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DUAL PHOTONS AND GRAVITONS

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+ work in progress

PLAN

- Study electric/magnetic dual in 4-dim
  - Maxwell theory
  - Linearized gravity
- Resolve duality relations by group theory methods to obtain dual fields
- Apply to AdS_4 and study holographic implications
- Extend to $SAdS_4$
- Discussion of directions

MAXWELL THEORY

The field equations and the Bianchi identities

$$\nabla_\nu F^{\mu\nu} = 0 = \nabla_\nu \tilde{F}^{\mu\nu}$$

are interchanged under $*$.

In terms of physical fields

$$E_a = F_{ta} \, , \quad B_a = \tilde{F}_{ta}$$

duality relations are

$$E_a \to B_a \, , \quad B_a \to -E_a \, .$$

For given gauge field A_μ define the dual configuration $\tilde{A}_\mu$ by

$$\tilde{F}_{\mu\nu}(A) = F_{\mu\nu}(\tilde{A})$$

$$\tilde{F}_{\mu\nu}(\tilde{A}) = -F_{\mu\nu}(A)$$

LINEARIZED GRAVITY

Likewise, for linearized gravity around maximally symmetric background,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

one finds a dual configuration

$$\tilde{g}_{\mu\nu} = g^{(0)}_{\mu\nu} + \tilde{h}_{\mu\nu}$$

satisfying Einstein's equation.

The dual graviton is defined by

$$\tilde{C}_{\mu\nu\rho\sigma}(g) = C_{\mu\nu\rho\sigma}(\tilde{g})$$

$$\tilde{C}_{\mu\nu\rho\sigma}(\tilde{g}) = -C_{\mu\nu\rho\sigma}(g)$$

where $C_{\mu\nu\rho\sigma}$ is on-shell Weyl curvature tensor.

In the presence of Λ we have

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{\Lambda}{3}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

and $g^{(0)}_{\mu\nu}$ is Mink_4, AdS_4 or dS_4 for $\Lambda = 0$, < 0 or > 0.

In terms of electric/magnetic fields

$$E_{ab} = C_{atbt}\,,\quad B_{ab} = \tilde{C}_{atbt}$$

we simply have

$$E_{ab} \to B_{ab}\,,\quad B_{ab} \to -E_{ab}$$

- E_{ab} and B_{ab} are 3×3 symmetric traceless matrices have 5 indept. compt. each.
- $g^{(0)}_{\mu\nu}$ is self-dual.

PROBLEM:

How to resolve duality relations in order to construct explicitly $\tilde{A}_\mu$, $\tilde{h}_{\mu\nu}$?

- these relations are non-local by they can be systematically resolved by group theoretical methods based on vector and tensor harmonics

DUAL PHOTON CONSTRUCTION

Consider Maxwell equations on spherically symmetic space-time

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 .$$

Under rotations the components of A_μ transform as

- two scalars : A_t, A_r
- one vector : (A_θ, A_ϕ)

Thus, we split the components as

$$A_\mu(t,r,\theta,\varphi) = \begin{pmatrix} A_t \\ A_r \\ \hline A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} S \\ \hline V \end{pmatrix} .$$

SCALAR HARMONICS:

- $Y_\ell^m(\theta,\phi)$ with parity $(-1)^\ell$

VECTOR HARMONICS:

- $\partial_i Y_\ell^m$ with parity $(-1)^\ell$
- $\epsilon_i{}^j \partial_j Y_\ell^m$ " " $(-1)^{\ell+1}$

TENSOR (RANK 2) HARMONICS:

- $\nabla_i \nabla_j Y_\ell^m$ with parity $(-1)^\ell$
- $\gamma_{ij} Y_\ell^m$ " " $(-1)^\ell$
- $\epsilon_i{}^k \nabla_k \nabla_j Y_\ell^m + \epsilon_j{}^k \nabla_k \nabla_i Y_\ell^m$

  with parity $(-1)^{\ell+1}$

where

$$\gamma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} ; \quad \epsilon^{\theta\phi} = -\epsilon^{\phi\theta} = \frac{1}{\sin\theta}$$

AXIAL GAUGE FIELD

They are configurations with parity $(-1)^{\ell+1}$. Thus,

$$S_{axial} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$V_{axial} = a(t,r) \begin{pmatrix} -\frac{1}{\sin\theta}\partial_\phi \\ \sin\theta\,\partial_\theta \end{pmatrix} Y_\ell^m$$

For axially symmetric configurations $(m=0)$, we have in particular

$$A_\mu^{axial} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a(r) \end{pmatrix} e^{-i\omega t} \sin\theta\,\partial_\theta P_\ell$$

by factorizing t and r dependence.

