Applications of AdS/CFT to condensed matter systems
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Effective Holographic Theories for low-T CM systems

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Bibliography

Based on current work with:

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and previous work

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The plan of the talk

- Introduction (Motivation, Tools, Goals, Strategy)

- Effective Holographic Theories

- On naked singularities

- Transport coefficients

- Holographic Dynamics at zero charge density (Solutions, thermodynamics, spectra and transport)

- Holographic Dynamics at finite charge density (Solutions, thermodynamics, spectra and transport)

- Outlook

Brief summary of results

• We will characterize the IR dynamics of strongly coupled theories at finite density driven by a leading relevant operator in terms of two real constants (γ, δ).

• For zero charge density we will scan the IR landscape and characterize theories by their spectra and their low temperature thermodynamics. Both 1st order and continuous transitions exist.

• At finite charge density we will find all near-extremal solutions and calculate the low-temperature conductivity, in order to characterize the dynamics. We will also analyze two families of exact solutions.

• We will find that some regions in the (γ, δ) will be excluded as unphysical.

• For all (γ, δ) except when $\gamma = \delta$ the entropy vanishes at extremality.

• There is a codimension-one space, where the IR resistivity is linear in the temperature

• When the scalar operator is not the dilaton, then in 2+1 dimensions, the IR resistivity has the same scaling as the entropy (and heat capacity).

• We will find the first holographic examples of Mott insulators at finite density.

• Generically the charge-induced entropy dominates the one without charge carriers.

The strategy

• Use EHT to search for strongly-coupled systems that realize the non-fermi liquid (strange metal) benchmark behavior.

• We will not worry about the superconducting instability.

• We must study both thermodynamics and transport.

• A holographic description assumes strongly interacting quasi-particles that are bound states of “partons”.

♠ We will parametrize the EHT

♠ Study the IR and implement “physicality criteria”

♠ Calculate then Thermodynamics and transport to characterize the $T \to 0$ physics.

Effective Holographic Theories

• In AdS/CFT we usually work in the 2d approximation: possible if there is a large gap in anomalous dimensions.

• In non-conformal theories the problem is more complicated but the approximation can be rephrased

♠ In EHTs we have a (low) UV cutoff and a finite number of operators (all the ones that are sources as well as important ones that obtain vevs).

♠ The cutoff can be very low, and only an IR scaling region is needed for the holographic calculation of IR dynamics.

• Sometimes there is no small parameter that justifies the EHT truncation but qualitative conclusions may be robust.

The strategy is:

1. Select the operators expected to be important for the dynamics

2. Write an effective (gravitational) holographic action that captures the (IR) dynamics.

3. Find the saddle points (classical solutions)

4. Study the physics around each acceptable saddle point.

• The bulk metric $g_{\mu\nu} \leftrightarrow T_{\mu\nu}$ is always sourced in any theory. In CFTs it captures all the dynamics of the stress tensor and the solution is AdS_{p+1}.

• In a theory with a conserved U(1) charge, a gauge field is also necessary, $A_\mu \leftrightarrow J_\mu$. If only $g_{\mu\nu}, A_\mu$ are important then we have an AdS-Einstein-Maxwell theory with saddle point solution=AdS-RN.

• The thermodynamics and CM physics of AdS-RN has been analyzed in detail in the last few years, revealing rich physical phenomena

Chamblin+Emparan+Johnson+Myers (1999), Hartnoll+Herzog (2008), Bak+Rey (2009),Cubrovic+Schalm+Zaanen (2009), Faulkner+Liu+McGreevy+Vegh (2009)

1. Emergent AdS_2 scaling symmetry

2. Interesting fermionic correlators

but

3. Is unstable (in N=4) to both neutral and charged scalar perturbations

Gubser+Pufu (2008), Hartnoll+Herzog+Horowitz (2008)

4. Have a non-zero (large) entropy at $T = 0$.

Einstein-Scalar-U(1) theory

• To go beyond RN, we must include the most important (relevant) scalar operator in the IR. This can capture the dynamics of the system (lattice as well as filled bands).

• The most general 2d action is

$$S = \int d^{p+1}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - Z(\phi)F^2\right]$$

involving two arbitrary functions of ϕ. Typically the potential is non-trivial. It may have an UV fixed point (not necessary).

• We assume it does not have an IR fixed point (otherwise back to RN).

• We will parametrize the IR asymptotics of V, Z using sugra intuition.

$$V(\phi) \sim e^{-\delta\phi} \quad , \quad Z(\phi) \sim e^{\gamma\phi} \quad , \quad \phi \to \pm\infty$$

• We must have $V(\phi) \to \infty$ in the IR (and the inverse in the UV). For Z in the IR

$$\begin{cases} Z \to \infty \quad , \quad \text{weak coupling} \quad , \quad \text{bulk U(1)'s} \\ \\ Z \to 0 \quad , \quad \text{strong coupling} \quad , \quad \text{tachyon condensation} \end{cases}$$

• From now on we set

$$V = \Lambda\ e^{-\delta\phi} \quad , \quad Z = e^{\gamma\phi}$$

• Solutions depend on ($\Lambda \to \Lambda e^{\delta\phi_0}$)

$$\phi_0 \quad , \quad Q \quad , \quad T$$

• If the U(1) originates on flavor branes the minimal system to study at finite density is the Einstein-Dilaton-U(1) system with DBI action

$$S = \int d^{p+1}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - Z(\phi)\left(\sqrt{\det(\delta^\mu{}_\nu + F^\mu{}_\nu)} - 1\right)\right]$$

On naked holographic singularities

• If no IR fixed points, all Poincaré invariant solutions end up in a naked IR singularity.

• In GR we abhor naked singularities.

• In holographic gravity some may be acceptable. The reason is that they do not signal a breakdown of predictability as is the case in GR. They could be resolved by stringy or KK physics, or they could be shielded for finite energy configurations.

Something similar happens in the “Liouville wall” of 2d gravity: all finite energy physics is not affected by the $e^{\phi} \to \infty$ singularity.

• An important task in EHT is to therefore ascertain when such naked singularities are acceptable and when are reliable (alias "good")

♠ Gubser gave the first criterion for good singularities: They should be limits of solutions with a regular horizon.

Gubser (2000)

• The second criterion amounts to having a well-defined spectral problem for fluctuations around the solution: The second order equations describing all fluctuations are Sturm-Liouville problems (no extra boundary conditions needed at the singularity).

Gursoy+E.K.+Nitti (2008)

• The singularity is "repulsive" (like the Liouville wall). It has an overlap with the previous criterion. It involves the calculation of "Wilson loops"

Gursoy+E.K.+Nitti (2008)

$$g^E_{\mu\nu} = e^{-k\phi} g^\sigma_{\mu\nu} \quad , \quad \text{for dilaton} \quad k = \sqrt{\frac{2}{p-1}}$$

• It is not known whether the list is complete. The 1st and 2-3rd criteria are non-overlapping.

Solutions at zero charge density

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Gursoy+Kiritsis+Mazzanti+Nitti (2009)

- The only parameter relevant for the solutions is $\delta \in R$. Take $p+1=4$.

- $0 \leq |\delta| < 1$. The T=0 solution has a "good" singularity.

The spectrum is continuous without gap.

At $T > 0$ there is a continuous transition to the BH phase (only one BH available).

- $|\delta| = 1$. This is a marginal case. It has a good singularity, a continuous spectrum and a gap. A lot of the physics of finite temperature transitions depends on subleading terms in the potential:

♠ If $V = e^{\phi}\ \phi^{P}$, with $P < 0$ this behaves as in $|\delta| < 1$. When $P > 0$ like $|\delta| > 1$.

♠ If $V = e^{\phi}\left[1 + C\ e^{-\frac{2\phi}{n-1}} + \cdots\right]$, then at $T = T_{min} = T_c$ there is an n-th order continuous transition (Not of the standard conformal kind).

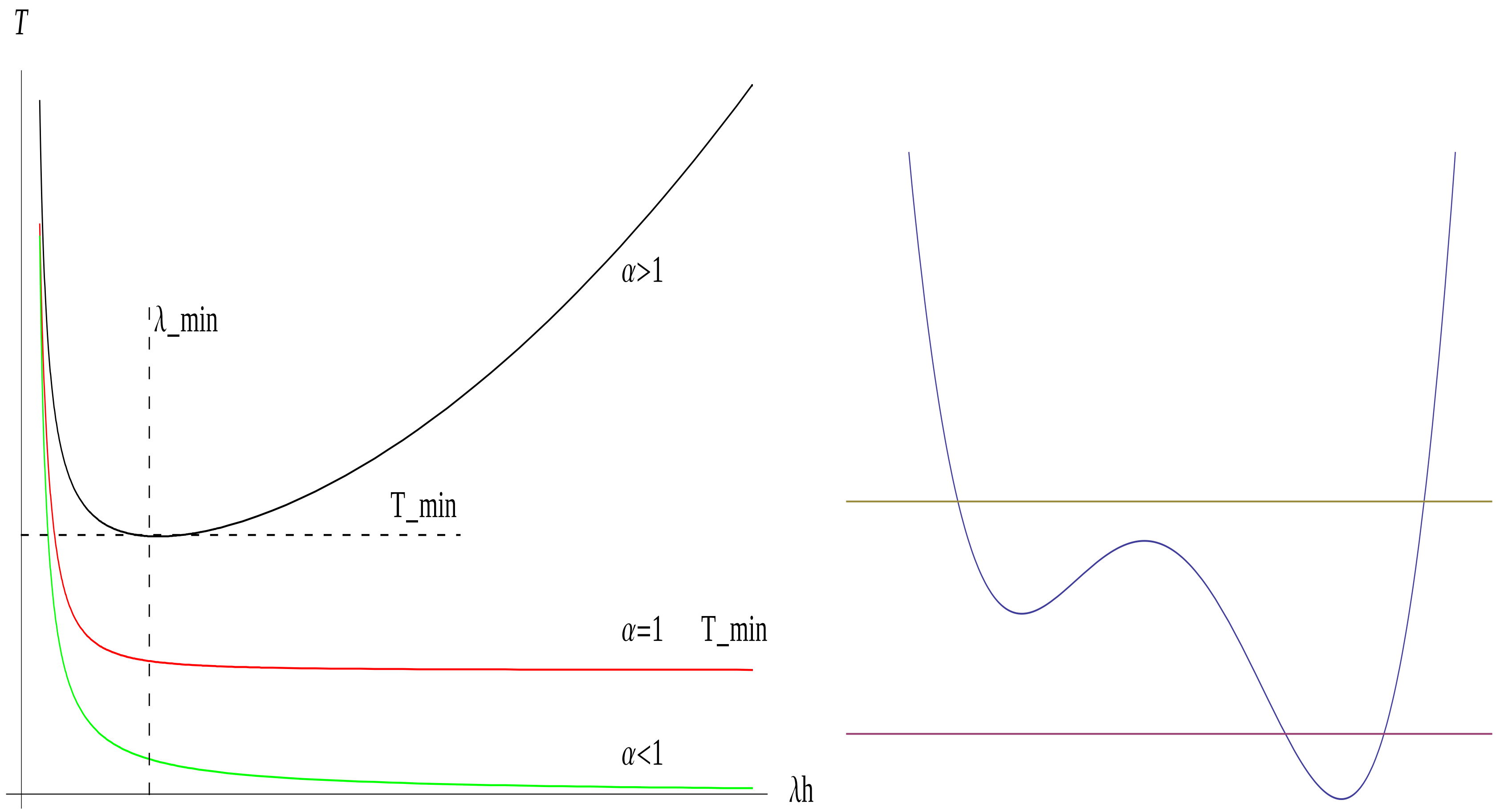
♠ If $V = e^{\phi}\left[1 + C/\phi^{k} + \cdots\right]$, then at $T = T_{min} = T_c$ there is a generalized KT phase transition

Gursoy (2010)

• When $1 < |\delta| < \sqrt{\frac{5}{3}}$ The naked singularity is stronger but still of the good kind. The spectrum is discrete and gapped.

