# Holographic C-theorems

with A. Sinha
(arXiv:1006.1263 & in progress)

## Overview:

1. Introductory remarks on c-theorem

2. Holographic c-theorem I: Einstein gravity

3. Holographic c-theorem II: Quasi-topological gravity

4. $a_d^*$, Entanglement Entropy and Beyond

5. Concluding remarks

## Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1, 2, 3, \cdots\}$ with beta-functions as "velocities"

- for unitary, renormalizable QFT's in two dimensions, there exists a positive-definite real function of the coupling constants $c(g)$:

1. monotonically decreasing along flows: $\frac{d}{dt}c(g) \le 0$
2. "stationary" at fixed points $g^i = (g^*)^i$ :

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}c(g) = 0$$

3. at fixed points, it equals central charge of corresponding CFT

$$c(g^*) = c$$

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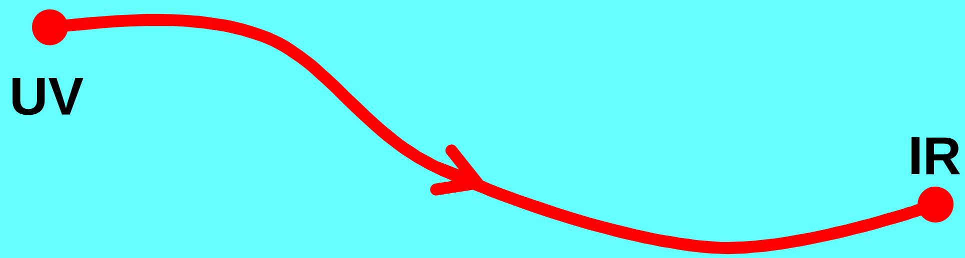
1. mono

2. “stati

3. at fix nding CFT

Consequence for any RG flow:




$$c_{UV} \; > \; c_{IR}$$

## C-theorems in higher dimensions??

d=2: $$\langle T_\mu{}^\mu \rangle = -\frac{c}{12} R$$

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R$$

$$I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$

- in 4 dimensions, have three central charges: $c,\ a,\ a'$
- do any of these obey a similar "c-theorem" under RG flows?

<u>$a'$-theorem</u>: $a'$ is scheme dependent (not globally defined)

<u>$c$ -theorem</u>: there are numerous counter-examples

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- numerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al)
- holographic field theories with gravity dual $(a = c)$
- no completely general proof
- counterexample proposed: Shapere & Tachikawa, 0809.3238

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## Counterexample to a-theorem:

- flow between two N = 2 superconformal gauge theories

**UV:** gauge group $SU(N_c+1)$ with $N_f=2N_c$ fundamental hyper's
**IR:** gauge group $SU(N_c)$ with $N_f=2N_c$ fundamental hyper's (m=0)

$$a_{UV} - a_{IR} = \frac{1}{72}\left(19\,N_c - 7\,N_c^2 + 15\right) \qquad \textbf{(}\ \leq 0 \ \text{ for } N_c \geq 4\ \textbf{)}$$

- loophole: accidental U(1) symmetry appears in the IR limit

- **counterexample is not valid:** UV fixed point does not exist for $N_f>2$ invalidating previous analysis

(Seiberg & Tachikawa)

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(Freedman, Gubser, Pilch & Warner, hep-th/9904017)
(Girardello, Petrini, Porrati and Zaffaroni, hep-th/9810126)

## Holographic RG flows:

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\,(R + \mathcal{L}_{\rm matter})$$

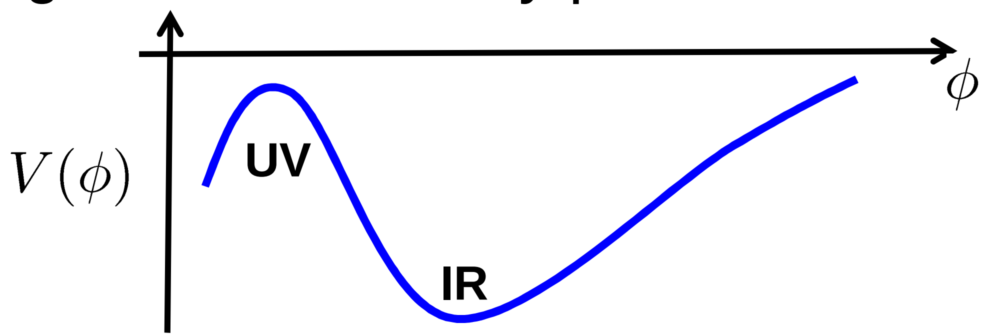
- assume stationary points: matter fields fixed and $\mathcal{L}_{\rm matter} = \frac{12}{L^2}\alpha_i^2$