POLAR GAUGE FIELD

They are configurations with parity $(-1)^{\ell}$. Thus,

$$S_{polar} = \begin{pmatrix} C(t,r) \\ D(t,r) \end{pmatrix} Y_\ell^m$$

$$V_{polar} = b(t,r) \begin{pmatrix} \partial_\theta \\ \partial_\phi \end{pmatrix} Y_\ell^m .$$

V_{polar} can be gauged away

$$A'_\mu = A_\mu - \partial_\mu \left[b(t,r) Y_\ell^m \right]$$

resulting to configurations (for $m=0$)

$$A_\mu^{polar} = \begin{pmatrix} C(r) \\ D(r) \\ 0 \\ 0 \end{pmatrix} e^{-i\omega t} P_\ell .$$

<u>FACT</u>: In either case, Maxwell eqs on space-time

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$$

reduce to an effective Schrödinger problem

$$\left(-\frac{d^2}{dr_*^2} + f(r)\frac{\ell(\ell+1)}{r^2}\right)\psi(r) = \omega^2\psi(r)$$

for appropriately chosen $\psi(r)$ wrt tortoise coordinate r_*:

$$dr_* = \frac{dr}{f(r)} \quad (\text{any } f(r))$$

▶ This realization then leads to the construction of $\tilde{A}_\mu$.

AXIAL SECTOR

$$F_{t\theta} = -i\omega a_\ell(r) e^{-i\omega t} \sin\theta \, \partial_\theta P_\ell$$

$$F_{r\phi} = a_\ell'(r) e^{-i\omega t} \sin\theta \, \partial_\theta P_\ell$$

$$F_{\theta\phi} = -\ell(\ell+1) a_\ell(r) e^{-i\omega t} \sin\theta \, P_\ell$$

with non-vanishing E_ϕ, B_r, B_θ.

Maxwell eqs $\nabla_\nu F^{\mu\nu} = 0$ are recast in Schrödinger form with "wave-function"

$$\psi_{axial}(r) = a_\ell(r).$$

Polar Sector

$$F_{tr} = -\left(C_\ell'(r) + i\omega D_\ell(r)\right) e^{-i\omega t} P_\ell$$

$$F_{t\theta} = -C_\ell(r)\, e^{-i\omega t}\, \partial_\theta P_\ell$$

$$F_{r\theta} = -D_\ell(r)\, e^{-i\omega t}\, \partial_\theta P_\ell$$

with non-vanishing E_r, E_θ, B_ϕ.

As before, Maxwell equations are recast in Schrödinger form with

$$\Psi_{polar}(r) = r^2\left(C_\ell'(r) + i\omega D_\ell(r)\right)$$

In fact, we have:

$$C_\ell(r) = \frac{1}{\ell(\ell+1)} \frac{d}{dr_*} \Psi_{polar}(r)$$

$$D_\ell(r) = -\frac{i\omega}{\ell(\ell+1) f(r)} \Psi_{polar}(r).$$

DUALITY RELATIONS

Then, it can be verified that

$$\tilde{F}^{axial}_{\mu\nu} = F^{polar}_{\mu\nu}$$

$$\tilde{F}^{polar}_{\mu\nu} = -F^{axial}_{\mu\nu}$$

provided that axial and polar "wave-functions" satisfy same boundary conditions (any).

Setting $\psi_{polar} = \ell(\ell+1)\,\psi_{axial}$

we obtain

$$\tilde{A}^{axial}_{\mu} = A^{polar}_{\mu}$$

$$\tilde{A}^{polar}_{\mu} = A^{axial}_{\mu}$$

E/M duality is axial/polar exchange

DUAL GRAVITON CONSTRUCTION

Next, consider linearized Einstein equations around

$$ds_0^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$$

with
$$f(r) = 1 - \frac{\Lambda}{3} r^2$$

for any value of cosmo constant.

