Coset CFTs, high spin sectors and non-abelian T-duality

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Motivation

Field equations in curved stringy backgrounds

In string theory want to go beyond (super)gravity:

- $\exists$ exact theories realizing this, particularly, coset (G/H) CFTs .
- When groups are non-abelian there are no isometries (generic).
- Solving the field equations is an impossible task with traditional methods, i.e. separation of variables.
- In physical applications this is precisely what is needed, i.e. propagating fluctuations, etc.

Understanding Non-abelian T-duality

Unlike abelian T-duality:

- Not well understood.
- Not likely to be an exact symmetry.
- Yet, what is it good for? Maybe for some effective description?

Outline

- Gauged WZW models and Non-abelian T-duality:
  - Gauged WZW models and their geometry.
  - Non-abelian WZW T-duals and their geometry.
  - Relating non-abelian T-duality to gauged WZW models.
- Solving field equations in coset CFT backgrounds.
- Example: $SU(2) \times SU(2)/SU(2)$
- The infinitely large spin limit and the effective non-abelian T-dual.
- Example: Non-abelian T-dual of $SU(2)$ WZW.
- Concluding remarks.
- Towards Non-Abelian T-duality in RR-backgrounds.

Gauged WZW models and Non-abelian T-duality

Gauged WZW models: Action

Let a group G and a subgroup $H \in G$.
Introduce $g \in G$ and gauge fields $A_\pm \in \mathcal{L}(H)$.

- The gauged WZW action is

$$S(g, A_\pm) = k \overbrace{I_0(g)}^{\text{WZW}} + \frac{k}{\pi} \int d\sigma^+ d\sigma^- \ \mathrm{Tr}\Big[A_- \partial_+ g g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_- A_+\Big] \ .$$

  Not minimally coupled gauged fields.

- Gauge invariance: For $\Lambda(\sigma^+, \sigma^-) \in H$

$$g \to \Lambda^{-1} g \Lambda \ , \qquad A_\pm \to \Lambda^{-1}(A_\pm - \partial_\pm)\Lambda \ .$$

Gauged WZW models: The geometry

- The gauge fields are non-dynamical and can be integrated out

$$A_+^a = -(M^{-1})^{ab}(\partial_+ g g^{-1})_b \ , \qquad A_-^a = (g^{-1}\partial_- g)_b (M^{-1})^{ba} \ ,$$

where $(a, b \in H)$

$$M_{ab} = \mathrm{Tr}(t_a g t_b g^{-1}) - \eta_{ab} \ .$$

- Gauge fix $\dim(H)$ parameters in g or, better, choose those that are H-singlets. That leaves $\dim(G/H)$ x^μ's.
- One obtains the σ-model of the form (only NS fields)

$$S = \frac{k}{\pi}\int (G_{\mu\nu} + B_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu$$

and a dilaton

$$\Phi = -\frac{1}{2}\ln\det(M) \ .$$

- Background solves beta-functions for conformal invariance.
- Isometries $G_L \times G_R$ of the WZW are generically broken.

Non-abelian WZW T-duals and their geometry

- The starting action is

$$S_{\text{NonAb}}(g, v, A_{\pm}) = S(g, A_{\pm}) - i\frac{k}{\pi}\int \underbrace{\mathrm{Tr}(vF_{+-})}_{\text{Lagrange mult.}} \quad .$$

- Gauge invariance: As before and in addition $v \to \Lambda^{-1} v \Lambda$.
- Gauge fix $\dim(H)$ parameters in g and v, leaving $\dim(G)$ variables X^M. Integrate out the $A_{\pm}$'s and get the σ-model.
- Properties:
  - Isometries $G_L \times G_R$ of the WZW generically broken.
  - Even if G is compact, (some) variables of its non-abelian dual appear non-compact.
  - Transformation is not invertible at the action level.
  - Drastically different than abelian T-duality.
- Is it useful? What does it describe?

Relating non-abelian T-duality to gauged WZW models

- Start with the gauged WZW action for $\frac{G_k \times H_\ell}{H_{k+\ell}}$ for two group elements $g \in G$, $h \in H$ and gauged field in $\mathcal{L}(H)$.
- Expand infinitesimally around the identity

$$h = \mathbb{I} + i\,\frac{k}{\ell}\,v + \mathcal{O}\left(\frac{1}{\ell^2}\right)$$

and take the limit $\ell \to \infty$.

- We get the non-abelian T-duality action, i.e. classically **[KS 94]**

$$\boxed{\left.\frac{G_k \times H_\ell}{H_{k+\ell}}\right|_{\ell\to\infty} = \text{ dual of } G_k \text{ with respect to } H}\,.$$

- Remarks:
  - In the limit some variables become non-compact.
  - A well defined limit can be taken on the geometric background.
  - Can this be the effective background describing a consistent sector of the parent theory?

