Holographic Renormalization as a Canonical Transformation:
A first step towards generalized holography

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A list of apparently unrelated problems...

- Conserved charges in supergravity with various asymptotics and in different dimensions: mass, angular momentum, free energy
- Stability theorems in supergravity with general boundary conditions (see Gary Horowitz's talk)
- General boundary conditions in QFT (see Mukund Rangamani's talk)
- Boundary conditions of the string sigma-model in general backgrounds - branes
- Non-compact brane embeddings in general backgrounds - boundary conditions for the DBI action
- AdS/CFT and generalizations (different asymtotics, higher derivatives)
- More examples from engineering and mathematics

Is there anything in common?

- There is a variational problem with boundary conditions imposed at infinity
- This variational problem is generically not automatically well defined:

  generic variations of the action with boundary conditions imposed at infinity do not imply the equations of motion
- The variational problem can be made well defined by the addition of a suitable boundary term
- This boundary term can be determined systematically in all such cases by the following general algorithm...

The algorithm

- Construct the space of general asymptotic solutions carrying a well defined symplectic form (i.e. repeat the analysis of Fefferman and Graham for asymptotically AdS gravity for the problem at hand)
- Formulate the variational problem for the degrees of freedom that parameterize the space of asymptotic solutions with a well defined symplectic form in a Hamiltonian language with Hamiltonian 'time' the coordinate emanating from the boundary at infinity
- Solve the Hamilton-Jacobi equation for this Hamiltonian system, at least in an asymptotic sense, for Hamilton's Principal function, $\mathcal{S}$
- The appropriate boundary term that renders the variational problem well defined is minus this solution of the Hamilton-Jacobi equation:

$$S_b = -\mathcal{S}$$

Holographic Renormalization?

- This algorithm has a priori nothing to do with holography or renormalization
- I nevertheless call it 'Holographic Renormalization' because:

1. That is exactly what it is in the case of supergravity in asymptotically locally AdS backgrounds.
2. This algorithm systematically reformulates the 'bulk' dynamics of the theory where it is applied in terms of degrees of freedom living in one dimension less, i.e. on the boundary at infinity. So, if there is a holographically dual theory to the bulk theory, then the observables of the dual theory should be related to the boundary variables that arise from the analysis of the variational problem.
3. For historical reasons.

There are many examples where this procedure has been implicitly carried out, at least partially, or where it can be applied:

Examples

- Supergravity in asymptotically locally AdS backgrounds
- AdS gravity in the first order Palatini formalism (see Tassos Petkou's talk)
- Holographic Wilson loops
- Holographic entanglement entropy (see Manuela Kulaxizi's talk)
- Dp branes in AdS and flat space
- Asymptotically Schrödinger backgrounds (see Marika Taylor's talk)
- Asymptotically Liftshitz backgrounds (Jerome Gauntlett's talk)
- Asymptotically flat backgrounds
- Asymptotically linear dilaton backgrounds
- Klebanov-Strassler and Maldacena-Núñez backgrounds
- Improved Holographic QCD

Outline

1. Holographic renormalization of point particles
2. Gravity in AdS
3. Conclusions

The action

$$S = \int_0^t dt' L = \int_0^t dt' \left(\frac{1}{2}\dot{q}^2 - V(q) \right).$$

We will take the potential to be arbitrary, subject to the conditions:

- $V(q) \to -\infty$ as $q \to \infty$, and
- The particle reaches $q = \infty$ at infinite time, i.e. the integral

  $$t - t_0 = \int_{q_0}^{q} \frac{dq'}{\sqrt{2(E - V(q'))}},$$

  diverges as $q \to \infty$.

The second condition ensures that even though the potential is unbounded from below, the Hamiltonian is still self adjoint and so it can be used to define a unitary time evolution operator.