As before, split perturbations as

$$h_{\mu\nu}(t,r,\theta,\phi) = \left(\begin{array}{cc:cc} h_{tt} & h_{tr} & h_{t\theta} & h_{t\phi} \\ h_{rt} & h_{rr} & h_{r\theta} & h_{r\phi} \\ \hdashline h_{\theta t} & h_{\theta r} & h_{\theta\theta} & h_{\theta\phi} \\ h_{\phi t} & h_{\phi r} & h_{\phi\theta} & h_{\phi\phi} \end{array}\right)$$

$$= \left(\begin{array}{c:c} S & V \\ \hdashline V^t & T \end{array}\right) \quad \text{wrt rotations}$$

AXIAL SECTOR

It contains perturbations $h_{\mu\nu}$ with parity $(-1)^{\ell+1}$, in which case

$$S_{axial} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

V_{axial} takes the general form

$$\left(\begin{array}{c|c} -h_0(t,r)\frac{1}{\sin\theta}\partial_\phi & h_0(t,r)\sin\theta\partial_\theta \\ \hline -h_1(t,r)\frac{1}{\sin\theta}\partial_\phi & h_1(t,r)\sin\theta\partial_\theta \end{array}\right) Y_\ell^m$$

whereas T_{axial} can be gauged away.

Thus, for $m=0$, we have

$$h_{\mu\nu}^{axial} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ \hline 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{array}\right) e^{-i\omega t} \sin\theta\,\partial_\theta P_\ell$$

POLAR SECTOR

It contains perturbations $h_{\mu\nu}$ with parity $(-1)^\ell$, in which case

$$S_{polar} = \left(\begin{array}{c|c} f(r)H_0(t,r) & H_1(t,r) \\ \hline H_1(t,r) & \frac{H_2(t,r)}{f(r)} \end{array}\right) Y_\ell^m$$

V_{polar} can be gauged away and T_{polar} can also be simplified.

In particular, for $m=0$, we have the following general parametrization

$$h^{polar}_{\mu\nu} = \begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & \frac{H_2(r)}{f(r)} & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} e^{i\omega t} P_\ell.$$

ANOTHER FACT: In either case, the linearized Einstein equations

$$\delta R_{\mu\nu} = \Lambda h_{\mu\nu}$$

for perturbations around

$$ds_0^2 = -\left(1-\frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1-\frac{\Lambda}{3}r^2} + r^2 d\Omega_2^2$$

reduce to an effective Schrödinger problem

$$\left(-\frac{d^2}{dr_*^2} + f(r)\frac{\ell(\ell+1)}{r^2}\right)\psi(r) = \omega^2\psi(r)$$

for appropriately chosen $\psi(r)$ w.r.t tortoise coordinate r_* which is <u>identical</u> to that of Maxwell theory for $f(r) = 1-\frac{\Lambda}{3}r^2$.

▶ it leads to construction of $\tilde{h}_{\mu\nu}$;

Axial Sector

Einstein equations yield the system

- $\frac{2}{r} h_0(r) - h_0'(r) = i \frac{f(r)}{\omega} \left(\frac{\omega^2}{f(r)} - \frac{(l-1)(l+2)}{r^2} \right) h_1(r)$

- $h_0(r) = i \frac{f(r)}{\omega} \left(f(r) h_1(r) \right)'$.

It transforms to Schrödinger problem for

$$\Psi_{axial}(r) = \frac{f(r)}{r} h_1(r)$$

in which case $h_0(r)$ is determined by

$$h_0(r) = \frac{i}{\omega} \frac{d}{dr_*} \left(r \Psi_{axial}(r) \right).$$

The electric and magnetic components of Weyl tensor can be computed explictly but they are lengthy.

POLAR SECTOR

Einstein equations yield the system

- $r K'(r) + \frac{K(r)}{f(r)} - H_0(r) - i\,\frac{\ell(\ell+1)}{2\omega r} H_1(r) = 0$

- $\left(f(r) H_0(r)\right)' - f(r) K'(r) + i\omega H_1(r) = 0$

- $\left(f(r) H_1(r)\right)' + i\omega \left(H_0(r) + K(r)\right) = 0$

together with two algebraic relations

- $H_0(r) = H_2(r)$

- $(\ell-1)(\ell+2) H_0(r) - \frac{2ir}{\omega}\left(\omega^2 + \frac{\Lambda}{6}\ell(\ell+1)\right) H_1$

  $= \left(\ell(\ell+1) - \frac{2}{f(r)}\left(\omega^2 r^2 + 1\right)\right) K(r).$

The latter can be regarded as first integral of ODE system.