At $T > 0$ there is a single BH solution that is unstable (it is a “small” BH), and never dominates the vacuum thermal solution. In this case one needs the full potential to ascertain what happens at finite temperature.

• There is a first order phase transition at T_c to a large BH.



• For more complicated potentials multiple phase transitions are possible.

Gursoy+Kiritsis+Mazzanti+Nitti (2009), Alanen+Kajantie+Tuominen (2010)

• For $|\delta| > \sqrt{\frac{5}{3}}$ the T=0 naked singularity is not repulsive and therefore the EHT is unreliable.

• For $|\delta| > \sqrt{3}$ the Gubser criterion is violated.

• We can also analyze the spectrum of current fluctuations that now depends on γ.

♠ $\frac{\gamma}{\delta} > \frac{3}{2}$ or $\frac{\gamma}{\delta} < -\frac{1}{2}$: When the UV dimension of the scalar $\Delta < 1$ then the potential diverges both in the UV and the IR and the spectrum is discrete and gapped. This resembles to an insulator. Otherwise it is a conductor.

♠ $-\frac{1}{2} < \frac{\gamma}{\delta} < \frac{3}{2}$. The spectral problem is unacceptable and therefore the spin-1 spectrum unreliable.

• The AC Conductivity at zero charge density:

When $|\delta| < 1$ the effective potential is

$$V_{eff} \simeq \frac{c}{z^2} \quad , \quad c = \frac{(\gamma\delta + 1 - \delta^2)\gamma\delta}{(1-\delta^2)^2} \quad , \quad \sigma \sim \omega^n \quad , \quad n = \sqrt{4c+1} - 1$$

• It becomes $n = -\frac{2}{3}$ iff

$$\gamma = \frac{\delta^2 - 1}{3\delta} \quad \text{or} \quad \gamma = \frac{2(\delta^2 - 1)}{3\delta}$$

• The DC conductivity can calculated to be

$$\sigma = e^{-k\phi_0}(\kappa T)^{\frac{2k\delta+2}{\delta^2-1}}\sqrt{\langle J^t\rangle^2 + e^{2(\gamma+k)\phi_0}(\kappa T)^{\frac{4[1+(\gamma+k)\delta]}{1-\delta^2}}},$$

$$\rho_{\text{light}} \sim T^{\frac{2\gamma\delta}{\delta^2-1}} \quad , \quad \rho_{\text{drag}} \sim \frac{T^{\frac{2k\delta+2}{1-\delta^2}}}{\langle J^t\rangle}$$

• In the first case we can attain linear resistivity when

$$\gamma = \gamma_{\text{linear}} \equiv \frac{\delta^2 - 1}{2\delta}.$$

Charged near extremal scaling solutions

$$ds^2 = r^{\frac{(\gamma-\delta)^2}{2}}\left[dx^2 + dy^2 - f(r)dt^2\right] + \frac{dr^2}{f(r)}$$

$$f(r) = \frac{16\,(-\Lambda)}{wu^2} e^{-\delta\phi_0} \quad r^{1-\frac{3}{4}(\gamma-\delta)^2} \quad \left(r^{\frac{wu}{4}} - 2m\right),$$

$$e^{\phi} = e^{\phi_0} r^{-(\gamma-\delta)} \quad , \quad \mathcal{A} = \frac{8}{wu}\sqrt{\frac{v\Lambda}{u}} e^{-\frac{(\gamma+\delta)}{2}\phi_0} \quad \left[r^{\frac{wu}{4}} - 2m\right] dt$$

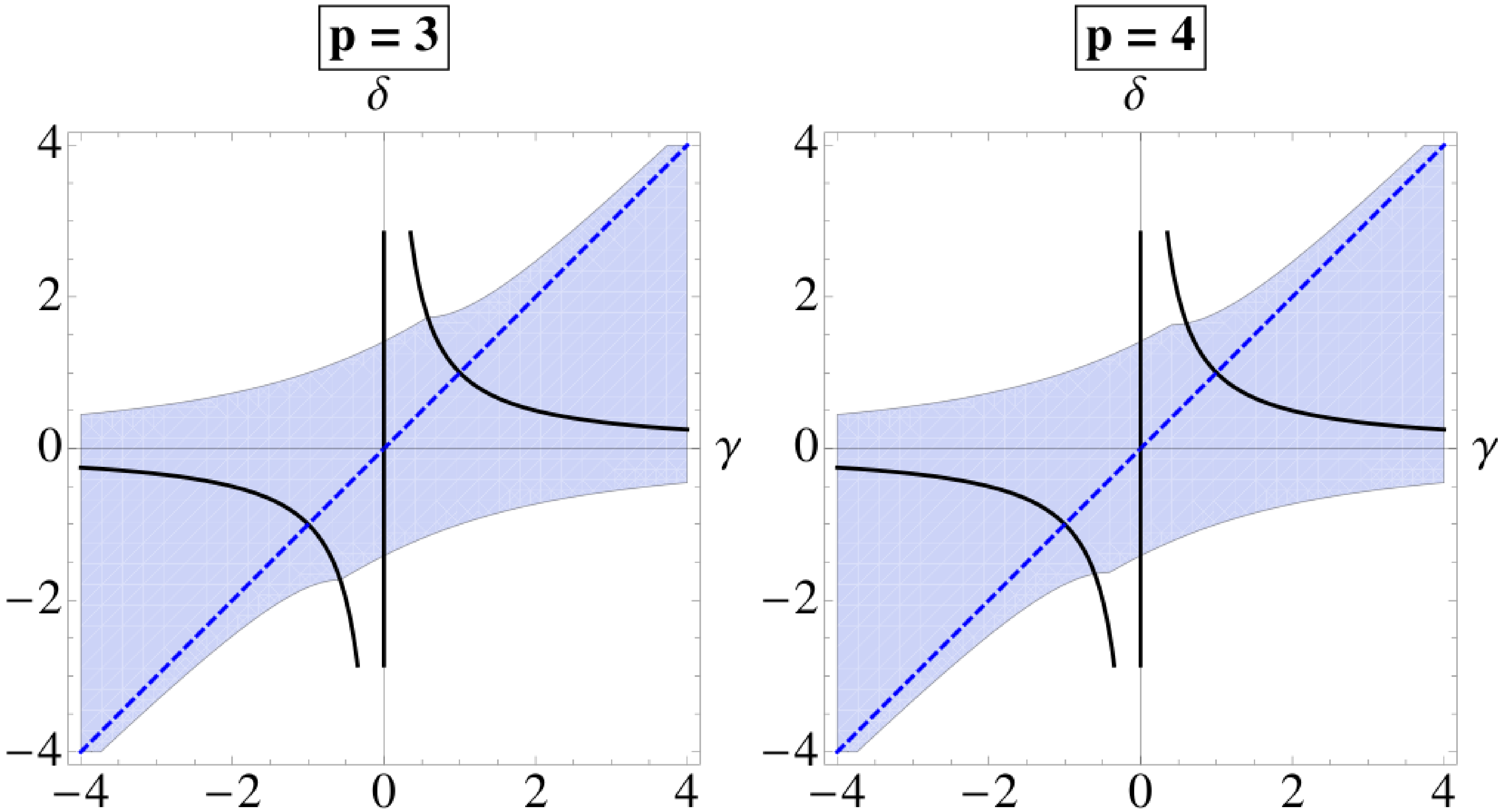
$$wu = 3\gamma^2 - \delta^2 - 2\gamma\delta + 4 > 0\,, \quad u = \gamma^2 - \gamma\delta + 2\,, \quad v = \delta^2 - \gamma\delta - 2\,, \delta^2 \le 3$$

- These are near extremal solutions (the charge density is fixed).
- When $\delta = 0$ they are Lifshitz solutions with $z = 1 + \frac{4}{\gamma^2}$.

