Entanglement Entropies in Holographic Field Theories

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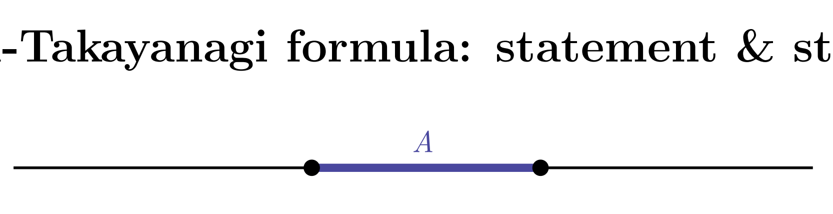
1 Ryu-Takayanagi formula: statement & status



Entanglement entropy (EE):

$$S_A = -\operatorname{tr}\rho_A \ln \rho_A$$

where

$$A = \text{subsystem (spatial region in QFT)}$$
$$\rho_A = \operatorname{tr}_{A^c}\rho = \text{reduced density matrix}$$

- Fundamental measure of quantum information in QFT
- Notoriously difficult to calculate (even in free QFTs)

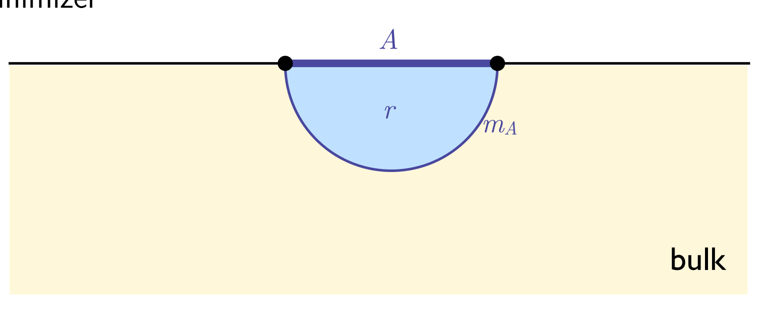
Ryu-Takayanagi formula for EE

- of a spatial region
- in a holographic theory
- dual to classical Einstein gravity ("large N, strong coupling")
- in a state described in the bulk by a static, classical field configuration ($\Rightarrow$ distinguished constant-time surfaces):

$$S_A = \frac{1}{4G_N}\min_{m\sim A}\left(\text{area}(m)\right)$$

where

- $m \sim A$ means $\exists$ region r s.t. $\partial r = m \cup A$
- area w.r.t. spatial, Einstein-frame metric
- $m_A \equiv$ minimizer



Simple, elegant, easy to compute!

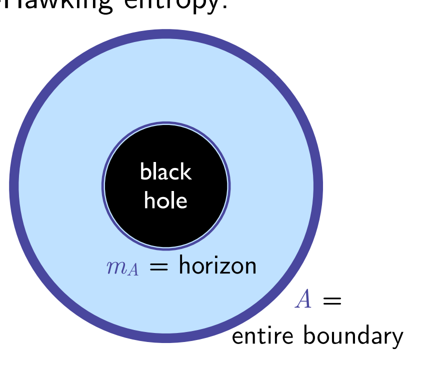
Checks:

- reproduces EE of interval in vacuum of $1+1$ CFT:

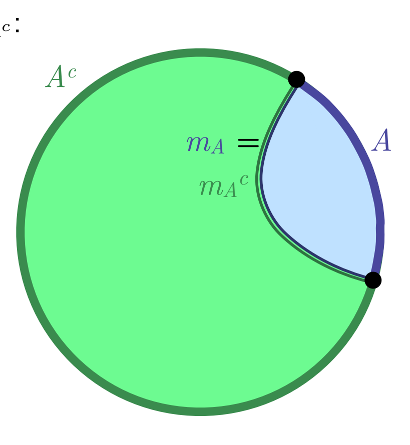
$$S_{[u,v]} = \frac{c}{3}\ln\left(\frac{v-u}{\epsilon}\right)$$

  on $\mathbf{R}$ (also on S^1) (Holzhey, Larsen, Wilczek '94)

- reproduces UV divergent part of EE (proportional to area of ∂A) in arbitrary dimension
- reproduces Bekenstein-Hawking entropy:



- pure state $\Rightarrow S_A = S_{A^c}$:



  (does not hold for mixed states)

- obeys strong subadditivity:

$$S_B + S_{ABC} \le S_{AB} + S_{BC}$$

  (MH + Takayanagi '07)

All evidence applies equally, whether A is connected or disconnected!

Fursaev's proof of RT formula (2006):

- completely general
- applied replica trick: computed entanglement Rényi entropy (ERE)

$$S_A^{(n)} = \frac{1}{1-n}\ln\operatorname{tr}\rho_A^n\,,$$

  analytically continued in n, took $n \to 1$, obtained RT

- computed ERE by finding action of gravitational Euclidean saddles w/appropriate boundary conditions
- unfortunately, the saddles aren't saddles (don't solve Einstein equation), so the ERE is wrong (e.g. for interval in $1+1$ CFT)
- not easy to fix: true saddles are in general complicated, no general formula
- however, proof is suggestive: seems to get topology of saddles right, suggests homology condition on m_A, makes prediction for structure of corrections, . . .