(eg, scalar field: $\mathcal{L}_{\rm matter} = -\frac{1}{2}(\partial\phi)^2 - V(\phi)$ )

- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$

- at stationary points, AdS$_5$ vacuum: $A(r) = r/\tilde{L}$ with $\tilde{L} = L/\alpha_i$

- RG flows are solutions starting at one stationary point and ending at another



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## Holographic RG flows:

- for general flow solutions, define: $a(r) \equiv \dfrac{\pi^2}{\ell_P^3 A'(r)^3}$

$$a'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) = -\frac{\pi^2}{\ell_P^3 A'(r)^4}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

Einstein equations — null energy condition

- at stationary points, $a(r) \to a^* = \pi^2 \tilde{L}^3/\ell_P^3$ and hence

$$\boxed{a^*_{UV} \geq a^*_{IR}}$$

- using holographic trace anomaly: $a^* = a$

(e.g., Henningson & Skenderis)

⟶ supports Cardy's conjecture

- for Einstein gravity, central charges equal $(a = c)$ : $c_{UV} \geq c_{IR}$

(Freedman, Gubser, Pilch & Warner, hep-th/9904017)

## Holographic RG flows:

$$I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g}\,(R + \mathcal{L}_{\rm matter})$$

- same story is readily extended to (d+1) dimensions

- defining: $a_d(r) \equiv \dfrac{\pi^{d/2}}{\Gamma\,(d/2)\,(\ell_P A'(r))^{d-1}}$

$$a'(r) \;=\; -\frac{(d-1)\pi^{d/2}}{\Gamma\,(d/2)\,\ell_P^{d-1} A'(r)^d} A''(r) = -\frac{\pi^{d/2}}{\Gamma\,(d/2)\,\ell_P^{d-1} A'(r)^d}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

**Einstein equations** **null energy condition**

- at stationary points, $a(r) \to a^* = \pi^{d/2}/\Gamma(d/2)\,(\tilde{L}/\ell_P)^{d-1}$ and so

$$\boxed{a^*_{UV} \geq a^*_{IR}}$$

- using holographic trace anomaly: $a^* \propto$ central charges

(for even d! what about odd d?) (e.g., Henningson & Skenderis)

## Improved Holographic RG Flows:

- add higher curvature interactions to bulk gravity action

⟶ provides holographic field theories with, eg, $a \neq c$
so that we can clearly distinguish evidence of a-theorem

(Nojiri & Odintsov; Blau, Narain & Gava)

⟶ more generally broadens class of dual CFT's

## Higher Curvature Terms in Derivative Expansion

- in strings, sugra action corrected by higher curvature terms

$\alpha'$ corrections: $\alpha'/L^2 \simeq 1/\sqrt{\lambda}$

string loops: $g_s \simeq \lambda/N_c$

- perturbing sugra theory with higher curvature terms provides insight into finite $N_c,\ \lambda$ corrections in gauge theory
- here I want to go beyond perturbative framework to study RG flows (i.e., want to consider finite values of new couplings)
- if we go to finite parameters where one of the higher curvature terms is important, expect all are important
- ultimately one needs to fully develop string theory for interesting holographic backgrounds

## Higher Curvature Terms without Derivative Expansion

- **instead** consider "toy models" with finite $R^n$ interactions (where we can maintain control of calculations)
- with AdS/CFT, higher curvature couplings become dials to adjust parameters characterizing the dual CFT
- note that any one $R^n$ interaction implicitly determines an infinite number of couplings in $T_{ab}$ correlators
- construct models to maintain control of calculations

## What about the swampland?

- constrain gravitational couplings with consistency tests (positive fluxes; causality; unitarity) and keep fingers crossed!
- seems an effective approach with Lovelock gravity

(eg, Brigante, Liu, Myers, Shenker & Yaida)

## Quasi-Topological gravity:

(Myers & Robinsion, 1003.5357)

$$I = \frac{1}{2\ell_P^3} \int d^5x \sqrt{-g} \left[ \frac{12}{L^2} + R + L^2 \frac{\lambda}{2} \chi_4 + L^4 \frac{7\mu}{4} \mathcal{Z}_5 \right]$$

with
$$\begin{aligned}
\chi_4 &= R^{abcd} R_{abcd} - 4 R_{ab} R^{ab} + R^2 \\
\mathcal{Z}_5 &= R_a{}^c{}_b{}^d R_d{}^e{}_c{}^f R_e{}^a{}_f{}^b + \frac{1}{56} \big( 21 R_{abcd} R^{abcd} R - 72 R_{abcd} R^{abc}{}_e R^{de} \\
&\qquad + 120 R_{abcd} R^{ac} R^{bd} + 144 R_a{}^b R_b{}^c R_c{}^a - 132 R_a{}^b R_b{}^a R + 15 R^3 \big)
\end{aligned}$$