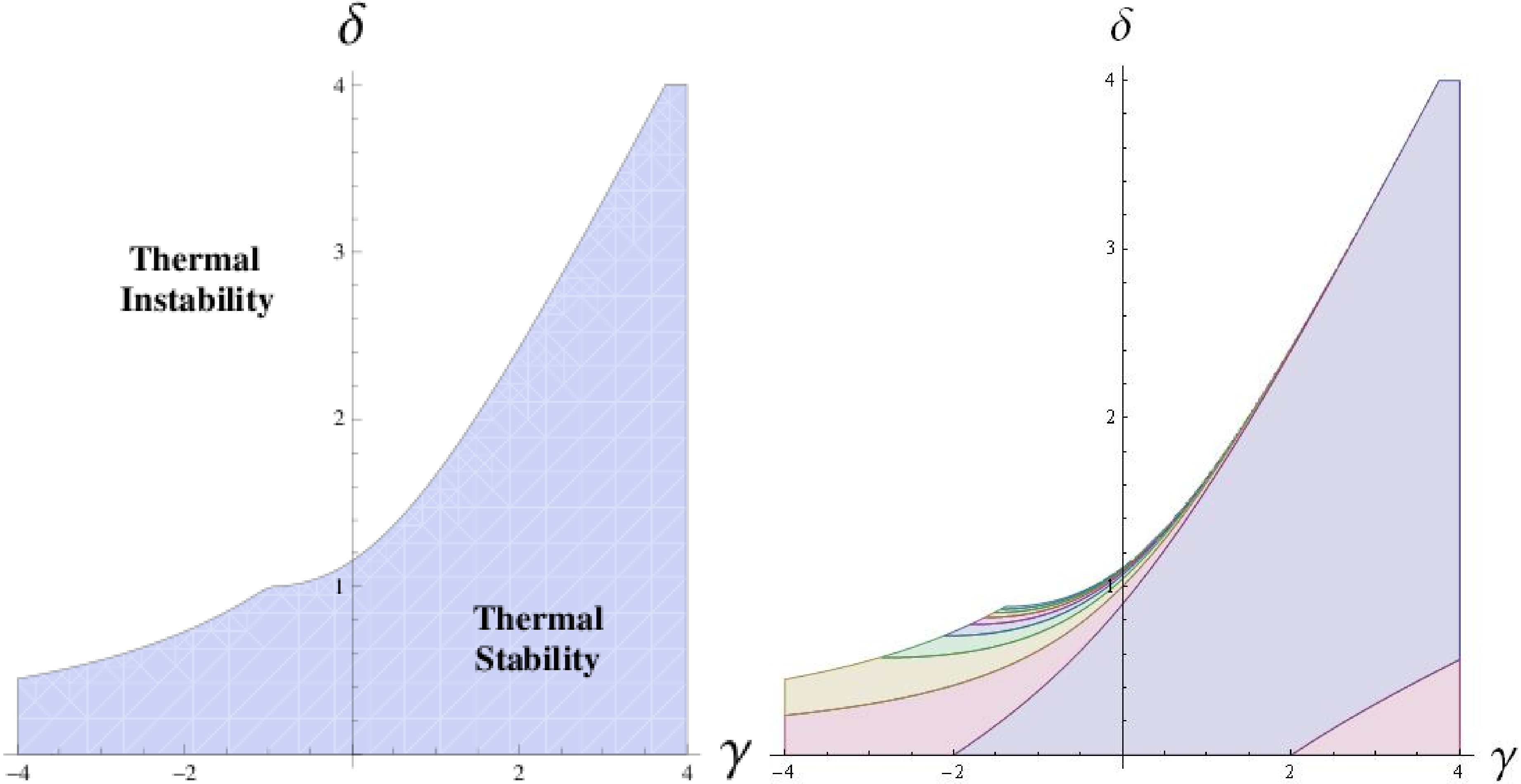
M. Taylor (2009)

$$T = \frac{1}{4\pi}\sqrt{-w\Lambda}\; e^{-\frac{\delta}{2}\phi_0}\; (2m)^{1-2\frac{(\gamma-\delta)^2}{wu}} \quad , \quad S \sim T^{\frac{2(\gamma-\delta)^2}{wu-2(\gamma-\delta)^2}}$$

- The Entropy vanishes at extremality if $\gamma \neq \delta$.
- If $\gamma = \delta$ the extremal solution is $AdS_2 \times R^2$.
- The charge entropy dominates the $Q = 0$ entropy almost everywhere.



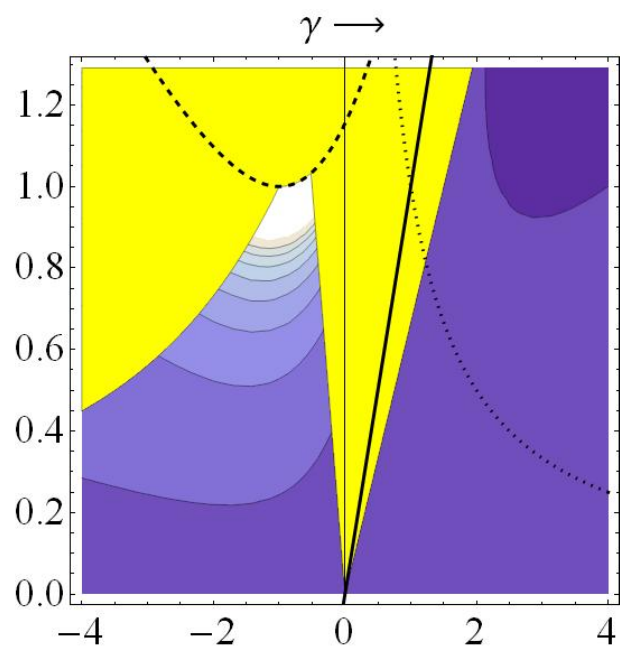
This graph shows the Gubser bounds on the near extremal solution on the whole of the (γ, δ) plane for $p = 3$ and $p = 4$. The blue regions are the allowed regions where the near extremal solutions are black-hole like. The white regions are solutions of a cosmological type and therefore fail the Gubser bound. The dashed blue line is the $\gamma = \delta$ solutions while the solid black line corresponds to the $\gamma\delta = 1$ solutions.



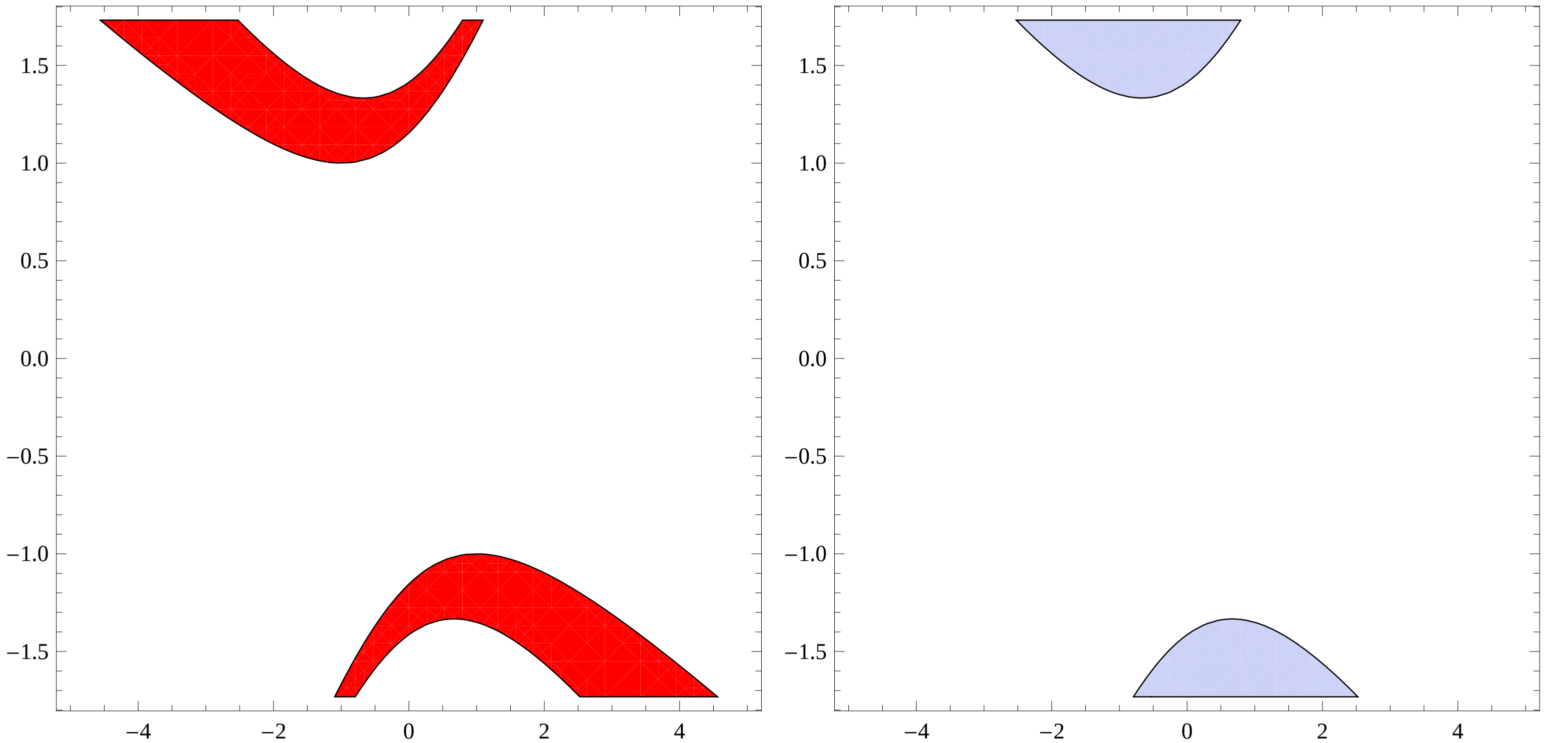
On the left: region of local stability of the near extremal black hole. Right: The variety of phase transitions of the near extremal black hole to the background at zero temperature. In the blue region continuous transitions occur, in the purple region adjacent to the blue one the transitions are of third-order. The stripes starting with yellow to the left of the blue and purple regions depicts transitions of fourth-(yellow) up to tenth-order. Above them all higher-order transitions also occur.

The extremal AC conductivity

$$\sigma \sim \omega^n, \qquad n = \left| \frac{(\delta-\gamma)(3\gamma+5\delta)-12}{(\delta-\gamma)(\gamma+3\delta)-4} \right| - 1\,.$$



Contour plot of the scaling exponent n in the (γ,δ) upper half plane for $p=3$ ($0 \le \delta \le \sqrt{\frac{5}{3}}$) Contours correspond to $n = 1.52, \ldots, 8.36$, starting with $n = 1.52$ in the upper right corner and increasing in steps of 0.76. The black solid line $\gamma = \delta$ is $n = 2$, and brighter colors correspond to larger n. The yellow regions are forbidden by several constraints (see text). The scaling exponent diverges to $+\infty$ along the dashed black line



Left: The region on the $(\gamma\delta)$ plane where the IR black holes are unstable and $c > 0$. Here the extremal finite density system has a mass gap and a discrete spectrum of charged excitations, when $\Delta < 1$. This resembles a Mott insulator and the figure provides the Mott insulator "islands" in the (γ, δ) plane. Right:The region where the IR black holes are unstable, and $c < 0$. In this region the extremal finite density system has a gapless continuous spectrum at zero temperature. In both figures the horizontal axis parametrizes γ, whereas the vertical axis δ.

The near-extremal DC conductivity

• For massive charge carriers

$$\rho \sim T^m, \qquad \frac{4k(\delta-\gamma)+2(\delta-\gamma)^2}{4(1-\delta(\delta-\gamma))+(\delta-\gamma)^2}$$

• The exponent becomes unity for two values of γ

$$\gamma_\pm = 3\delta + 2k \pm 2\sqrt{1+(\delta+k)^2}\,.$$

• For a non-dilatonic scalar, $k=0$ and the temperature dependence of the entropy and the resistivity are the same. Therefore, the entropy also scales linearly with T.

• For the Lifshitz solutions, we must take $\delta=0$ and $\gamma=-\sqrt{\frac{4}{(z-1)}}$. In this case we obtain that

$$m_p = \frac{2+k\sqrt{4(z-1)}}{z}\,,$$

• When $k=0$ this is in agreement with Hartnoll+Polchinski+Silverstein+Tong

Exact charged solutions

- Charmousis+Gouteraux+Soda (2009) found solutions for $\gamma\delta = 1$ and $\gamma = \delta$. For $\gamma\delta = 1$ it is the general solution.

