$\delta S'_{\text{ren.}}\Big|_{\text{os}} = \frac{3\sigma_\perp}{8\pi G\ell}\int_\partial \epsilon_{\alpha\beta\gamma}F^\alpha_{[3]} \wedge E^\beta \wedge \delta E^\gamma + \mathcal{O}(e^{-t/\ell})$$

$$\tau_\alpha \equiv \frac{\delta S'_{\text{ren.}}}{\delta E^\alpha} = \frac{3\sigma_\perp}{8\pi G\ell}\epsilon_{\alpha\beta\gamma}F^\beta_{[3]} \wedge E^\gamma = \frac{\sigma_\perp \ell}{8\pi G}\lim_{t\to+\infty} e^{t/\ell}\tilde{W}_\alpha$$

$$\langle T_{ij}\rangle_s = E^\alpha{}_i\left(\tilde{*}\tau_\alpha\right)_j = \frac{3}{8\pi G\ell}F_{[3]\;ij}$$

KOUTERTERMS

WE WISH TO CANONICALLY TRANSFORM E TO THE QUANTITY W SUCH THAT

$$\Sigma_\alpha = \ell^2 \tilde{W}_\alpha - \ell^2 \rho_\alpha + \frac{\ell^2}{2}\epsilon_{\alpha\beta\gamma} K^\beta \wedge K^\gamma$$

USING W INSTEAD OF E, THE ACTION BECOMES

$$\begin{aligned} S_{\mathrm{EH}} \mapsto S'_{\mathrm{EH}} &= -\frac{\sigma_\perp \ell^2}{8\pi G}\int_{\mathcal{M}} \left[\tilde{W}_\alpha \wedge K^\alpha + \tilde{D}K_\alpha \wedge B^\alpha\right] \\ &\quad + \frac{\sigma_\perp \ell^2}{8\pi G}\int_\partial \left[\rho_\alpha \wedge K^\alpha - \frac{1}{6}\epsilon_{\alpha\beta\gamma} K^\alpha \wedge K^\beta \wedge K^\gamma\right] \end{aligned}$$

VANISHES AT THE BOUNDARY

THE BOUNDARY TERM IS MINUS THE EULER DENSITY

$$\chi = -\frac{\sigma_\perp \ell^2}{64\pi G}\int \epsilon_{abcd} R^{ab} \wedge R^{cd} = -\frac{\sigma_\perp \ell^2}{8\pi G}\int_\partial \left[\rho_\alpha \wedge K^\alpha - \frac{1}{6}\epsilon_{\alpha\beta\gamma} K^\alpha \wedge K^\beta \wedge K^\gamma\right]$$

SUBTRACTING THE EULER DENSITY, GIVES

$$S_{\mathrm{ren.}}\Big|_{os} = S'_{\mathrm{EH}} + \chi = -\frac{\sigma_\perp \ell^2}{8\pi G}\int_{\mathcal{M}} \tilde{W}_\alpha \wedge dK^\alpha$$

ITS VARIATION YIELDS THE SAME BOUNDARY E.M. TENSOR

NOTE: K IS HELD FIXED

REMARKABLY, THE ON-SHELL ACTION COINCIDES WITH THE MM ACTION

$$S_{\mathrm{MM}} = -\frac{\sigma_\perp \ell^2}{64\pi G}\int \epsilon_{abcd} W^{ab} \wedge W^{cd}$$

CONSIDER THE GENERIC AADS4 BLACK HOLE

$$ds^2 = \sigma_\perp \frac{dr^2}{V(r)} - \sigma_\perp V(r) d\tau^2 + r^2 d\Omega_\kappa^2 \qquad V(r) = \sigma_\perp \kappa - \frac{2M}{r} + \frac{r^2}{\ell^2} \qquad \kappa = 0, \pm 1$$

$$d\Omega_\kappa^2 = e^{2\gamma} dw \bar{w}\,, \qquad e^\gamma = (1 + \kappa |w|^2/4)^{-1}\,,\; w = x + iy$$

THE BULK DREIBEIN, ELECTRIC AND MAGNETIC FIELDS ARE

$$e^0 = V(r)^{-1/2} dr\,, \qquad \tilde{e}^3 = V(r)^{1/2} d\tau\,, \qquad \tilde{e}^\bullet = r e^\gamma dw$$

$$K^3 = -\left(\frac{M}{r^2} + \frac{r}{\ell^2}\right) dz\,, \qquad K^\bullet = -V(r)^{1/2} e^\gamma dw$$

$$B^3 = -i\left(\partial\gamma dw - \bar{\partial}\gamma d\bar{w}\right)\,, \qquad B^\bullet = 0$$

THE TENSOR P RELATED TO THE WEYL TENSOR IS

$$\mathcal{P}^3 = -\sigma_\perp \frac{2M}{r^3} \tilde{e}^3\,, \qquad \mathcal{P}^\bullet = \sigma_\perp \frac{M}{r^3} \tilde{e}^\bullet$$

SINCE HERE WE HAVE A NON-TRIVIAL LAPSE, WE NEED TO BE MORE CAREFUL DEFINING THE BOUNDARY QUANTITIES.