Conjecture for corrections (α' = classical higher-derivative; G_N = quantum):

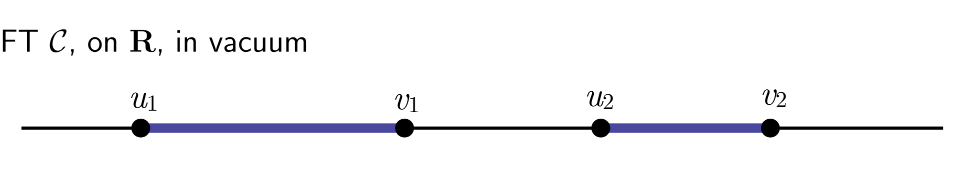
$$S_A = \frac{1}{4G_N}\min_{m\sim A}\left(\text{area}(m) + \mathcal{O}(\alpha')\right) + \mathcal{O}(G_N^0)$$

Precise formulas have not been proposed yet

2 Two intervals: predictions & tests

(MH '10)

$1+1$ CFT $\mathcal{C}$, on $\mathbf{R}$, in vacuum



EE of 2 intervals: much harder to compute from first principles than of 1 interval; depends on $\mathcal{C}$, not just c

Mutual information (MI) is UV-finite & conformally invariant:

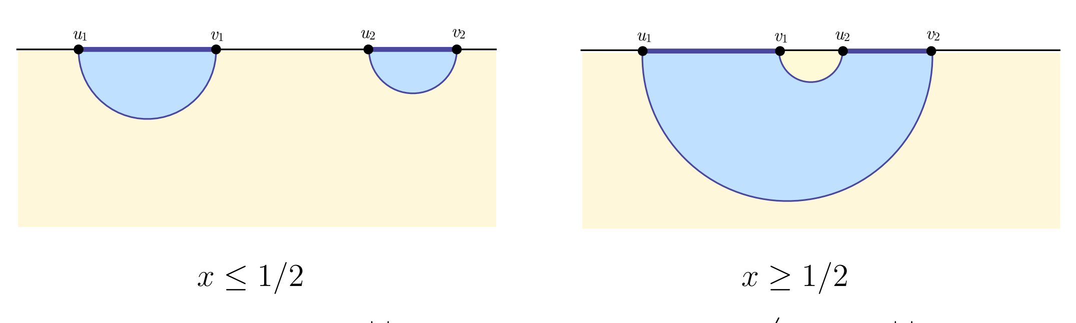
$$I_{[u_1,v_1],[u_2,v_2]} = I(x) \equiv S_{[u_1,v_1]} + S_{[u_2,v_2]} - S_{[u_1,v_1]\cup[u_2,v_2]}$$
$$x \equiv \frac{(v_1-u_1)(v_2-u_2)}{(u_2-u_1)(v_2-v_1)}$$

(In general, MI bounds correlators between operators in A & B:

$$\left(\langle\mathcal{O}_A\mathcal{O}_B\rangle - \langle\mathcal{O}_A\rangle\langle\mathcal{O}_\mathcal{B}\rangle\right)^2 \le I_{A,B}$$

(Wolf, Verstraete, Hastings, Cirac '07))

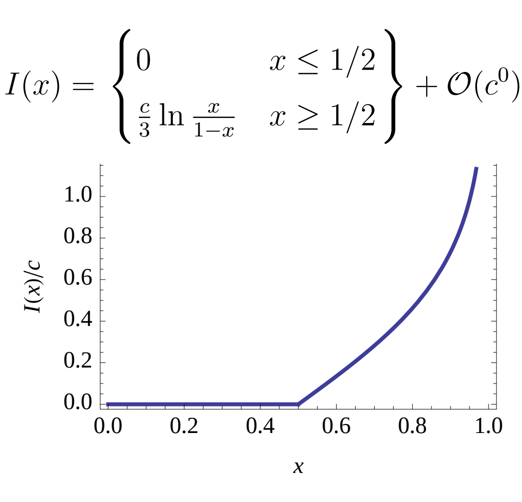
Prediction from RT formula:



$x \le 1/2$ | $x \ge 1/2$

$m_{[u_1,v_1],[u_2,v_2]} = m_{[u_1,v_1]} \cup m_{[u_2,v_2]}$ | $m_{[u_1,v_1],[u_2,v_2]} \ne m_{[u_1,v_1]} \cup m_{[u_2,v_2]}$

$$I(x) = \begin{Bmatrix} 0 & x \le 1/2 \\ \frac{c}{3}\ln\frac{x}{1-x} & x \ge 1/2 \end{Bmatrix} + \mathcal{O}(c^0)$$



(Recall $G_N \sim 1/c$.) Qualitative features:

- phase transition at $x = 1/2$
- $I(x) = 0$ for $x \le 1/2$

(Expect both features to be preserved by higher-derivative classical corrections, not by quantum corrections)

Let's attempt to compute Rényi entropies holographically

Mutual Rényi information (MRI):