- $\delta^2 < 3$ otherwise the solutions are Anti-De-Sitter like (cf Gubser).

- For $\gamma\delta = 1$ there are three distinct classes of dynamics:
$$\delta^2 \in [0,1] \cup [1, 1+\tfrac{2}{\sqrt{3}}] \cup [1+\tfrac{2}{\sqrt{3}}, 3)$$

- For $\gamma = \delta$ only two $\delta^2 \in [0,1] \cup [1,3]$

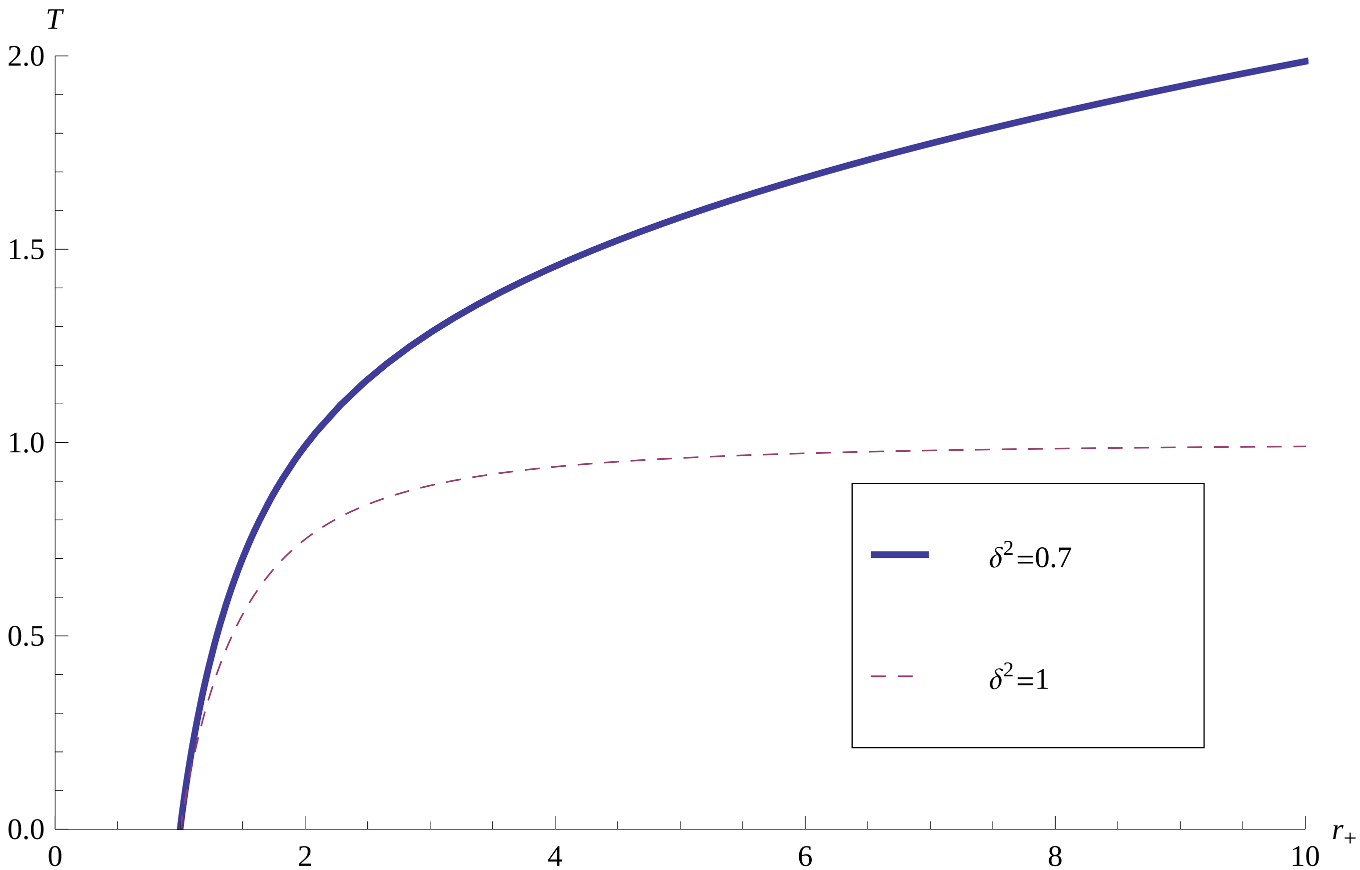
- At $Q = 0$ all $|\delta| > 1$ systems were insulators. Now this range is split in two in $\gamma\delta = 1$.

- In $\gamma = \delta$ the system is always a conductor at finite density.

- $\gamma\delta = 1$ has zero entropy but $\gamma = \delta$ has finite entropy at extremality .

$\gamma\delta = 1$ solutions

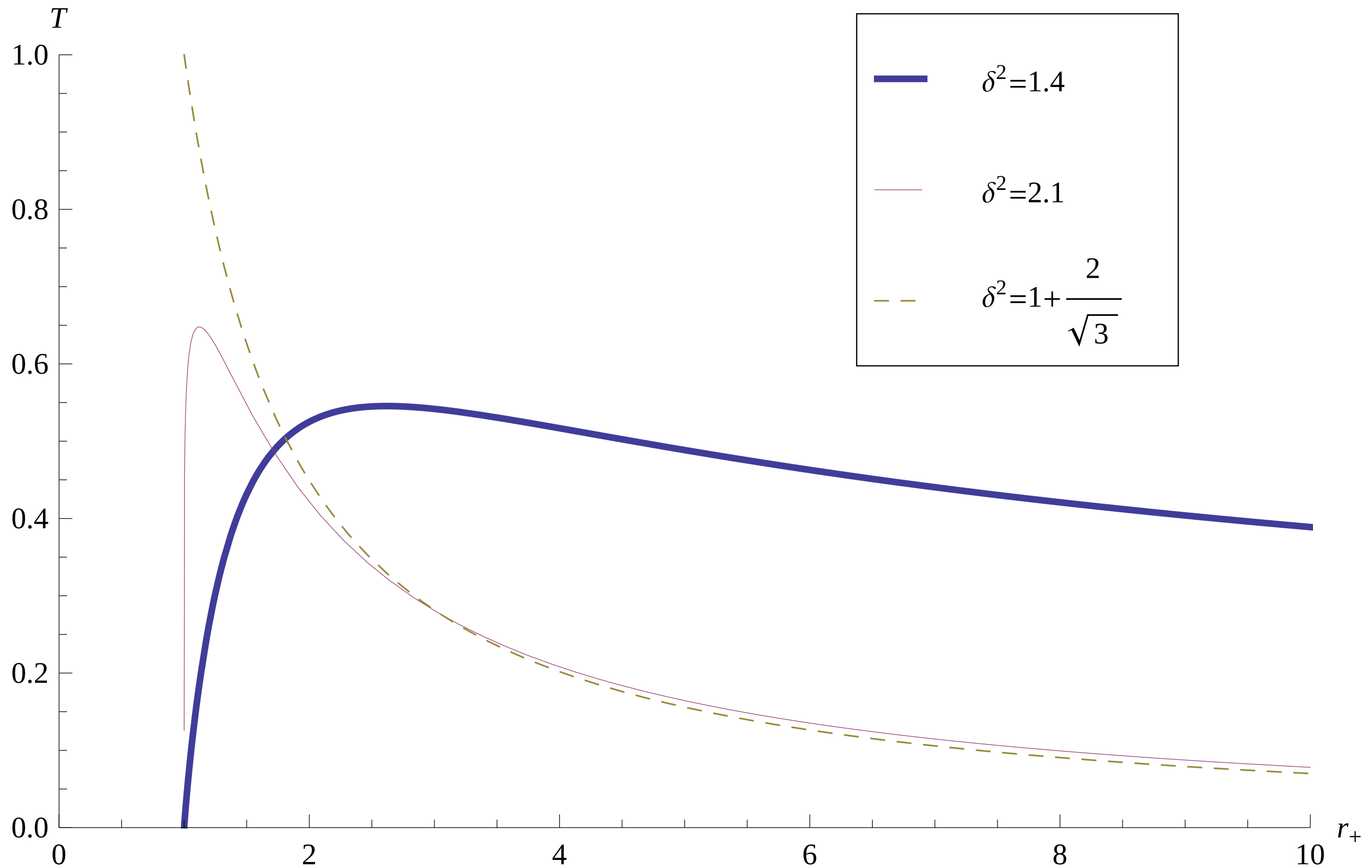
- $0 \leq |\delta| < 1$



- A single branch of BH that dominate at $T > 0$. The transition at $T = 0^+$ is between 2nd and 3rd order.

- The system is a conductor.

• $\delta^2 \in [1, 1+\frac{2}{\sqrt{3}}]$



• There are two black holes at a given temperature $T < T_{max}$.

• At $T_{max} > T > 0$ it is the small black hole branch that dominates thermodynamically. The transition at $T = 0^+$ is continuous of any order. The transition at T_{max} is 0-th order (Will be modified).

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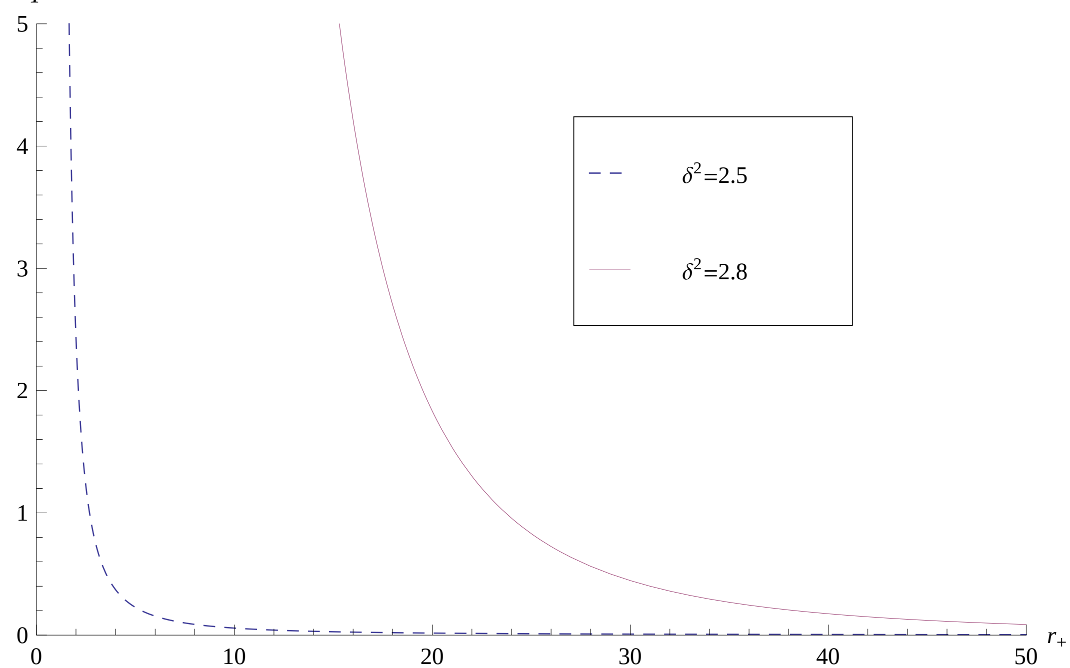


• The BH solution is unstable and never dominant. This is like the $\delta^2 > 1$ case at zero density.