GENERICALLY, FOR A METRIC OF THE FORM

$$ds^2 = \sigma_\perp N(\rho)^2 d\rho^2 + h_{ij}(\rho, \vec{x}) dx^i dx^j \, , \; N(\rho) = 1 + \zeta(\rho) \, , \, \zeta(\rho) \to 0 \, , \, \rho \to \infty$$

WE CAN DEFINE THE FG VARIABLE T AS:

$$N(\rho) d\rho = dt \, , \; e^{t/\ell} = e^{\rho/\ell} \left[1 + \epsilon(\rho)\right] \, , \; \frac{\zeta(\rho)}{\ell} = \frac{\epsilon'(\rho)}{1 + \epsilon(\rho)}$$

IN SADS4 BLACK-HOLE THIS CAN BE DONE AND WE OBTAIN THE USUAL

$$E^3 = d\tau \, , \qquad E^\bullet = \ell \, e^\gamma dw$$

$$F^3_{[3]} = -\frac{2M\ell^2}{3} E^3 \, , \, F^\bullet_{[3]} = \frac{M\ell^2}{3} E^\bullet \; \Rightarrow \; \langle T_{33} \rangle_s = \sigma_\perp \frac{M\ell}{4\pi G} \, , \, \langle T_{\bullet\bar{\bullet}} \rangle_s = \frac{M\ell}{8\pi G}$$

IN THE CASE OF TAUB-NUT-ADS4

$$ds^2 = \sigma_\perp \frac{dr^2}{V(r)} - \sigma_\perp V(r) \left(d\tau + \sigma\right)^2 + \left(r^2 + n^2\right) e^{2\gamma} dw d\bar{w}$$

$$V(r) = \left(\sigma_\perp \kappa + \frac{4n^2}{\ell^2}\right) \frac{r^2 - n^2}{r^2 + n^2} - \frac{2Mr}{r^2 + n^2} + \frac{r^2 + n^2}{\ell^2} \qquad \sigma = -i \frac{\sigma_\perp n}{2} e^\gamma \left(\bar{w} dw - w d\bar{w}\right)$$

THE VARIOUS QUANTITIES ARE:

$$\tilde{e}^3 = V(r)^{1/2} \left(d\tau + \sigma\right) \,, \qquad \tilde{e}^\bullet = (r^2 + n^2)^{1/2} e^\gamma dw$$

$$K^3 = -\frac{1}{2} V'(r) V(r)^{-1/2} \tilde{e}^3 \,, \qquad K^\bullet = -\frac{r}{r^2 + n^2} V(r)^{1/2} \tilde{e}^\bullet$$

$$B^3 = i \frac{\kappa}{4} e^\gamma \left(\bar{w} w - w \bar{w}\right) - \frac{n}{r^2 + n^2} V(r)^{1/2} \tilde{e}^3 \,, \; B^\bullet = \frac{n}{r^2 + n^2} V(r)^{1/2} \tilde{e}^\bullet$$

MOREOVER

$$\mathcal{P}^3 = -\sigma_\perp 2F(r)\tilde{e}^3\,, \qquad \mathcal{P}^\bullet = \sigma_\perp F(r)\tilde{e}^\bullet$$

$$F(r) = \frac{Mr\left(r^2-3n^2\right)+n^2\left(\sigma_\perp\kappa+\frac{4n^2}{\ell^2}\right)\left(3r^2-n^2\right)}{(r^2+n^2)^3}$$

THE BOUNDARY DREIBEIN IS THEN:

$$E^3 = d\tau + \sigma\ , \qquad E^\bullet = \ell\, e^\gamma dw$$

$$\kappa = 1 \ \Rightarrow\ g_{\phi\phi} = \sin^2\theta - 16\frac{n^2}{\ell^2}\sin^4\frac{\theta}{2} < 0$$

CONFORMALLY FLAT

BUT NONTRIVIAL

THE BOUNDARY SPACETIME HAS CLOSED

TIMELIKE CURVES:

IT MAY DESCRIBE SUPERFLUID VORTICES

$$F^3_{[3]} = -\frac{2M\ell^2}{3}E^3\,, \qquad F^\bullet_{[3]} = \frac{M\ell^2}{3}E^\bullet$$

THE SAME BOUNDARY E.M.

TENSOR

SELF-DUALITY IN ADS4

MORALLY SPEAKING, WE DID NOT DO HOLOGRAPHY SO FAR.

HOLOGRAPHY WOULD EMERGE FROM GRAVITY FLUCTUATIONS AROUND THOSE BACKGROUNDS WITH THE IMPOSITION OF THE APPROPRIATE B.C.

IN THE HOLOGRAPHIC LANGUAGE, THOSE BACKGROUND CONFIGURATIONS CORRESPOND TO THE VACUUM OF THE BOUNDARY THEORY:

THEY PROVIDE THE BOUNDARY METRIC AND AN INDEPENDENT EXPECTATION VALUE OF THE E.M. TENSOR

USUALLY, ONE DOES NOT THINK OF THE BOUNDARY METRIC AS A SOURCE - FLUCTUATIONS OF THE METRIC SOURCE THE BOUNDARY E.M. TENSOR.