The variational problem

$$
\begin{aligned}
\delta S &= -\int_0^t dt' \left(\ddot{q} + V'(q)\right) \delta q + L\delta t + p\delta q \\
&= -\int_0^t dt' \left(\ddot{q} + V'(q)\right) \delta q + (p\dot{q} - H)\delta t + p\delta q.
\end{aligned}
$$

- The usual Dirichlet BVP is set up by keeping the location of the 'boundary' fixed, i.e. $t = t_o$ fixed, and requiring

$$
\delta q|_{t_o} = 0.
$$

- However, if we want to set up the BVP at $t = \infty$, setting $\delta t = 0$ at $t = \infty$ does not make sense. So, unless $L \to 0$ as $t \to \infty$, the variational problem does not imply the equations of motion.
- In order to even ask what is the value of L asymptotically, we must evaluate it on a prescribed class of asymptotic solutions of the equations of motion.
- For some purposes it may be sufficient to restrict to variations of q within the space of asymptotic solutions such that $L \to 0$ as $q \to \infty$, if such solutions exist.
- However, this is not the most general solution to the problem, and in the context of holography this is inadequate...

- More generally, we can add a boundary term $S_b(q)$ such that a generic variation of the action takes the form:

$$\delta(S+S_b) = -\int_0^t dt'\,\left(\ddot{q}+V'(q)\right)\delta q + \left(L+\dot{S}_b\right)\delta t + (p+S_b'(q))\delta q.$$

- The variational problem is then well defined provided we can find a boundary term $S_b(q)$ such that when q asymptotically approaches generic solutions of the equations of motion

$$\frac{d}{dt}(S+S_b) \xrightarrow{q\to\infty} 0.$$

- Since q is asymptotically evaluated on-shell, we can replace the on-shell action S in this expression with Hamilton's principal function, $\mathcal{S}$, i.e. a solution of the Hamilton-Jacobi equation which is a function of q on the 'boundary'.
- One then has to solve the Hamilton-Jacobi equation (to sufficiently high order) to obtain

$$S_b = -\mathcal{S}.$$

- Note that the above requirement for the variational problem to be well defined is related to the requirement that the on-shell action be finite.

- The present case is simple enough that we can determine the boundary term S_b directly from the above condition, without resorting to the Hamilton-Jacobi equation.
- From energy conservation we have

$$\dot{q} = \pm\sqrt{2(E - V(q))} = \pm\sqrt{-2V(q)}\left(1 - \frac{E}{2V(q)} + \cdots\right).$$

- Picking the plus sign since we demand $q \to +\infty$ as $t \to \infty$ and inserting this in the above condition for S_b we get

$$\left(\sqrt{-2V(q)} + S_b'(q)\right)\left(\frac{E}{\sqrt{-2V(q)}} + \sqrt{-2V(q)}\right) \xrightarrow{q\to\infty} 0.$$

- We conclude that the desired boundary term is given by

$$S_b(q) = -\int^q dq' \sqrt{-2V(q')}.$$

Canonical transformations

- What is the significance of this boundary term?
- The canonical momentum p conjugate to q is obtained from Hamilton's principal function, $\mathcal{S}$, by
$$p = \frac{d\mathcal{S}}{dq}.$$
- After adding the boundary term $S_b(q)$ we obtain instead
$$P = \frac{d(\mathcal{S} + S_b)}{dq} = p + S_b'(q).$$
- The boundary term, therefore, implements the phase space transformation
$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} P \\ Q \end{pmatrix} := \begin{pmatrix} p + S_b'(q) \\ q \end{pmatrix}$$
- This transformation is canonical since
$$PdQ - pdq = dS_b(q) = dS_b(Q).$$
- In particular, the symplectic form is left invariant:
$$\Omega = dp \wedge dq = dP \wedge dQ.$$

From phase space to the space of asymptotic solutions

- Let $\mathcal{P}$ denote the phase space parameterized by (p, q) and $\mathcal{C}$ denote the space of asymptotic solutions parameterized by the integration constants (E, t_o).
- For every t there is a canonical bijective map

$$\begin{aligned} \varphi_t : \mathcal{C} &\quad\to\quad \mathcal{P} \\ (E, t_0) &\quad\mapsto\quad (p, q) \end{aligned}$$