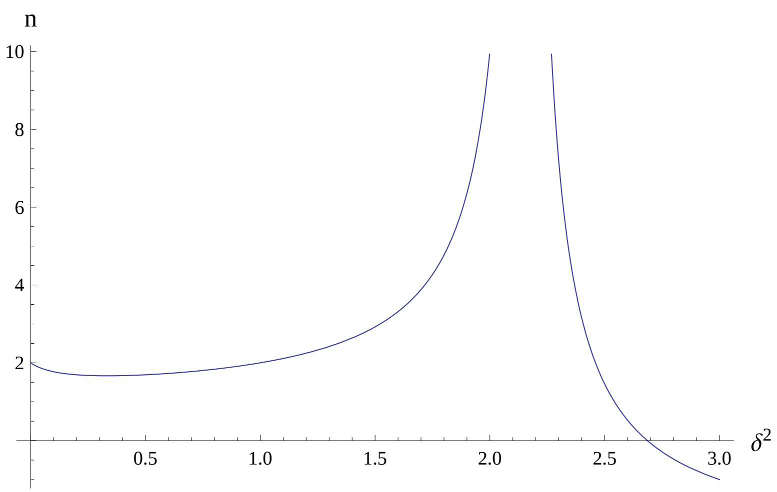
• For $1 + \frac{2}{\sqrt{3}} \leq \delta^2 \leq \frac{5+\sqrt{33}}{4}$ the system has a mass gap and discrete spectrum in the current correlator if $\Delta < 1$. It is a Mott-like insulator.

• For a modified potential a RN-like new stable BH solution is expected to appear for $T > T_{min}$. There will be a first or second order phase transition to a conducting phase at $T_c > T_{min}$.

• For $\frac{5+\sqrt{33}}{4} \leq \delta^2 < 3$ The system has a continuous spectrum and is again a conductor.

In the first two regimes $0 \le \delta^2 \le 1 + \frac{2}{\sqrt{3}}$ the AC conductivity is

$$\sigma(\omega) \simeq \omega^n\,, \qquad n = \frac{(3-\delta^2)(5\delta^2+1)}{|3\delta^4 - 6\delta^2 - 1|} - 1\,.$$

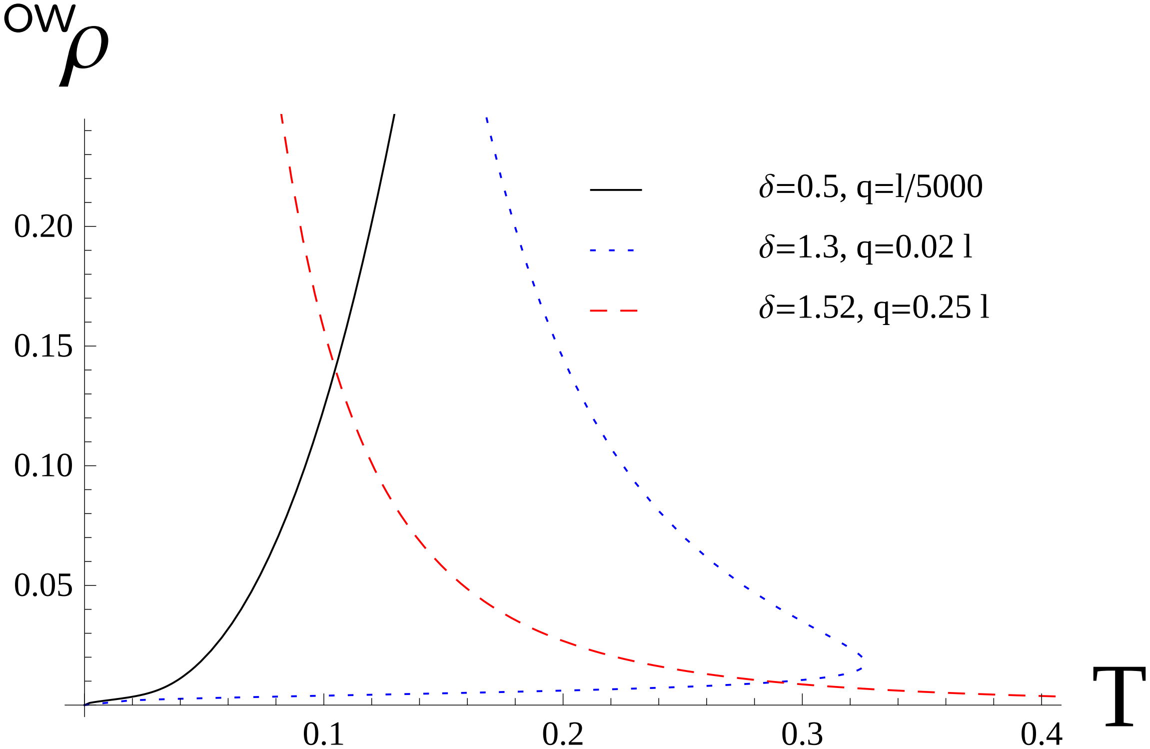


• The exponent is always larger than 5/3 in the region, $0 \le \delta^2 < 1 + \frac{2}{\sqrt{3}}$ and diverges at $\delta^2 = 1 + \frac{2}{\sqrt{3}}$.

• The system behaves as a conductor.

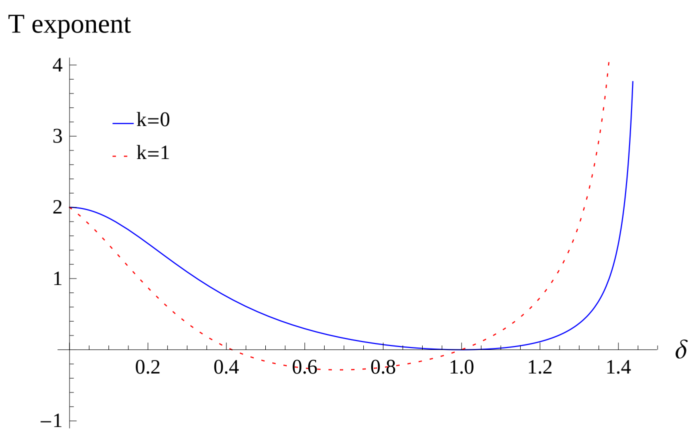
• The system is again conducting for $\frac{1}{4}(5+\sqrt{33}) < \delta^2 < 3$.

The DC resistivity is plotted below.



The leading behavior at low temperature is

$$\rho_{\text{leading}} \sim \frac{T_f}{J^t}\left(\frac{q}{\ell}\right)^{\frac{2\delta(\delta(3-\delta^2)+(1+\delta^2)k)}{1+6\delta^2-3\delta^4}}\ (\ell T)^{\frac{2(\delta^2-1)(\delta^2-1+2k\delta)}{1+6\delta^2-3\delta^4}}$$



- It is one at $\delta^2 = 1 + \frac{2}{\sqrt{5}}$.

Outlook 1

• We have analyzed parts of the landscape of strongly coupled EHT with a scalar, a graviton and a vector.

• Without doping, depending on the dynamics we can have a low-energy phase that is quasi-conformal (continuum without a gap) or with discrete spectrum and a gap (insulator). The finite temperature phase is a liquid with standard quasi-normal modes.

• With doping and important back-reaction in cases we analyzed we have a low-energy phases that are conducting but with unusual equations of state (non-Fermi liquids). At higher temperatures there are phase transitions to RN-like phases

• Several families indicate benchmark transport coefficients like linear resistivity, T^{-3} Hall conductivity etc.

• We also found systems that resemble Mott insulators: the charge carriers are gapped, due to strong interactions.

• For the generic theory we found near-extremal solutions but we cannot yet dress the full phase diagram.

Outlook 2

- We have implemented the essence of EHT: scan the IR behaviors, then match to AdS behavior.

- We have found many scaling solutions. The ones that pass the physical tests represent universality classes.

- For some cases we analyzed all relevant charged solutions and found a non-trivial and unusual phase structure.

- Further analysis is needed in order to elucidate the viability and nature of these solutions.

- Apply more general techniques to find other classes of solutions.

- Attempt matching to CM systems.

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THANK YOU

Introduction

• In CM physics many interesting systems are strongly coupled:

1. Materials at the border with magnetism (Cuprate high-Tc superconductors, pnictides, heavy fermion metals, Al-Mn alloys etc)

2. Quantum Hall systems

3. Graphene

• Almost always, sign-problems and critical behavior make numerical simulation prohibitive.

• In the UV we have a well understood theory=electrons+ions+photons. Generically

$$\text{potential energy} \gg \text{kinetic energy} \quad \to \quad \text{Strong Coupling}$$

• By "luck" sometimes dressed electrons (quasiparticles) are weakly coupled $\to$ Landau theory of Fermi-liquids $\to$ standard metals.