HOWEVER, LETS US MOMENTARILY VIEW THE BOUNDARY METRIC - OR RATHER ITS CONFORMAL CLASSES - AS SOURCES FOR THE BOUNDARY E.M. TENSOR. IN PRINCIPLE, SUCH A PROCEDURE COULD CLASSIFY ALL 3D VACUA ON NON-TRIVIAL BACKGROUNDS - AT ZERO TEMPERATURE.

IN VIEW OF ALL THE ABOVE, AN EQUIVALENT VIEW OF 3D VACUA IS THE CLASSIFICATION OF POSSIBLE "POSITIONS" (METRICS) AND "VELOCITIES" (E.M. TENSORS) THAT DEVELOP INTO TO WELL-DEFINED BULK METRICS.

A SIMPLE EXAMPLE: ZERO BOUNDARY E.M. TENSOR **[K. Skenderis & S. Solodukhin (99)]**

THIS MEANS $\boxed{F^{\alpha}_{[3]} = 0}$

AS A RESULT, THE BULK DREIBEIN HAS A FINITE FG EXPANSION, THE BOUNDARY DREIBEIN IS CONFORMALLY FLAT AND THE BOUNDARY COTTON TENSOR VANISHES. THE BOUNDARY GEOMETRICAL DATA SATISFY THE E.O.M. OF A SO(3,2) OR SO(4,1) CHERN-SIMONS THEORY WHOSE SOLUTIONS ARE CONFORMALLY FLAT CONNECTIONS.

THE RESULTING BULK METRIC HAS ZERO WEYL TENSOR AND IS TORSIONLESS.

THE E.O.M. OF EINSTEIN GRAVITY IN THE PRESENCE OF A C.C. READ:

$$\hat{*}W^a{}_b \wedge e^b = 0\,, \qquad T^a = 0 \quad \text{implies} \quad W^a{}_b \wedge e^b = 0$$

HENCE, THEY ARE SOLVED BY SELF-DUAL CONFIGURATIONS

$$W^{ab} = \pm \hat{*}W^{ab} \qquad\qquad W^{0\alpha} = \pm \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma} W^{\beta\gamma} = \mp W^\alpha$$

IN TERMS OF THE ELECTRIC AND MAGNETIC FIELDS, THEY IMPLY:

$$\begin{aligned} (K^\alpha \pm B^\alpha)^{\cdot} = -\Lambda \tilde{e}^\alpha\,, \\ \tilde{d}\left(K^\alpha \pm B^\alpha\right) \pm \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma}\left(K^\beta \pm B^\beta\right) \wedge \left(K^\gamma \pm B^\gamma\right) = \pm \Lambda \Sigma^\alpha \end{aligned}$$

THEIR ON-SHELL WEYL TENSOR HAS THE EXPANSION

$$\tilde{W}^\alpha = -e^{-t/\ell}\frac{3}{\ell^2}\epsilon^\alpha{}_{\beta\gamma} F^\beta_{[3]} \wedge E^\gamma - e^{-2t/\ell}\frac{8}{\ell^2}\epsilon^\alpha{}_{\beta\gamma} F^\beta_{[4]} \wedge E^\gamma + \mathcal{O}\left(e^{-3t/\ell}\right)$$

RECALL THAT

$$\langle T_{ij}\rangle_s = E^{\alpha}{}_i \left(\tilde{*}\tau_\alpha\right)_j = -\frac{3}{8\pi G\ell} F_{[3]\;ij}$$

WE CAN SEE THAT THE SELF-DUALITY CONDITIONS IMPLY:

$$\frac{1}{\ell}\left[(k+2)^2 - 1\right] F^{\alpha}_{[k+3]} = \mp (k+2) B^{\alpha}_{[k+2]}$$

HENCE, THEY IMPLY A PARTICULAR RELATIONSHIP THAT THE "INITIAL VELOCITY" HAS WRT TO THE "INITIAL POSITION"

$$\boxed{F^{\alpha}_{[3]} = \mp\frac{2}{3}\ell B^{\alpha}_{[2]} = \pm\frac{\ell^3}{3}\tilde{*}C^{\alpha} \quad \Rightarrow \quad \langle T_{ij}\rangle = \mp\frac{\ell^2}{8\pi G}\tilde{*}C_{ij}}$$

THE EFFECTIVE ACTION OF THE BOUNDARY THEORY IS THE GRAVITATIONAL CHERN-SIMONS

CONCLUSIONS-PROSPECTS

- HOLOGRAPHY IS INTIMATELY RELATED TO HAMILTONIAN DYNAMICS

REVERSING THE LOGIC IT SEEMS NATURAL TO EXPECT THAT MANY INTERESTING DYNAMICAL SYSTEMS (INTEGRABLE SYSTEMS, CHAOS, TURBULENCE) SHOULD HAVE GRAVITATIONAL DESCRIPTIONS.

- "HOLOGRAPHIC RENORMALIZATION" = CANONICAL TRANSFORMATIONS.