- three dimensionless couplings, $L/\ell_P\,,\ \lambda\,,\ \mu\,,$ allow us to explore dual CFT's with most general three-point function $\langle T_{ab}\, T_{cd}\, T_{ef} \rangle$

"maintain control of calculations"

- analytic black hole solutions
- linearized eom in $AdS_5$ are second order (in fact, Einstein eq's!)
- can be extended to higher dimensions (D≥7)

## Quasi-Topological gravity:

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- so calculate!
- curvature in AdS$_5$ vacuum: $\dfrac{1}{\tilde{L}^2} = \dfrac{f_\infty}{L^2}$ ,

where $\quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- holographic trace anomaly:

(Myers, Paulos & Sinha, 1004.2055)

$$a = \pi^2\frac{\tilde{L}^3}{\ell_P^3}\left(1 - 6\lambda f_\infty + 9\mu f_\infty^2\right)\,,\qquad c = \pi^2\frac{\tilde{L}^3}{\ell_P^3}\left(1 - 2\lambda f_\infty - 3\mu f_\infty^2\right)$$

# RG flows in Quasi-Topological gravity:

- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$

$\longrightarrow$ AdS$_5$ vacua: $A(r) = r/\tilde{L}$

- natural to define "flow functions":

$$a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}\left(1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4\right)$$

$$c(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}\left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right)$$

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- in general flows:

$$\begin{aligned} a'(r) &= -\frac{3\pi^2}{\ell_P^3 A'(r)^4}\,A''(r)\left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right) \\ &= -\frac{\pi^2}{\ell_P^3 A'(r)^4}\left(T^t{}_t - T^r{}_r\right) \geq 0 \end{aligned}$$

assume null energy condition

gravitational equations of motion

## RG flows in Quasi-Topological gravity:

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- can try to be more creative in defining c(r) but we were unable to find a expression where flow is guaranteed to be monotonic
- our toy model seems to provide support for Cardy's “a-theorem” in four dimensions

**Higher Dimensions:** $D = d+1 \ (d \geq 6)$

- straightforward to reverse engineer “a-theorem” flows
- eq’s of motion:

$$T^t{}_t - T^r{}_r = (d-1)\, A''(r)\, \left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right)$$

- expression with natural flow:

$$a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)\, (\ell_P A'(r))^{d-1}} \left(1 - \frac{2(d-1)}{d-3}\lambda L^2 A'(r)^2 - \frac{3(d-1)}{d-5}\mu L^4 A'(r)^4\right)$$

$$\Longrightarrow \quad a'_d(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1} A'(r)^d}\left(T^t{}_t - T^r{}_r\right) \geq 0$$

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- flow between stationary points (where $a_d^* \equiv a_d(r)|_{AdS}$ )

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

**What is $a_d^*$ ??**

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$$a_d^* = \frac{\pi^{d/2}\tilde{L}^{d-1}}{\Gamma(d/2)\ell_P^{d-1}} \left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right)$$

where AdS curvature: $\frac{1}{\tilde{L}^2} = \frac{f_\infty}{L^2}\,, \quad \alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

- $a_d^*$ is NOT $C_T$ , coefficient of leading singularity in

$$\langle\, T_{ab}(x)\, T_{cd}(0)\,\rangle = \frac{C_T}{x^{2d}}\,\mathcal{I}_{ab,cd}(x)$$

- $a_d^*$ is NOT $C_S$ , coefficient in entropy density: $s = C_S\, T^{d-1}$

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- trace anomaly for CFT's with even d:

$$\langle\, T_\mu{}^\mu \,\rangle = \sum B_i(\text{Weyl invariant})_i - 2(-)^{d/2} A\,(\text{Euler density})_d$$

- verify that we have precisely reproduced central charge

$$\boxed{a_d^* = A}$$

(Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz)

$\longrightarrow$ agrees with Cardy's proposal (1988)

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**What is $a_d^*$ for odd d??** **(One moment!)**

## RG flows in Quasi-Topological gravity:

Comment:

- "c-theorem" still assume null energy condition

$\longrightarrow$ construct a toy model with reasonable physical properties

creative gravity

status quo matter

$$V(\phi^*) = -\frac{12}{L^2}\alpha^2$$

- natural to consider more general models:

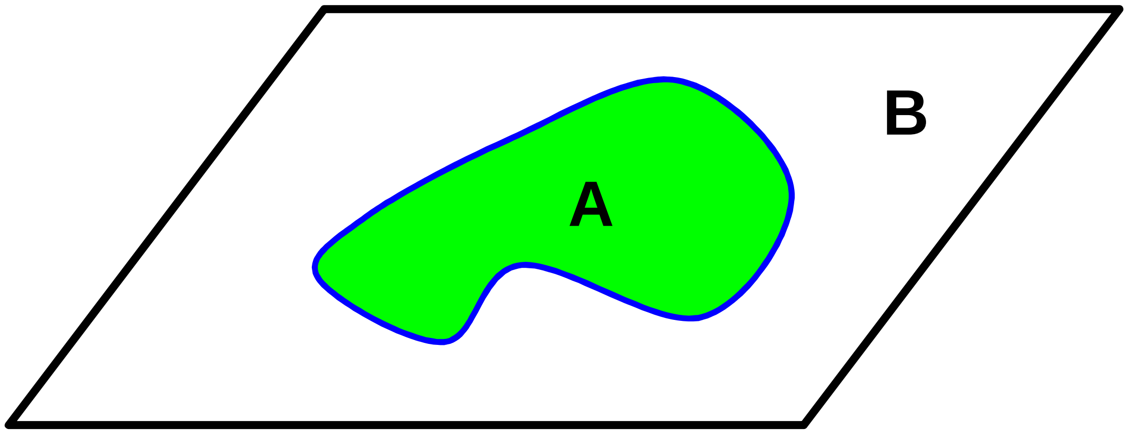
$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\,\Big[-V(\phi)\,+R+\frac{L^2}{2}\lambda(\phi)\,\chi_4+\frac{7L^4}{4}\mu(\phi)\,\mathcal{Z}_5$$
$$+L^2\gamma(\phi)\,R^{ab}\partial_a\phi\,\partial_b\phi+L^4\gamma'(\phi)R^2\nabla^2\phi+\cdots\Big]$$

**What are the rules??**

## $a_d^*$ and Entanglement Entropy

- introduce a(n arbitrary) boundary dividing the system in two
- integrate out degrees of freedom in outside region
- remaining dof are described by a density matrix $\rho_A$

$\longrightarrow$ entanglement entropy: $S = -Tr\left[\rho_A\, \log \rho_A\right]$




- full result sensitive to UV physics: $S \, = \, c_0 \dfrac{A_{boundary}}{\delta^{d-2}} \, + \, \cdots$
- universal information appears in subleading terms:

$$S \, = \, \cdots \, + \, c_d \log\left(R/\delta\right) \, + \, \cdots \qquad \text{for even d}$$

## $a_d^*$ and Entanglement Entropy

- in 1003.5357, studied black hole thermodynamics for quasi-topological gravity with various horizons: $R^{d-1}$, $S^{d-1}$, $H^{d-1}$
- allows for the following observation:
- place CFT on hyperbolic hyperplane (ie, R X $H^{d-1}$)

$\longrightarrow$ ground-state energy density is now negative

- heat system up until energy density is precisely zero, $\rho_E = 0$

$\longrightarrow$ entropy density:

$$s = (4\pi)^{d/2}\,\Gamma\left(d/2\right)\; a_d^* \; T^{d-1}$$
$$= \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\;\frac{a_d^*}{\tilde{L}^{d-1}}$$