Solving field equations in coset CFT backgrounds

For the background corresponding to

$$\boxed{G_k \times H_\ell / H_{k+\ell}} \ ,$$

we would like to solve the scalar equation

$$-\frac{1}{e^{-2\Phi}\sqrt{G}}\ \partial_\mu e^{-2\Phi}\sqrt{G}G^{\mu\nu}\partial_\nu\Psi = E\Psi\ .$$

- We will obtain its general solution from that of the scalar equation for the WZW model for $G \times H$.
- Start with Reps of $G \times H$. The eigenstates are

  $$R_{\alpha\beta}(g)\ r_{\mu\nu}(h)\ ,$$

  which are, relatively, easy to construct.
- The eigenvalues (semiclassically, for $k, \ell \gg 1$) are

  $$E(R,r) = \frac{C_2(R)}{k} + \frac{C_2(r)}{\ell}\ ,$$

  where the C_2's are the Casimirs.

- Under the vector H-transf they transform as

$$(R \times r) \times (\bar{R} \times \bar{r}) = (r_1 \oplus r_2 \oplus \cdots) \otimes (\bar{r}_1 \oplus \bar{r}_2 \oplus \cdots) \,.$$

- We decompose $R \times r$ and its conjugate into Reps r_i of H.
- We get a singlet from all products of the form $r_i \times \bar{r}_i$.
- These singlets, representing the coset eigenstates, are

$$\boxed{\psi_{R,r;r_i}(g,h) = \sum_{a;\alpha,\beta,\mu,\nu} \underbrace{C^a_{\alpha\mu}(R,r;r_i)\, C^a_{\beta\nu}(R,r;r_i)}_{\text{Clebsch}-\text{Gordan}} \underbrace{\overbrace{R_{\alpha\beta}(g)\, r_{\mu\nu}(h)}^{G\times H}}_{\text{gauged fixed}}} \,.$$

- Conclusion: The states in the $G \times H/H$ coset theory are the H-singlet combinations of the states in the WZW model $G \times H$ as this is dictated by group theory.

Remarks:

- The eigenvalues get shifted as

$$E(R, r; r_i) = \frac{C_2(R)}{k} + \frac{C_2(r)}{\ell} - \frac{C_2(r_i)}{k+\ell} \ .$$

  This is in accordance to the algebraic coset construction [Goddard-Kent-Olive, 85].
- The coset background fields receives $1/k$ corrections. They become simple in the semiclassical limit for $k \gg 1$.
- Remarkably, the eigenstates do not depend on $\alpha' \sim 1/k$, only the eigenvalues do (indicated expressions are for $k \gg 1$).

Example: $SU(2) \times SU(2)/SU(2)$

Parametrization

We parametrize $g_1 \times g_2 \in SU(2) \times SU(2)$ as

$$g_1 = \begin{pmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{pmatrix} , \quad g_2 = \begin{pmatrix} \beta_0 + i\beta_3 & \beta_2 + i\beta_1 \\ -\beta_2 + i\beta_1 & \beta_0 - i\beta_3 \end{pmatrix} ,$$

where from unitarity

$$\alpha_0^2 + \vec{\alpha}^2 = 1 , \qquad \beta_0^2 + \vec{\beta}^2 = 1 .$$

- Under the diagonal $SU(2)$ they transform as vectors

$$\delta\alpha_i = \epsilon_{ijk}\alpha_j\epsilon_k , \qquad \delta\beta_i = \epsilon_{ijk}\beta_j\epsilon_k ,$$

- The $SU(2)$-singlets are α_0, β_0 and $\gamma = \vec{\alpha} \cdot \vec{\beta}$, obeying

$$0 \leqslant \alpha_0, \beta_0 \leqslant 1 , \qquad |\gamma| \leqslant \sqrt{1 - \alpha_0^2}\sqrt{1 - \beta_0^2} .$$

and represent the three compact geometrical coordinates.

The background fields

Following the general procedure outlined above...

- The metrics is

$$
\begin{aligned}
ds^2 &= \frac{k_1 + k_2}{(1-\alpha_0^2)(1-\beta_0^2)-\gamma^2}\left(\Delta_{\alpha\alpha} d\alpha_0^2 + \Delta_{\beta\beta} d\beta_0^2 + \Delta_{\gamma\gamma} d\gamma^2\right.\\
&\quad \left. + 2\Delta_{\alpha\beta} d\alpha_0 d\beta_0 + 2\Delta_{\alpha\gamma} d\alpha_0 d\gamma + 2\Delta_{\beta\gamma} d\beta_0 d\gamma\right) .
\end{aligned}
$$

  The Δ's are functions of $\alpha_0, \beta_0, \gamma$ and of $r = k_1 / k_2$.