• In other cases we may expect emergent IR degrees of freedom, that are strongly coupled and YM-like:

1. In spin/fermion systems

Laughlin 80's, Sachdev(2010)

2. Non-abelian CS seems to emerge in several contexts. Coupled to matter $\rightarrow$ M2 class of theories.

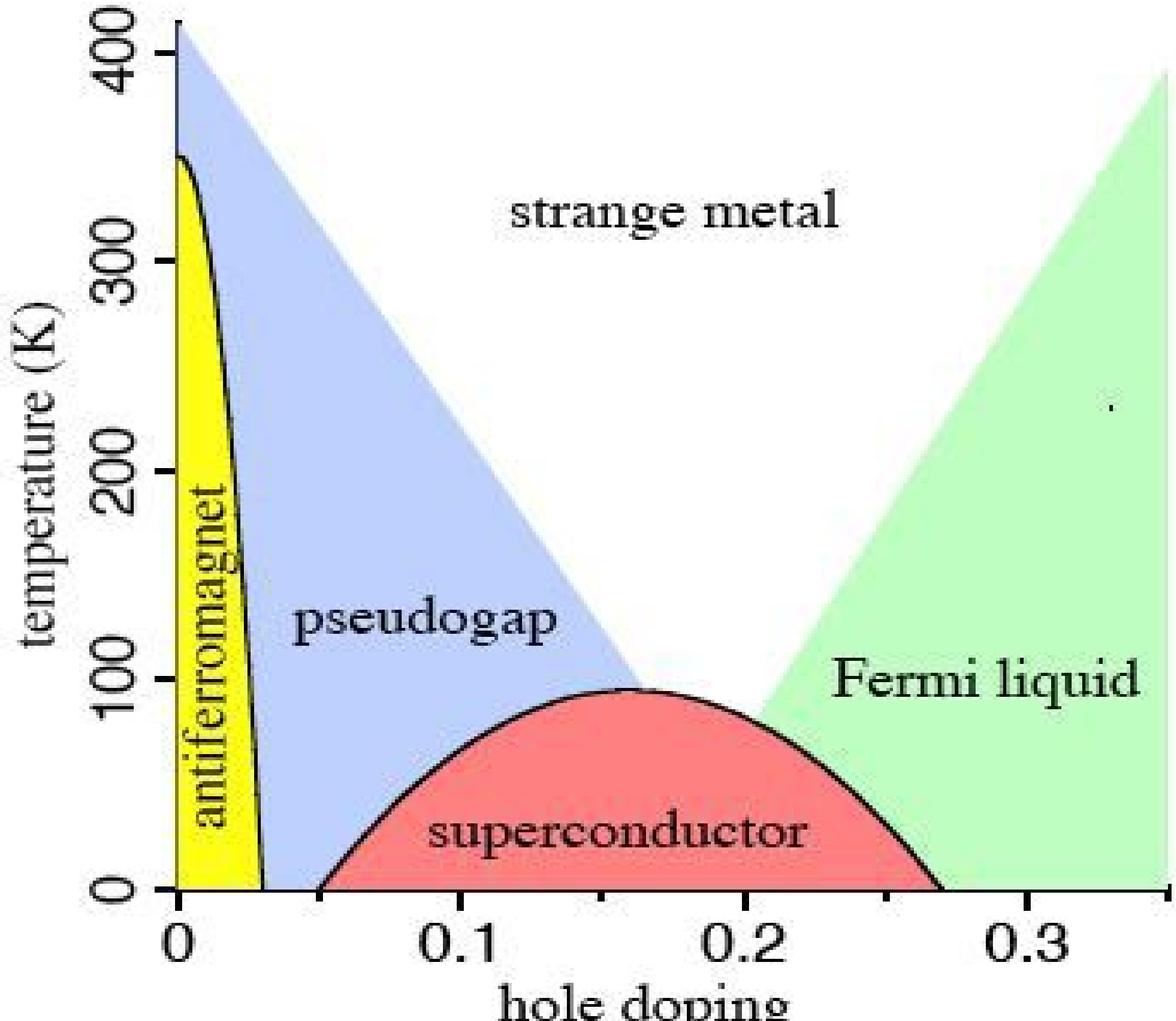
3. Massless 3d fermions+EM seems to have a non-abelian large N structure

S. C. Lee (2009), Metlitski+Sachdev (2010), Mross+McGreevy+Liu+Senthil (2010)

• The behavior generated is known as strange metal (non-fermi liquid), and exists in all systems at the border with magnetism.

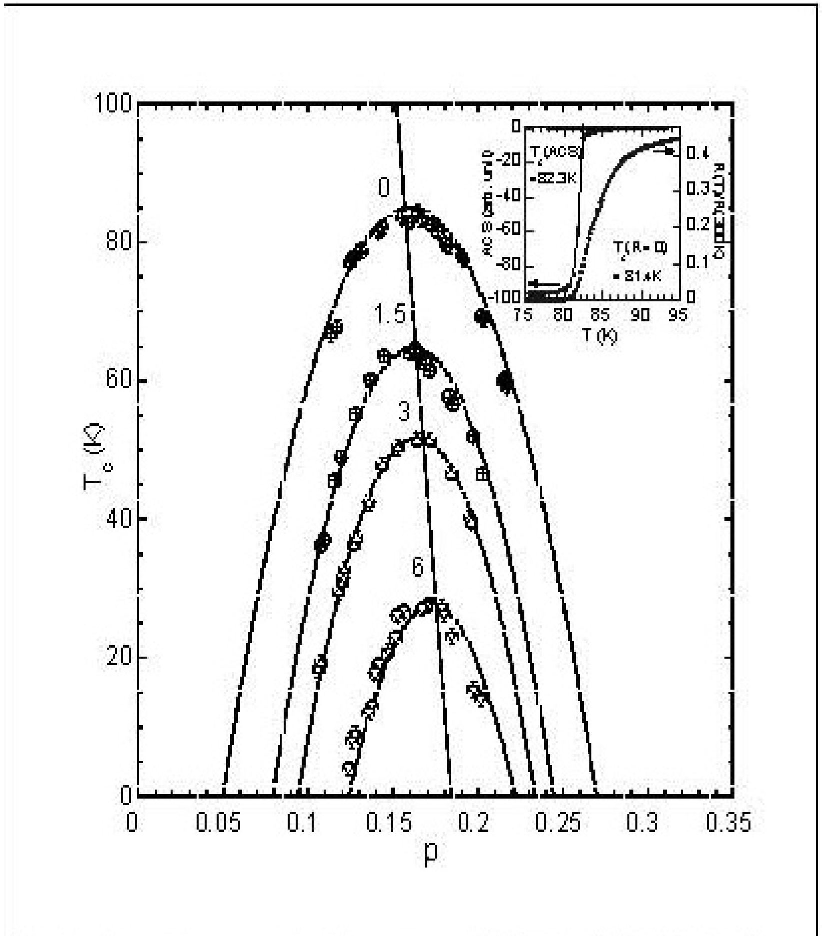
• They are several benchmarks of non-fermi liquid behavior: 2d-behavior, linear resistivity, linear electronic heat capacity, power scaling of the AC conductivity, etc.

A typical phase diagram

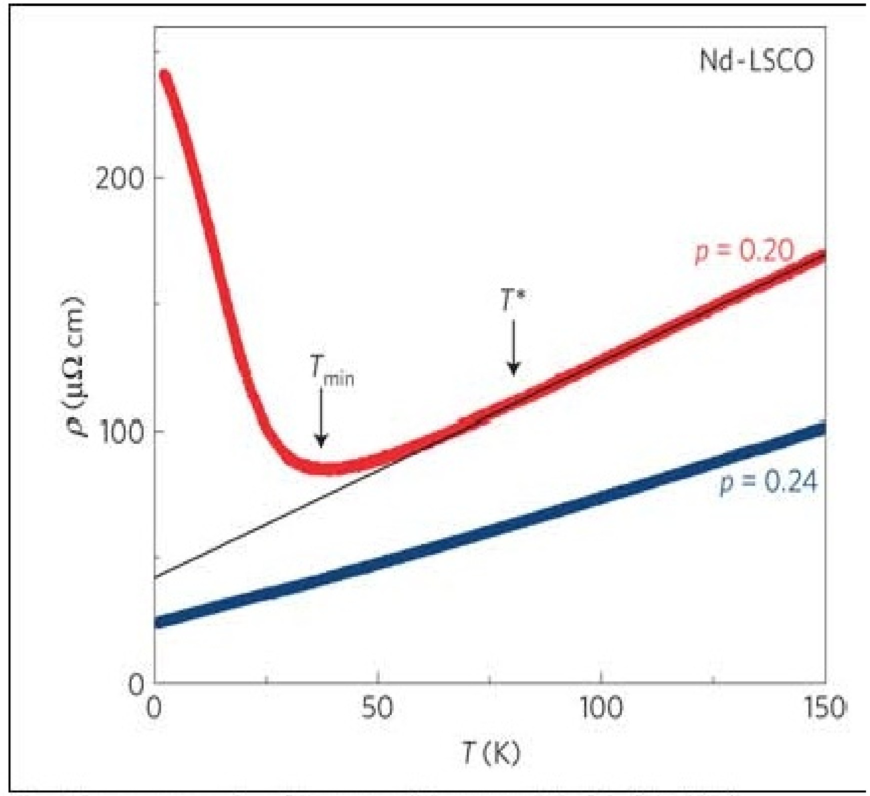


Phase diagram of hole-doped cuprates.In other systems the pseudogap region is much smaller, the superconducting region can shrink to almost nothing etc.

Linear Resistivity



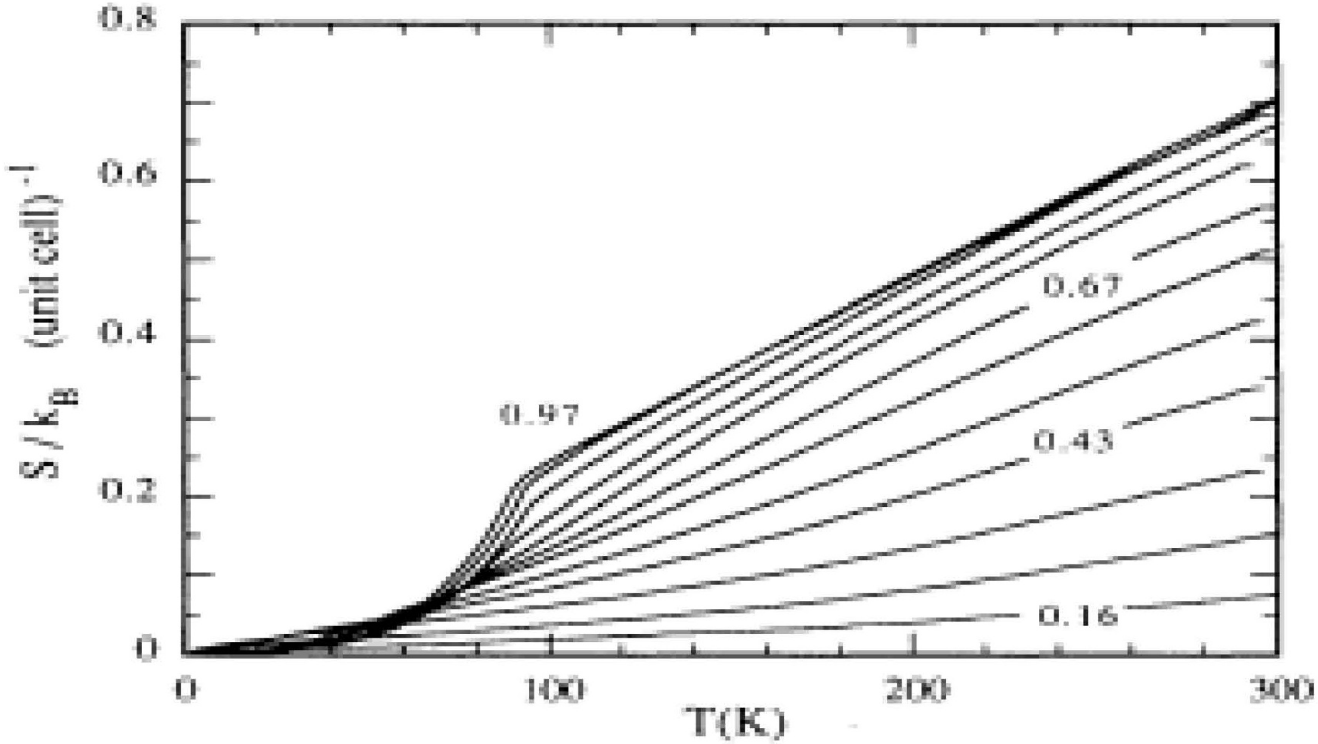
S. H. Naqib et. al., Physica C 387, 365 (2003)