**Why entanglement entropy?**

## $a_d^*$ and Entanglement Entropy

- CFT on hyperbolic hyperplane $H^{d-1}$ at finite T tuned to $\rho_E = 0$

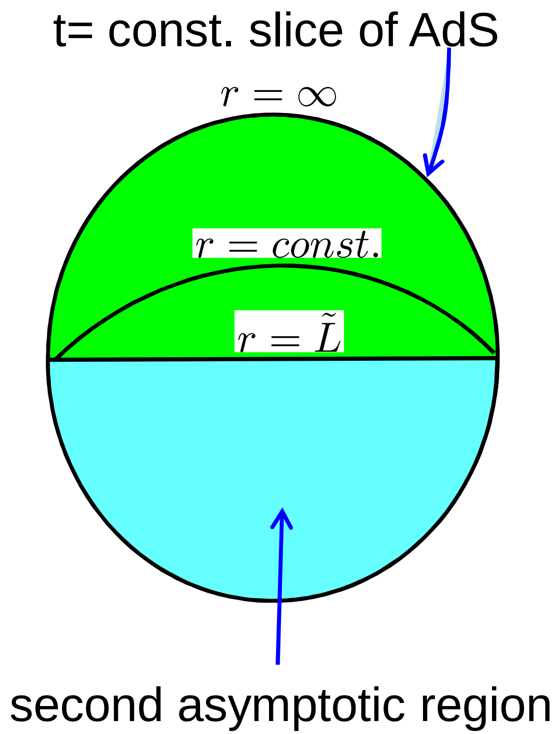
  ⟶ bulk spacetime is pure $AdS_{d+1}$

$$ds^2 = \frac{dr^2}{\left(\frac{r^2}{\tilde{L}^2} - 1\right)} - \left(\frac{r^2}{\tilde{L}^2} - 1\right) dt^2 + r^2\, d\Sigma_2^{d-1}$$

- so why is there entropy at all??

- $r \to \infty$ only reaches half boundary surface

- hyperbolic foliation divides boundary sphere into two halves and entropy is entanglement entropy of system



entropy density: $s = \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\ \frac{a_d^*}{\tilde{L}^{d-1}}$

total entropy: $S = \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\ \frac{a_d^*}{\tilde{L}^{d-1}}\, V\left(H^{d-1}\right)$

$$ds^2 = \tilde{L}^2\left[\frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}\right]$$

$$S = a_d^*\,\frac{2\pi}{\pi^{d/2}}\,\underbrace{\frac{\Gamma(d/2)}{d-2}\,\Omega_{d-2}\left(\frac{\tilde{L}}{\delta}\right)^{d-2}}_{\text{“area law” for d-dimensional CFT}} + \cdots$$

entropy density: $s = \frac{2}{\pi} \frac{a_d^*}{\tilde{L}^{d-1}}$

total entropy: $S = \frac{2}{\pi} \frac{a_d^*}{\tilde{L}^{d-1}} V(H^{d-1})$

$$ds^2 = \tilde{L}^2 \left[ \frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2} \right]$$

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \, \log\left(\tilde{L}/\delta\right) + \cdots \quad \text{for even d}$$

$$\cdots + \underbrace{(-)^{\frac{d-1}{2}} 2\pi\, a_d^*}_{\text{universal contribution}} + \cdots \quad \text{for odd d}$$

(Note: can be derived with conventional definition of $S_{entangle}$)

**Conjecture:**

- place CFT on $S^{d-1}$ X R and divide sphere in half along equator
- entanglement entropy of ground state has universal contribution

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^* \log(L/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\; 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}$$

(any gravitational action)
(any CFT in even d)

- in RG flows between fixed points

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

("unitary" models)

⟶ gives framework to consider c-theorem for **odd** or even *d*

⟶ behaviour discovered for holographic model but conjecture that result applies generally (outside of holography)

## and Beyond:

- Susskind & Witten: density of degrees of freedom in N=4 SYM connected to area of holographic screen at large R in $AdS_5$

$$N_{dof} \sim N_c^2 \times \frac{V_3}{\delta^3} \sim \frac{A(R)}{\ell_P^3}$$

cut-off scale defined by regulator radius: $\frac{1}{\delta} = \frac{R}{L^2}$

- given higher curvature bulk action, natural extension is to evaluate Wald entropy on holographic screen at large R

$$S = -2\pi \oint d^{d-1}x\sqrt{h}\ \hat{\varepsilon}^{ab}\ \hat{\varepsilon}_{cd}\ \frac{\partial \mathcal{L}_{bulk}}{\partial R^{ab}{}_{cd}}$$

- straightforward evaluate:

$$\boxed{N_{dof} = \frac{2\pi}{\pi^{d/2}}\Gamma(d/2)\ a_d^*\ \frac{V_{d-1}}{\delta^{d-1}}}$$

for any covariant action: $\mathcal{L}_{bulk} = \mathcal{L}_{bulk}\left(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \cdots\right)$

## Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories
- toy theories with higher-R interactions extend class of CFT's

$\longrightarrow$ maintain calculational control with LL or quasi-top. gravity

- consistency (causality & positive fluxes) constrains couplings
- provide interesting insights into RG flows
- naturally support Cardy's version of a-theorem with d even
- suggests extension of a-theorem to d odd
- $a_d^*$ seems to play a privileged role in holography
- further implications for holographic dualities??

**Lots to explore!**