- The field $B_{\mu\nu} = 0$ and the dilaton

$$
\Phi = -\frac{1}{2} \ln \left((1-\alpha_0^2)(1-\beta_0^2) - \gamma^2 \right) .
$$

- Background is a bit complicated, with no isometries.

Solution of the eigenvalue problem

A general $SU(2)$ spin j Rep has matrix elements

$$R^j_{m_1,m_2}(a,b,c,d) = \sum_k A^j_{m_1,m_2,k}\ a^{j-m_1-k} d^{j+m_2-k} b^k c^{k+m_1-m_2}\ ,$$

The general state is

$$\boxed{\Psi^j_{j_1,j_2} = \sum_m \sum_{m_2,n_2=-j_2}^{j_2} \underbrace{C^{j,m}_{j_1,m-m_2,j_2,m_2}\ C^{j,m}_{j_1,m-n_2,j_2,n_2}}_{\text{Clebsch-Gordan}}\ \underbrace{\overbrace{R^{j_1}_{m-m_2,m-n_2}(g_1)\,R^{j_2}_{m_2,n_2}(g_2)}^{SU(2)\times SU(2)\ \text{d-functions}}}_{\text{gauged fixed}}}\ .$$

Examples:

- For $j_2 = 0$ and thus $j_1 = j$:

$$\Psi^j_{j,0} = \overbrace{\sum_{m=-j}^{j} R^j_{m,m}(g_1)}^{\text{Character}} = \underbrace{\overbrace{U_{2j}(\alpha_0)}^{\text{Chebyshev Pol.}}}_{\text{2nd kind}}\ .$$

- For $(j_1, j_2) = (1, 1/2)$:

$$\Psi^{1/2}_{1,1/2} = (4\alpha_0^2 - 1)\beta_0 + 4\alpha_0\gamma \ , \quad \Psi^{3/2}_{1,1/2} = (4\alpha_0^2 - 1)b_0 - 2\alpha_0\gamma \ .$$

For $(j_1, j_2) = (1, 1)$:

$$\begin{aligned} \Psi^0_{1,1} &= 4(\alpha_0\beta_0 + \gamma)^2 - 1 \ , \\ \Psi^1_{1,1} &= 6\alpha_0^2\beta_0^2 + 4\alpha_0\beta_0\gamma - 2\gamma^2 - 2(\alpha_0^2 + \beta_0^2) + 1 \ , \\ \Psi^2_{1,1} &= 26\alpha_0^2\beta_0^2 - 20\alpha_0\beta_0\gamma + 2\gamma^2 - 6(\alpha_0^2 + \beta_0^2) + 1 \ . \end{aligned}$$

- Due to luck of isometries, there is no factorization.
- With increasing $j_{1,2}$ expressions become complicated. Impossible to obtain with other methods.
- What about the high spin limit, i.e. when $j_1, j \gg 1$?

Large spins and the effective non-abelian T-dual

High spin limit in the $G_k \times H_\ell / H_{k+\ell}$ theory?

- Let Reps in $\mathcal{L}(H)$ with highest weight (spin) $j \gg 1$. The Reps in the tensor product with those in $\mathcal{L}(G)$ have also large spin.
- We may expand as

$$C_2(r) = a(r) j^2 + b(r) j + \mathcal{O}(1) \ .$$

- Similarly for $C_2(r_i)$, with j replaced by $j + n$ ($n =$finite).
- Keeping the eigenenergies finite requires the correlated limit

$$\ell = \frac{k}{\delta}\, j \to \infty \ , \qquad \delta = \text{positive real} \ .$$

  The limit of the eigenfunction is delicate. It involves the limiting behaviour of the Clebsch–Gordans.
- But, $\ell \to \infty$ is associated to the non-abelian T-dual of G_k.
- Hence:
  Non-abelian T-duality provides an effective description of the high spin sector of the parent theory.

Example: Non-abelian T-dual of $SU(2)$ WZW

The background fields

- Non-abelian T-dual of the $SU(2)$ WZW model w.r.t. $SU(2)$

  $$ds^2 = d\psi^2 + \frac{\cos^2\psi}{x_3^2}dx_1^2 + \frac{\left(x_3 dx_3 + (\sin\psi\cos\psi + x_1 + \psi)dx_1\right)^2}{x_3^2\cos^2\psi} ,$$

  plus a dilaton

  $$\Phi = -\frac{1}{2}\ln(x_3^2\cos^2\psi) .$$

  - A bit complicated with no isometries.
  - ψ is periodic and x_1, x_3 are non-compact.
- What do the eigenfunctions and eigenenergies look like? They should effectively describe the large spin sector of the $SU(2) \times SU(2)/SU(2)$ coset.

Solution of the eigenvalue problem

We will take the limit in the states and eigenvalues of the coset.