It all transforms to a Schrödinger problem for

$$\Psi_{polar}(r) = \frac{2r}{(l-1)(l+2)}\left(K(r) - i\frac{f(r)}{\omega r}H_1(r)\right)$$

in which case the metric compts are determined as

- $H_0(r) = H_2(r) = \left(\frac{l(l+1)}{2r} - \frac{\omega^2 r}{f(r)} + \frac{d}{dr}\right)\Psi_{pol}(r)$

- $H_1(r) = -\frac{i\omega}{f(r)}\left(1 + r\frac{d}{dr_*}\right)\Psi_{polar}(r)$

- $K(r) = \left(\frac{l(l+1)}{2r} + \frac{d}{dr_*}\right)\Psi_{polar}(r).$

Explicit computation also yields the electric and magnetic compts of Weyl tensor (complementary)

Duality Relations

The end result can be simply stated as

$$E_{ab}^{polar} = B_{ab}^{axial}$$

$$B_{ab}^{polar} = - E_{ab}^{axial}$$

using the identification

$$\Psi_{axial}(r) = \frac{i\omega}{2} \Psi_{polar}(r)$$

Thus, E/M duality in linearized gravity is realized as an axial/polar exchange, leading to dual graviton construction

$$\tilde{h}_{\mu\nu}^{axial} = h_{\mu\nu}^{polar} \; ; \; \tilde{h}_{\mu\nu}^{polar} = h_{\mu\nu}^{axial} .$$

HOLOGRAPHIC REALIZATION

For AdS_4 background, the effective Schrödinger problem is

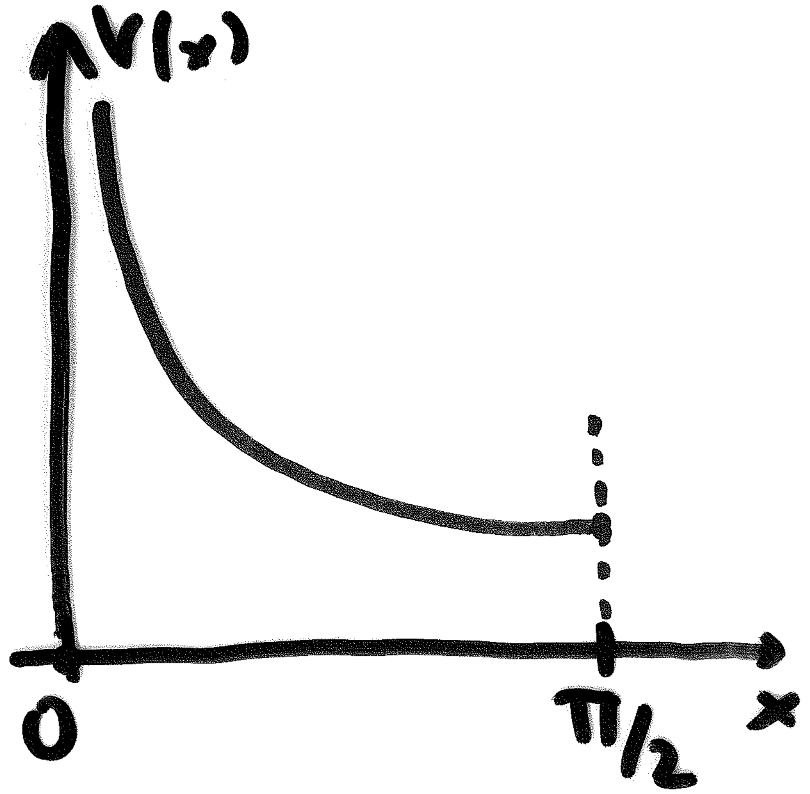
$$\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{\sin^2 x}\right)\psi(x)=\Omega^2\psi(x)$$

where

$$\tan\left(\sqrt{-\frac{\Lambda}{3}}\,r_*\right)=\sqrt{-\frac{\Lambda}{3}}\,r$$

and

$$x=\sqrt{-\frac{\Lambda}{3}}\,r_* \;;\quad \Omega=\sqrt{-\frac{3}{\Lambda}}\,\omega$$



$x: 0 \ldots \frac{\pi}{2}$

as

$r: 0 \ldots \infty$

The normalizable solution with $\psi(0)=0$ is

$$\psi(x)=\cos x \sin^{l+1} x \; F(a,b;c;\sin^2 x)$$

with coefficients

$$a=\frac{1}{2}(l+2+\Omega),\; b=\frac{1}{2}(l+2-\Omega),\; c=l+\frac{3}{2}$$

Asymptotic expansion as $r\to\infty$ is

$$\psi(r)=I_0+\frac{I_1}{r}+\frac{I_2}{r^2}+\cdots.$$

with

$$I_0=\Gamma^{-1}\left(\frac{1}{2}(l+2+\Omega)\right)\Gamma^{-1}\left(\frac{1}{2}(l+2-\Omega)\right)$$

$$I_1=-2\sqrt{-\frac{3}{\Lambda}}\,\Gamma^{-1}\left(\frac{1}{2}(l+1+\Omega)\right)\Gamma^{-1}\left(\frac{1}{2}(l+1-\Omega)\right).$$

Fixing

$$\frac{I_0}{I_1}=\frac{\Lambda}{3}\cdot\frac{\psi}{d\psi/dr_*}\Big|_{r=\infty}$$

determines spectrum of allowed ω.