R. Daou et. al., Nature Physics 5, 31 (2009)
& R. A. Cooper, et. al., Science 323, 603(2009)
Nicolas Doiron-Leyraud, et. al., arXiv:0905.0964

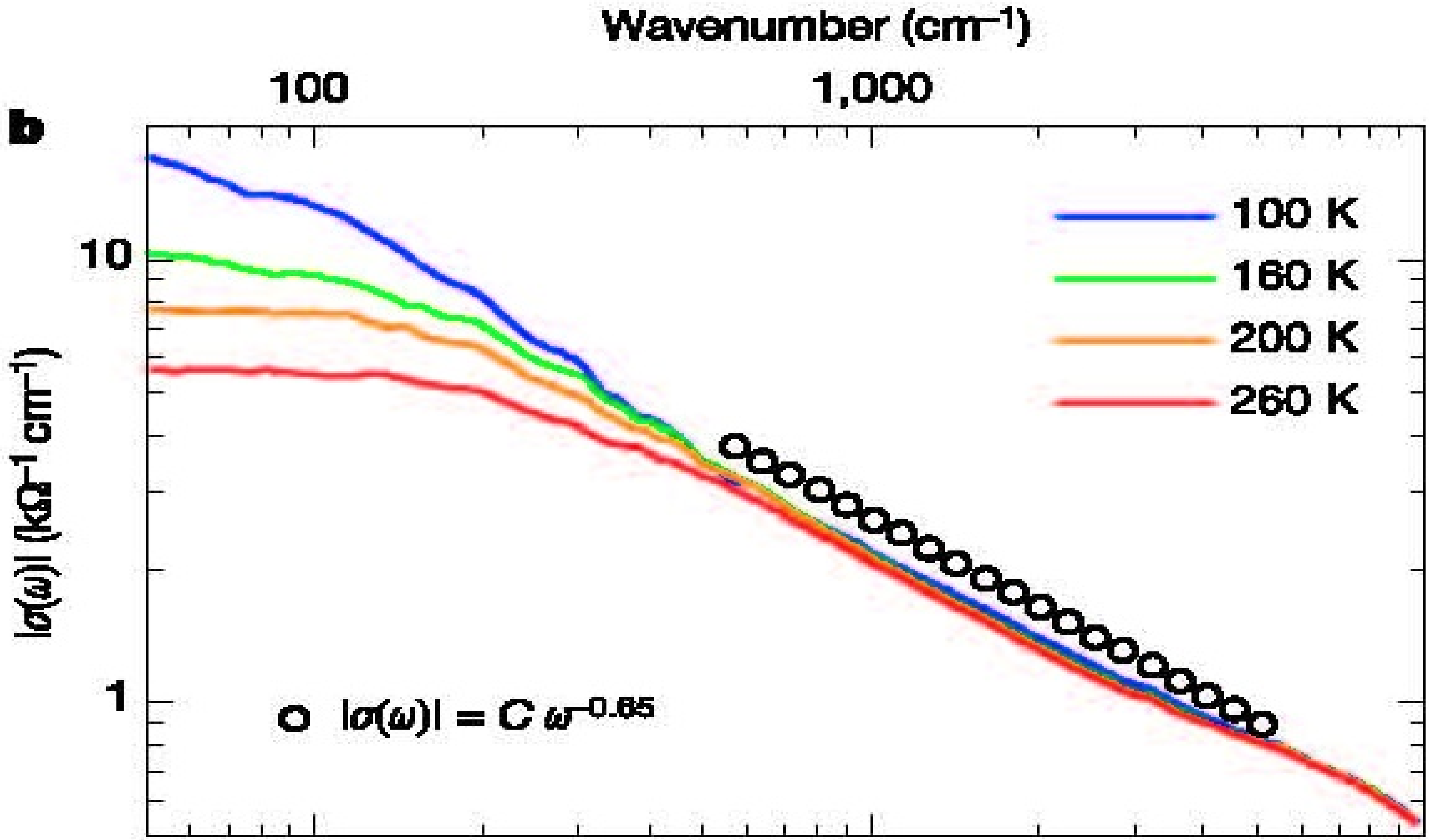
- Suppress superconducting dome with *Zn* substitution or large magnetic field
- Linear temperature dependence of resistivity around the critical point

Linear Heat Capacity



[Loram et. al. PRL 71, 11, 1993]

AC conductivity



van der Marel+Molegraaf+Zaanen+Nussinov+Carbone+Damascelli+Eisaki+Greven+Kes+Li, Nature 425 (2003) 271

Conductivity

- It is main characteristic transport coefficient in a finite density system.

$$J^i(\omega,\vec{k}) = \sigma^{ij}(\omega,\vec{k})\ E_j(\omega,\vec{k})$$

- Can be calculated from a Kubo formula

$$\sigma^{ij}(\omega,\vec{k}) = \frac{G_R^{ij}(\omega,\vec{k})}{i\omega} \quad , \quad G_R^{ij} \equiv \langle J^i\ J^j \rangle$$

- Various limits are of experimental importance

$$\vec{k} \to 0 \quad \to \quad \sigma^{ij}(\omega,T) \quad \to \quad \text{AC conductivity}$$

$$\omega \to 0 \quad \text{and} \quad \vec{k} \to 0 \quad \to \quad \sigma^{ij}(T) \quad \to \quad \text{DC conductivity}$$

- The limits $\omega \to 0$ and $\vec{k} \to 0$ do not commute.

Romatchke+Son (2009)

- We can use the drag calculation to calculate the DC conductivity for massive carriers

$$\rho = \frac{T_f}{J^t}\ g^E_{xx}(r_h)\ e^{k\phi(r_h)}$$

AC Conductivity: derivation

To compute the frequency depended current correlator we perturb we start with a general diagonal metric ansatz

$$ds^2 = -D(r)dt^2 + B(r)dr^2 + C(r)(dx_i dx^i) \quad , \quad A_t' = q\frac{\sqrt{D(r)B(r)}}{Z(\phi)C(r)^{\frac{p-1}{2}}}$$

In the backreacted case we must turn on perturbations

$$A_i = a_i(r)e^{i(\omega t)} \quad , \quad g_{ti}(r,t) = z_i(r)e^{i\omega t}$$

From the r, x_i Einstein equation we obtain

$$z_i' - \frac{C'}{C}z_i = -ZA_t'\ a_i$$

while from the gauge field equations

$$\partial_r\left(ZC^{\frac{p-3}{2}}\sqrt{\frac{D}{B}}a_i'\right) + ZC^{\frac{p-3}{2}}\sqrt{\frac{B}{D}}\omega^2\ a_i = \frac{q}{C}\left(z_i' - \frac{C'}{C}z_i\right)$$

Substituting we obtain

$$\partial_r\left(ZC^{\frac{p-3}{2}}\sqrt{\frac{D}{B}}a_i'\right) + ZC^{\frac{p-3}{2}}\left(\sqrt{\frac{B}{D}}\omega^2 - \frac{q^2\sqrt{DB}}{ZC^{p-1}}\right)a_i = 0$$

We can map to a Schrödinger problem

$$\frac{dz}{dr} = \sqrt{\frac{B}{D}} \quad , \quad a_i = \frac{\Psi}{\sqrt{\bar{Z}}} \quad , \quad \bar{Z} = ZC^{\frac{p-3}{3}}$$

$$-\frac{d^2\Psi}{dz^2}+V_{eff}\Psi=\omega^2\Psi \quad , \quad V_{eff}=\frac{q^2 D}{ZC^{p-1}}+\frac{1}{4}\left(\frac{\partial_z\bar{Z}}{\bar{Z}}\right)^2+\frac{1}{2}\partial_z\frac{\partial_z\bar{Z}}{\bar{Z}}$$

Near an AdS boundary the potential asymptotes to

$$V_{eff}\simeq\frac{(p-1)(p-3)}{4z^2}+\frac{q^2}{Z_b}\left(\frac{z}{\ell}\right)^{2(p-2)}+\cdots$$

When $p=3$ the leading behavior is given by

$$V_{\perp,p=3}=-\frac{k}{2}\Delta(2\Delta-1)\;r^{2\Delta-2}+\cdots$$

The frequency dependent conductivity is given by

$$\sigma(\omega)=\frac{1-\mathcal{R}}{1+\mathcal{R}}-\frac{i}{2\omega}\frac{\dot{Z}}{Z}\Big|_{\text{boundary}}$$

Roberts+Horowitz (2009), Goldstein+Kachru+Prakash+Trivedi (2009)

At extremality, near the singularity at $r=r_0$, $D=c_D(r-r_0)^2$, $B=c_B/(r-r_0)^2$ and

$$V\simeq\frac{\nu^2-\frac{1}{4}}{z^2}+\cdots \quad , \quad \nu^2-\frac{1}{4}=\frac{q^2 c_B}{Z_0 C_0^{p-1}}$$

Calculation of the reflection coefficient then gives

$$\sigma\sim\omega^{2\nu-1}$$

Goldstein+Kachru+Prakash+Trivedi (2009)

Drag calculation of DC conductivity

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Gubser (2005), Karch+O'Bannon (2007)

$$S_{NG} = T_f \int d^2\xi \sqrt{\widehat{g}} + \int d\tau A_\mu \dot{x}^\mu \,, \qquad \widehat{g}_{\alpha\beta} = g_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \,,$$

In a direction with translation invariance we have the following world-sheet Poincaré conserved currents

$$\pi^\alpha_\mu = \bar{\pi}^\alpha_\mu + A_\mu \eta^{\alpha\tau} = T_f \sqrt{\widehat{g}}\widehat{g}^{\beta\alpha} g_{\nu\mu}\partial_\beta x^\nu + A_\mu \eta^{\alpha\tau} \,,$$

The bulk and boundary equations are

$$\partial_\alpha \bar{\pi}^\alpha_\mu = 0 \quad , \quad T_f \sqrt{\widehat{g}}\widehat{g}^{\sigma\beta} g_{\mu\nu}\partial_\beta x^\nu + qF_{\mu\nu}\dot{x}^\nu = 0 \,.$$

We now consider a space-time metric in a generic coordinate system and a bulk gauge field

$$ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^i dx^i \quad , \quad A_{x^1} = -Et + h(r) \quad , \quad A_t(r)$$