- This in turn induces the pullback map between the corresponding cotangent bundles

$$\begin{aligned} \varphi_t^* : \mathcal{T}^*\mathcal{P} &\quad\to\quad \mathcal{T}^*\mathcal{C} \\ \begin{pmatrix} dp \\ dq \end{pmatrix} &\quad\mapsto\quad \begin{pmatrix} dE \\ dt_0 \end{pmatrix}. \end{aligned}$$

- Applying this to the symplectic form, $\Omega_{\mathcal{C}} = \varphi_t^* \Omega_{\mathcal{P}}$, then gives

$$\begin{aligned} &\varphi_t^* \Omega_{\mathcal{P}} = \varphi_t^* (dp \wedge dq) = d\sqrt{2(E - V(q))} \wedge dq \\ &= dE \wedge \frac{dq}{\sqrt{2(E - V(q))}} = -dE \wedge dt_0 \equiv \Omega_{\mathcal{C}}. \end{aligned}$$

- So the space $\mathcal{C}$ of asymptotic solutions parametrized by E and t_o inherits a well defined symplectic form from that on phase space.

Diagonalizing the symplectic map

- Let us now look at the effect the canonical transformation corresponding to the addition of the boundary term has on the canonical map $\varphi_t^* : \mathcal{T}^*\mathcal{P} \to \mathcal{T}^*\mathcal{C}$.
- Before the canonical transformation:

$$\begin{aligned} dp &\sim \frac{1}{\sqrt{-2V(q)}} dE + V'(q) dt_0, \\ dq &\sim -\sqrt{-2V(q)} dt_0. \end{aligned}$$

- After the canonical transformation:

$$\begin{aligned} dP = dp + S_b''(q) dq &\sim \frac{1}{\sqrt{-2V(q)}} dE \\ dQ &\sim -\sqrt{-2V(q)} dt_0. \end{aligned}$$

- Hence, after the canonical transformation the map $\varphi_t^* : \mathcal{T}^*\mathcal{P} \to \mathcal{T}^*\mathcal{C}$ is diagonal:

$$\varphi_t^* \to \begin{pmatrix} \frac{1}{\sqrt{-2V(q)}} & 0 \\ 0 & -\sqrt{-2V(q)} \end{pmatrix}.$$

- Therefore, variations of the canonically transformed phase space variables diagonally map to variations of the symplectic variables parameterizing the space of asymptotic solutions.

Summary up to now...

- The following three facts:
  1. the variational problem at infinity is not well defined
  2. the on-shell action is divergent
  3. the symplectic map $\varphi_t^* : \mathcal{T}^*\mathcal{P} \to \mathcal{T}^*\mathcal{C}$ is not diagonal

  are different manifestations of the same problem.

- They are all simultaneously solved by a suitable canonical transformation that corresponds to the addition of a certain boundary term to the action.

- This boundary term is determined dynamically by solving the Hamilton-Jacobi equation.

AdS gravity

Einstein-Hilbert + Gibbons-Hawking

$$S=-\frac{1}{2\kappa^2}\int_{\mathcal{M}}d^{d+1}x\sqrt{g}(R-2\Lambda)-\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}}d^dx\sqrt{\gamma}2K,$$

- Let us now see how applying the algorithm outlined above for the point particle to AdS gravity reproduces the standard procedure of Holographic renormalization.
- Starting from the Einstein-Hilbert action, we formulate the variational problem in a radial Hamiltonian formalism with the radial coordinate as Hamiltonian ‘time’...