Using the electric/magnetic components of Weyl tensor wrt radial ADM decomposition

$$\mathcal{E}_{ij} = C_{irjr} \, , \; \mathcal{B}_{ij} = \tilde{C}_{irjr}$$

it turns out that the holographic stress-energy tensor is

$$T_{ij} = \lim_{r\to\infty} \left(\frac{\Lambda}{3} r^3 \mathcal{E}_{ij} \right)$$

and the Cotton tensor of $\mathcal{J}$ is

$$C_{ij} = \lim_{r\to\infty} \left(\frac{\Delta^2}{9} r^3 \mathcal{B}_{ij} \right).$$

Thus, for any I_0/I_1 we have

$$C_{ij}^{\text{axial}} = T_{ij}^{\text{polar}} \; ; \; C_{ij}^{\text{polar}} = T_{ij}^{\text{axial}}$$

as boundary manifestation of dual graviton relation.

GENERALIZATION TO $SAdS_4$

For $SAdS_4$ $\quad f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2$
there is no direct analogue of electric/magnetic duality for linearized Einstein equation ($SAdS_4$ is not self-dual).
However, there is a curious relation among the corresponding axial and polar perturbations.
The field equations reduce to effective Schrödinger problems with $V_{axial}(r)$ and $V_{polar}(r)$ being partner potentials, as in SUSY quantum mechanics in r_*

Skipping the intermediate steps, we get

$$\left(-\frac{d^2}{dr_*^2} + V_\pm(r)\right)\Psi_\pm(r) = \omega^2 \Psi_\pm(r)$$

with
$$V_\pm(r) = W^2(r) \pm \frac{dW(r)}{dr_*} + \omega_s^2$$

letting
$$W(r) = \frac{6mf(r)}{r\big((\ell-1)(\ell+2)r + 6m\big)} + i\omega_s$$

and
$$\omega_s = -\frac{i}{12m}(\ell-1)\ell(\ell+1)(\ell+2).$$

V_- is axial and V_+ is polar
This in turn leads to stress-energy / Cotton tensor duality on $\mathcal{I}$ for a <u>previledged</u> set of boundary conditions.

The axial and polar potentials are isospectral provided that SUSY partner boundary conditions are imposed at $r = \infty$

$$\left(\mp \frac{d}{dr_*} + W(r) \right) \Psi_{\pm} = i(\omega_s \pm \omega) \Psi_{\mp}$$

However, this is not still enough to obtain duality relations at $\mathcal{I}$.

Requiring

$$\delta T_{ij}^{axial} = C_{ij}^{polar}$$

selects only

$$\delta g_{\mu\nu}^{axial} \Big|_{\mathcal{I}} = 0$$

and SUSY partner b.c. for $\delta g_{\mu\nu}^{polar}$

Likewise, requiring

$$\delta T_{ij}^{\text{polar}} = C_{ij}^{\text{axial}}$$

selects only

$$\delta g_{\mu\nu}^{\text{polar}} \Big|_{\partial} = 0$$

and SUSY partner b.c. for $\delta g_{\mu\nu}^{\text{axial}}$.

Furthermore, when stress-energy / Cotton tensor duality holds we obtain for the shear viscosity

$$\eta_{\text{axial}} = \eta_{\text{polar}}$$

in which case $\frac{\eta}{s} = \frac{1}{4\pi}$ (KSS)

(and conversely).

Thus, stress-energy / Cotton tensor duality acts as symmetry of KSS bound on η/s for AdS_4 black hole hydrodynamics

It can be explicitly verified that this selects uniquely the hydrodynamic modes

$$\omega_s = -i \frac{(\ell-1)(\ell+2)}{3 r_h}$$

$$\omega_\pm = \pm \sqrt{-\frac{\Lambda}{6}\ell(\ell+1)} - i \frac{(\ell-1)(\ell+2)}{6 r_h}$$