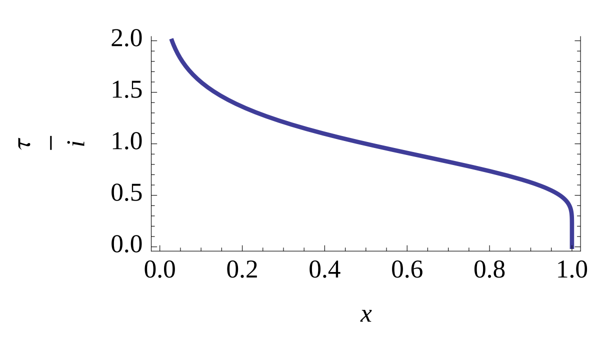
$$\begin{aligned} I^{(n)}(x) &\equiv S^{(n)}_{[u_1,v_1]} + S^{(n)}_{[u_2,v_2]} - S^{(n)}_{[u_1,v_1]\cup[u_2,v_2]} \\ &= \frac{1}{n-1}\ln\frac{\langle\sigma_1(0)\sigma_{-1}(x)\sigma_1(1)\sigma_{-1}(\infty)\rangle}{\langle\sigma_1(0)\sigma_{-1}(x)\rangle\langle\sigma_1(1)\sigma_{-1}(\infty)\rangle} \end{aligned}$$

$\sigma_{1,-1}$ = twist operators in $\mathcal{C}^n/S_n$

$n = 2$: Lunin-Mathur '00:

$$\frac{\langle\sigma_1(0)\sigma_{-1}(x)\sigma_1(1)\sigma_{-1}(\infty)\rangle}{\langle\sigma_1(0)\sigma_{-1}(x)\rangle\langle\sigma_1(1)\sigma_{-1}(\infty)\rangle} = \left(\frac{2^8(1-x)}{x^2}\right)^{-c/12} Z_{\text{torus}}(\tau)$$

where



$$x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

Maldacena-Strominger '98:

$$\ln Z_{\text{torus}}(\tau) = \frac{2\pi c}{12}\begin{Bmatrix} (\tau/i)^{-1}\,, & \tau/i \ge 1, & x \le 1/2 \\ \tau/i\,, & \tau/i \le 1\,, & x \ge 1/2 \end{Bmatrix} + \mathcal{O}(c^0)$$

Hawking-Page transition $\Rightarrow$ phase transition in $I^{(2)}(x)$ at $x = 1/2$ (smoothed out by non-perturbative corrections)

$n > 2$? Formulas for $\langle\sigma_1(0)\sigma_{-1}(x)\sigma_1(1)\sigma_{-1}(\infty)\rangle$ in holographic theories still missing. But: dependence of Euclidean saddle points on $x \Rightarrow$ phase transition in $I^{(n)}(x)$ at $x = 1/2$ for all n

Strong support for phase transition in $I(x)$ at $x = 1/2$

Also, $\ln Z_{\text{torus}}(\tau)$ (hence $I^{(2)}(x)$) is same for $N \to \infty$ limit of symmetric-product orbifold CFT,

$$\mathcal{C} = \mathcal{C}^N/S_N$$

Is MRI $I^{(n)}(x)$ same for all large-N CFTs?

Second approach: to get $I^{(n)}(x)$ for all n, first expand in x

OPE:

$$\begin{aligned} \frac{\langle\sigma_1(0)\sigma_{-1}(x)\sigma_1(1)\sigma_{-1}(\infty)\rangle}{\langle\sigma_1(0)\sigma_{-1}(x)\rangle\langle\sigma_1(1)\sigma_{-1}(\infty)\rangle} &= \sum_{\mathcal{O}_m\in\mathcal{C}^n/S_n} c^\sigma{}_{\sigma m}c^m{}_{\sigma\sigma}x^{2d_m} \\ &= 1 + \frac{(n^2-1)^2c}{144n^3}x^2 + \mathcal{O}(x^3) \end{aligned}$$

(from identity & stress tensor)

$$\Rightarrow \quad I^{(n)}(x) = \frac{(n+1)^2(n-1)c}{144n^3}x^2 + \mathcal{O}(x^3)$$
$$\Rightarrow \quad I(x) = \lim_{n\to 1} I^{(n)}(x) = \mathcal{O}(x^3)$$

Higher order in x: Assume that, in large-c limit,

- number of operators stays finite
- three-point functions stay finite

Analyzing conformal blocks in $\mathcal{C}^n/S_n$, can show:

- order-c^1 part of $I^{(n)}(x)$ is universal (same for any $\mathcal{C}$)
- order-c^1 part of $I(x) = 0$ (strong support for RT formula)

(at least to $\mathcal{O}(x^6)$)

3 Take-home messages

RT formula:

- has not been proved
- is supported by very strong evidence
- applies equally to connected & disconnected regions
- associates to each spatial region of the boundary a unique spatial region of the bulk
- receives classical & quantum corrections that are highly constrained but have not been worked out yet

Also: In certain cases, entanglement structure (e.g. MRIs) appears to be identical in all large-N theories

4 Advertisement