Radial Hamiltonian formalism

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j,$$

$$L = -\frac{1}{2\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} N \left(R[\gamma] - 2\Lambda + K^2 - K^i_j K^j_i \right).$$

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2} \sqrt{\gamma} (K \gamma^{ij} - K^{ij}),$$

$$H = \int_{\Sigma_r} d^d x \pi^{ij} \dot{\gamma}_{ij} - L = \int_{\Sigma_r} d^d x \left(N\mathcal{H} + N_i \mathcal{H}^i \right),$$

$$\mathcal{H} = 2\kappa^2 \gamma^{-\frac{1}{2}} \left(\pi^i_j \pi^j_i - \frac{1}{d-1} \pi^2 \right) + \frac{1}{2\kappa^2} \sqrt{\gamma} \left(R[\gamma] - 2\Lambda \right), \quad \mathcal{H}^i = -2D_j \pi^{ij}.$$

$$\mathcal{H} = 0, \quad \mathcal{H}^i = 0.$$

Symplectic form

$$\Omega = \int_{\Sigma_r} d^d x \delta\pi^{ij} \wedge \delta\gamma_{ij},$$

$$\left\{ \gamma_{ij}(r,x), \pi^{kl}(r,x') \right\} = \delta_i^{(k} \delta_j^{l)} \delta^{(d)}(x - x'),$$

Constraints as generators of diffeomorphisms

$$C[\xi] = \int_{\Sigma_r} d^d x \left(\xi \mathcal{H} + \xi^i \mathcal{H}_i \right),$$

$$\{C[\xi], \gamma_{ij}\} = \delta_{\tilde{\xi}} \gamma_{ij}, \quad \left\{C[\xi], \pi^{ij}\right\} = \delta_{\tilde{\xi}} \pi^{ij},$$

$$\tilde{\xi}^\mu = \left(\xi/N, \xi^i - \xi N^i/N \right),$$

Algebra of constraints

$$\left\{C[\xi], C[\xi']\right\} = C[\xi''],$$

$$\xi''^\mu = \left(\xi^i \partial_i \xi' - \xi'^i \partial_i \xi, \quad \xi^i \partial_i \xi^j - \xi'^i \partial_i \xi^j - (\xi D^j \xi' - \xi' D^j \xi) \right).$$

The physical phase space is $\mathcal{P}/\mathrm{Diff}(\mathcal{M})$.

The space of asymptotic solutions

Fefferman-Graham expansion

$$\gamma_{ij}(r,x) =$$
$$e^{2r}\left(g_{(0)ij}(x) + e^{-2r}g_{(2)ij}(x) + \cdots + e^{-dr}\left(-2rh_{(d)ij}(x) + g_{(d)ij}(x)\right) + \cdots\right),$$

Constraints

$$D_{(0)}{}^{i}\mathcal{T}_{ij}(x) = 0, \quad \mathcal{T}^{i}_{i}(x) = \mathcal{A}(x),$$

$$\mathcal{T}_{ij} = \frac{d}{2\kappa^2}\left(g_{(d)ij} - g_{(0)}{}^{kl}g_{(d)kl}g_{(0)ij}\right) + X_{ij}[g_{(0)}],$$

Symplectic form

$$\Omega_{\mathcal{C}} = \varphi_r^*\Omega_{\mathcal{P}} = \int d^dx\,\delta\pi_{(d)}{}^{ij} \wedge \delta g_{(0)ij}, \quad \pi_{(d)}{}^{ij} \equiv -\frac{1}{2}\sqrt{g_{(0)}}\mathcal{T}^{ij}$$

Diffeomorphisms leaving FG expansion form invariant

- Generic infinitesimal diffeomorphism

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu .$$

- Requiring that this diffeomorphism preserves the gauge choice $N = 1$, $N^i = 0$ we get

$$\begin{aligned}
\delta_\xi g_{rr} &= -\mathcal{L}_\xi g_{rr} = -2\dot{\xi}^r = 0, \\
\delta_\xi g_{ri} &= -\mathcal{L}_\xi g_{ri} = -\gamma_{ij}(\dot{\xi}^j + \partial^j \xi^r) = 0.
\end{aligned}$$

- Solving these conditions we obtain

$$\begin{aligned}
\xi^r &= -\sigma(x), \\
\xi^i &= \xi^i_o(x) - \partial_j \sigma(x) \int_r^\infty dr' \gamma^{ji}(r', x).
\end{aligned}$$

where $\sigma(x)$ and $\xi^i_o(x)$ are arbitrary.