- Consider the high spin-level limit

$$j_1 = j - n\ , \quad k_1 = \frac{k_2}{\delta}\, j\ , \quad j_2, n = \text{finite}\ , \quad j \gg 1\ .$$

- The energy eigenvalues $E^{j}_{j_1,j_2} = \frac{j_1(j_1+1)}{k_1} + \frac{j_2(j_2+1)}{k_2} - \frac{j(j+1)}{k_1+k_2}$, remain finite

$$E_{j_2,n,\delta} = \lim_{j\to\infty} E^{j}_{j_1,j_2} = \frac{j_2(j_2+1)}{k_2} + \frac{\delta - 2n}{k_2}\,\delta\ .$$

- In the high spin limit the Clebsch–Gordan coefficients

$$\lim_{j\to\infty} C^{j,m}_{j-n,m-m_2,j_2,m_2} = d^{j_2}_{m_2,n}(\zeta)\ , \quad \cos\zeta = \frac{m}{j}\ .$$

  - They get associated with an auxiliary $SU(2)$ rotation.
  - Expected for a classical body given extra angular momentum.

- At the end we obtain the finite sum

$$\boxed{\Psi_{j_2,n,\delta}(x_1, x_3, \psi) = \lim_{j\to\infty} \Psi^{j}_{j-n,j_2} = \sum_{m_2=-j_2}^{j_2} \Gamma_{j_2,m_2,n,\delta}(x_3)\ \underbrace{R^{j_2}_{m_2,m_2}(g_2)}_{\text{gauged fixed}}}\ ,$$

where

$$\Gamma_{j_2,m_2,n,\delta}(x_3) = \int_0^{\pi} d\zeta \sin\zeta \left(d^{j_2}_{m_2,n}(\zeta)\right)^2 e^{-2i\delta v_3 \cos\zeta}\ .$$

- Explicit expressions become complicated fast, as j_2 increases.
- Fair to say:
Solution would have never been found without using this method.

Examples of states:

- Define

$$v_3 = \sqrt{(x_1+\psi)^2 + x_3^2}\ , \qquad \beta_0 = \sin\psi\ ,$$
$$\beta_1 = \frac{x_3 \cos\psi}{\sqrt{(x_1+\psi)^2 + x_3^2}}\ , \qquad \beta_3 = \frac{(x_1+\psi)\cos\psi}{\sqrt{(x_1+\psi)^2 + x_3^2}}\ .$$

- For instance, for $j_2 = 1$ (and $\delta = 1$):

$$\Psi_{1,\pm 1} = \frac{\beta_1^2 - 2\beta_3(\beta_3 \mp 2\beta_0 v_3)}{2v_3^2}\ \cos 2v_3$$
$$+ \frac{2\beta_3^2 - \beta_1^2 + \mp 4\beta_0\beta_3 v_3 + 4(\beta_0^2 - \beta_3^2)v_3^2}{4v_3^3}\ \sin 2v_3\ ,$$
$$\Psi_{1,0} = \frac{2\beta_3^2 - \beta_1^2}{v_3^2}\ \cos 2v_3 + \frac{\beta_1^2 - 2\beta_3^2 + 2(1-2\beta_1^2)v_3^2}{2v_3^3}\ \sin 2v_3\ .$$

Concluding remarks

- Using group theoretical methods one may solve field equations for general G/H, for which:
  - Generically, there are no isometries.
  - Conventional techniques are not applicable.
- Non-abelian duality generates solutions that:
  - effectively describe high spin sectors.
  - Taking the limit is a delicate procedure, but nevertheless the only way to solve field equation of the T-dual background.
- Method works in other occasions with non-abelian isometries. For instance, when the symmetry group acts from on side, i.e. in Principal Chiral Models.
- Use non-compact groups leading to Minkowski signature spacetimes, i.e. $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/SL(2,\mathbb{R})$. Explore physical applications, i.e. in cosmology.

Towards Non-Abelian T-duality in RR-backgrounds

with D. Thompson (Vrije Universiteit Brussels – Solvay Institute)

- So far Non-abelian T-duality has been formulated only in pure NS-NS backgrounds.
- What about backgrounds with RR-fluxes?
  - Abelian T-duality: From type-IIA to type-IIB and vice versa.
  - Natural expectation: Non-abelian T-duality changes (remains in the same) type-II theory if the dimension of the isometry group is odd (even).
- A natural formulation is within the pure spinor formalism allowing the description of the superstring in general curved backgrounds with non-trivial RR sectors [Berkovits 07].

Example: The near horizon of the D1-, D5-brane system.

- $AdS_3 \times S^3$, but with RR-fluxes (proportional to volume forms).
- NS-NS sector as in Principal Chiral Model for $SL(2,\mathbb{R}) \times SU(2)$.
- Some evidence that non-abelian T-duality w.r.t. $SU(2)$ gives a solution of the massive IIA Romans theory.