- Inserting the FG expansion into the second of these expressions gives

$$\xi^i = \xi^i_o(x) - \frac{1}{2} e^{-2r} \left({g_{(0)}}^{ij} - \frac{1}{2} e^{-2r} {g_{(2)}}^{ij} + \mathcal{O}(e^{-4r}) \right) \partial_j \sigma(x).$$

- Under such diffeomorphisms the induced metric then transforms as

$$\delta_\xi \gamma_{ij} = -\mathcal{L}_\xi g_{ij} = -\left(L_\xi \gamma_{ij} + 2K_{ij}\xi^r \right) = -\left(D_i \xi_j + D_j \xi_i - 2K_{ij}\sigma \right).$$

- It follows that

$$\delta_\xi g_{(0)ij} = -\left(D_{(0)i}\xi_{oj} + D_{(0)j}\xi_{oi} \right) + 2\sigma g_{(0)ij}.$$

$$\begin{aligned} \delta_\xi {\pi_{(d)}}^{ij} \quad &= \quad -\left(D_{(0)k}\left({\pi_{(d)}}^{ij}\xi_o^k \right) - {\pi_{(d)}}^{ik} D_{(0)k}\xi_o^j - {\pi_{(d)}}^{jk} D_{(0)k}\xi_o^i \right) \\ &\quad -2\sigma(x){\pi_{(d)}}^{ij} - \frac{\delta}{\delta g_{(0)ij}} \int d^d x \sqrt{g_{(0)}} \mathcal{A}\sigma. \end{aligned}$$

- These transformations are generated by the constraints

$$C[\xi_o, \sigma] = \int d^d x \sqrt{g_{(0)}} \left(\xi_o^i(x) {D_{(0)}}^j \mathcal{T}_{ij} + \sigma(x) \left(\mathcal{T}_i^i - \mathcal{A} \right) \right),$$

- Namely,

$$\left\{ C[\xi_o, \sigma], g_{(0)ij}(x) \right\} = \delta_\xi g_{(0)ij}(x), \quad \left\{ C[\xi_o, \sigma], {\pi_{(d)}}^{ij}(x) \right\} = \delta_\xi {\pi_{(d)}}^{ij}(x).$$

- Moreover, the constraints close onto themselves

$$\left\{C[\xi_o,\sigma],C[\xi'_o,\sigma']\right\}=C[\xi''_o,\sigma''],$$

$$\xi''^i_o=\xi^j_o\partial_j\xi'^i_o-\xi'^j_o\partial_j\xi^i_o,\quad \sigma''=\xi^j_o\partial_j\sigma'-\xi'^j_o\partial_j\sigma.$$

- At this point we have a complete description of the space of general asymptotic solutions of AdS gravity, $\mathcal{C}/\mathrm{Diff}_o(\mathcal{M})$.

- This space is parameterized in terms of the variables ${\pi_{(d)}}^{ij}$ and $g_{(d)ij}$, subject to the above constraints.

- The Fefferman-Graham expansion implements the map

$$\varphi_r:\mathcal{C}/\mathrm{Diff}_o(\mathcal{M})\to\mathcal{P}/\mathrm{Diff}(\mathcal{M})$$

- We will next see how the canonical transformation implemented by the boundary term required to make the variational problem well defined diagonalizes the map

$$\varphi_r^*:\mathcal{T}^*(\mathcal{P}/\mathrm{Diff}(\mathcal{M}))\to\mathcal{T}^*(\mathcal{C}/\mathrm{Diff}_o(\mathcal{M})).$$

A small digression...

- So far we have just constructed the space of general asymptotic solutions of AdS gravity, inheriting a well defined symplectic form from phase space.
- There has been no reference to holography so far!
- However, the structure of the symplectic space of asymptotic solutions maps directly into physical statements about the dual QFT:

The holographic dictionary

- $g_{(d)\,ij}$: boundary metric
- ${\pi_{(d)}}^{ij}$: dual stress tensor
- ${D_{(0)}}^{i}\mathcal{T}_{ij}(x) = 0$: Diff. Ward identity
- $\mathcal{T}_i^i(x) = \mathcal{A}(x)$: Trace Ward identity

Hamilton-Jacobi equation and the boundary term

- The same argument as for the point particle leads again to the conclusion that the variational problem can be made well defined provided we add a boundary term $S_b = -\mathcal{S}$, where $\mathcal{S}$ solves the Hamilton-Jacobi equation

Hamilton-Jacobi equation

$$2\kappa^2\gamma^{-\frac{1}{2}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - 2\Lambda\right),$$

$$D_j\left(\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\right) = 0.$$

- The solution is well known and depends on the dimension. For $d = 4$

$$\mathcal{S} = -\frac{1}{\kappa^2}\int d^4x\sqrt{\gamma}\left(3 + \frac{1}{4}R[\gamma] - \log e^{-2r}\frac{1}{16}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right)\right).$$

Canonical transformation

- Now, as for the point particle, the addition of this term induces the transformation

$$\Pi^{ij} = \pi^{ij} + \frac{\delta S_b}{\delta \gamma_{ij}}.$$

- This transformation is again canonical since

$$\int d^d x \delta \left(\frac{\delta S_b}{\delta \gamma_{ij}} \right) \wedge \delta \gamma_{ij} = \int d^d x \delta_1 \left(\frac{\delta S_b}{\delta \gamma_{ij}} \right) \delta_2 \gamma_{ij} - (1 \leftrightarrow 2) =$$
$$\delta_1 \int d^d x \frac{\delta S_b}{\delta \gamma_{ij}} \delta_2 \gamma_{ij} - (1 \leftrightarrow 2) = (\delta_1 \delta_2 - 1 \leftrightarrow 2) S_b = 0.$$

- Hence, the symplectic form is preserved:

$$\Omega_{\mathcal{P}} = \int d^d x \delta \Pi^{ij} \wedge \delta \gamma_{ij} = \int d^d x \delta \pi^{ij} \wedge \delta \gamma_{ij}.$$

Diagonalization of the symplectic map

- Moreover, this canonical transformation again diagonalizes the map $\varphi_r^*: \mathcal{T}^*\mathcal{P} \to \mathcal{T}^*\mathcal{C}$:

$$\varphi_r^* \begin{pmatrix} \delta\Pi^{ij} \\ \delta\gamma_{ij} \end{pmatrix} = \begin{pmatrix} e^{-2r}\delta\pi_{(d)}{}^{ij} + \cdots \\ e^{2r}\delta g_{(0)\,ij} + \cdots \end{pmatrix}.$$

Summary & Conclusions

The algorithm

- Construct the space of general asymptotic solutions carrying a well defined symplectic form (i.e. repeat the analysis of Fefferman and Graham for asymptotically AdS gravity for the problem at hand)
- Formulate the variational problem for the degrees of freedom that parameterize the space of asymptotic solutions with a well defined symplectic form in a Hamiltonian language with Hamiltonian 'time' the coordinate emanating from the boundary at infinity
- Solve the Hamilton-Jacobi equation for this Hamiltonian system, at least in an asymptotic sense, for Hamilton's Principal function, $\mathcal{S}$
- The appropriate boundary term that renders the variational problem well defined is minus this solution of the Hamilton-Jacobi equation:

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Summary & Conclusions

- Can be applied to any Hamiltonian system
- In the presence of a holographic duality systematically builds the space of observables of the boundary theory from the bulk theory
- Work in progress: application to Improved Holographic QCD and asymptotically flat gravity