We choose a static gauge with $\sigma = r$ and $\tau = t$ and make the ansatz

$$x^1 = X = vt + \xi(r) \,,$$

which is motivated by the expectation that the motion of the string will make it have a profile that is dragging on one side as it lowers inside the bulk space.
The boundary equation for $\mu = t$ and $\mu = x$ are equivalent and become

$$T_f \frac{\widehat{g}_{\sigma\tau}}{\sqrt{-\widehat{g}}} g_{tt} + Ev = 0 \quad \to \quad \bar{\pi}_x = E \,.$$

Solving we obtain

$$\xi' = \sqrt{\frac{g_{rr}}{g_{tt}g_{xx}}}\frac{\sqrt{g_{tt}-g_{xx}v^2}}{\sqrt{T_f^2 g_{tt}g_{xx}-\bar{\pi}_x^2}}\bar{\pi}_x\,.$$

To ensure we have a real solution, there must be a turning point at $r = r_s$

$$v^2 = \frac{g_{tt}(r_s)}{g_{xx}(r_s)}\,,\qquad \bar{\pi}_x = -T_f\sqrt{g_{tt}(r_s)g_{xx}(r_s)}$$

Finally as v is constant we obtain

$$T_f\sqrt{g_{tt}(r_s)g_{xx}(r_s)} = -E \quad , \quad \frac{dp}{dt} = -\bar{\pi}_x + qE\,,$$

and the steady state solution is $\bar{\pi}_x = E$. For small velocities we obtain

$$\bar{\pi}_x \simeq -T_f g_{xx}(r_h)\ v + \mathcal{O}(v^2) \quad , \quad J^x = J^t\ v \simeq \frac{J^t\bar{\pi}_x}{T_f g_{xx}(r_h)} \simeq \frac{J^t}{T_f g_{xx}(r_h)}E\,,$$

and we obtain the DC conductivity and related resistivity as

$$\sigma \simeq \frac{J^t}{T_f g_{xx}(r_h)}\,,\qquad \rho \simeq \frac{T_f g_{xx}(r_h)}{J^t} = \frac{T_f g^E_{xx}(r_h)e^{k\phi(r_h)}}{J^t}\,.$$

In the case that $k = 0$

$$\frac{\rho(T)}{S(T)^{\frac{2}{p-1}}} = \text{constant}\,.$$

Vacuum solutions in the Einstein-Dilaton theory

$$V(\lambda) \sim V_0 \lambda^{2Q} \quad , \quad \lambda \equiv e^{\phi} \to \infty$$

- The solutions can be parameterized in terms of a fake superpotential

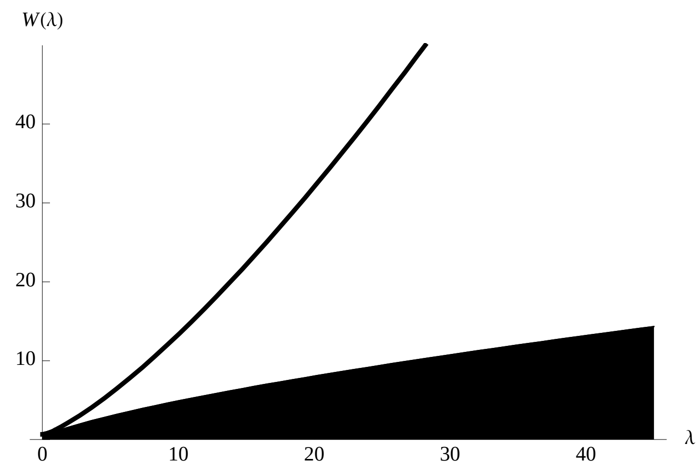
$$V = \frac{64}{27} W^2 - \frac{4}{3} \lambda^2 \, W'^2 \quad , \quad W \geq \frac{3}{8}\sqrt{3V}$$

The crucial parameter resides in the solution to the diff. equation above. There are three types of solutions for $W(\lambda)$:

Gursoy+E.K.+Mazzanti+Nitti

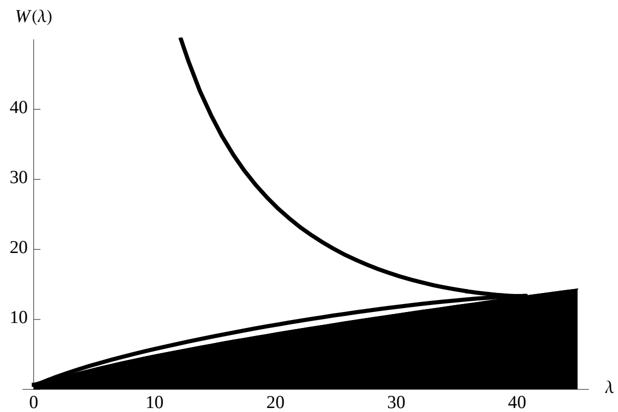
1. Generic Solutions (bad IR singularity)

$$W(\lambda) \sim \lambda^{\frac{4}{3}} \quad , \quad \lambda \to \infty$$



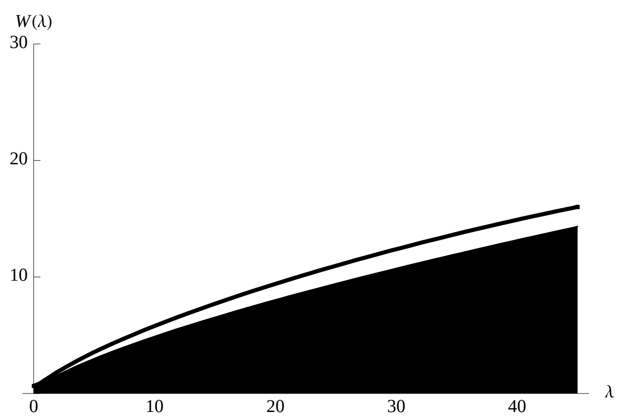
2. Bouncing Solutions (bad IR singularity)

$$W(\lambda) \sim \lambda^{-\frac{4}{3}} \quad , \quad \lambda \to \infty$$



3. The "special" solution.

$$W(\lambda) \sim W_\infty \lambda^{Q} \quad , \quad \lambda \to \infty \quad , \quad W_\infty = \sqrt{\frac{27V_0}{4(16 - 9Q^2)}}$$



Good+repulsive IR singularity if $Q < \frac{4\sqrt{2}}{3}$

• For $Q > \frac{4}{3}$ all solutions are of the bouncing type (therefore bad).

• There is another special asymptotics in the potential namely $Q = \frac{2}{3}$. Below $Q = \frac{2}{3}$ the spectrum changes to continuous without mass gap.

In that region a finer parametrization of asymptotics is necessary

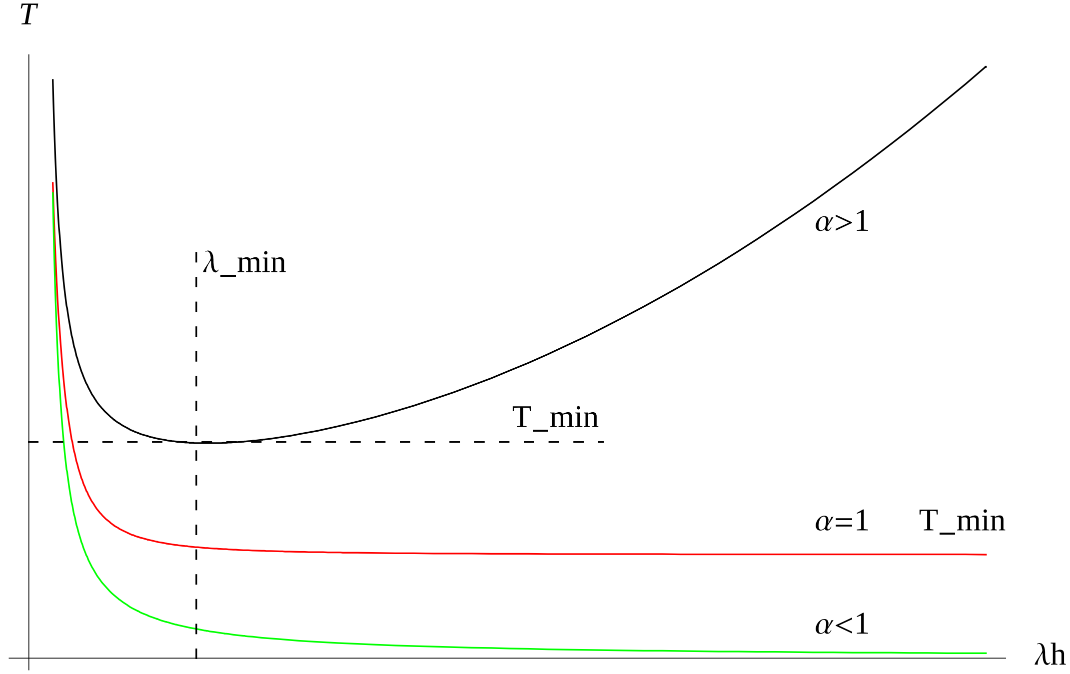
$$V(\lambda) \sim V_0 \; \lambda^{\frac{4}{3}} \; (\log \lambda)^P$$

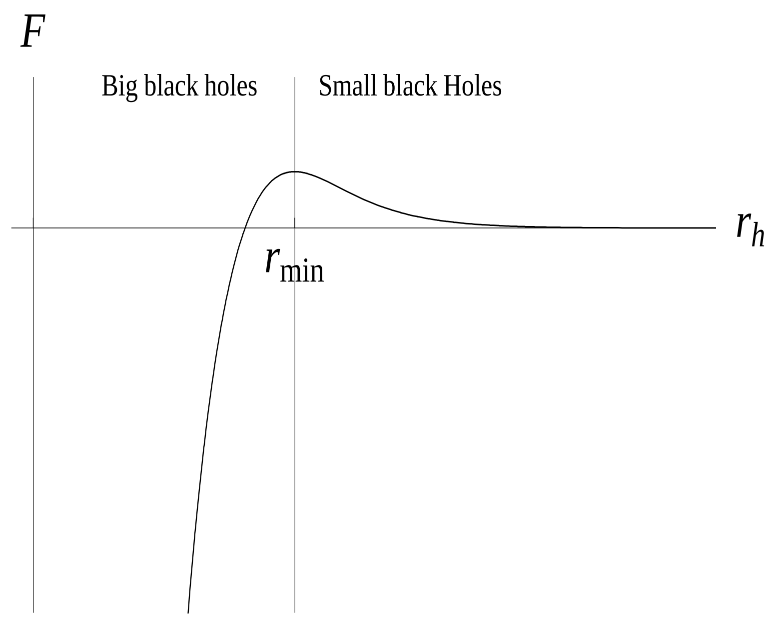
• For $P > 0$ there is a mass gap, discrete spectrum and confinement of charges. There is also a first order deconfining phase transition at finite temperature.

• For $P < 0$, the spectrum is continuous, without mas gap, and there is a transition at T=0 (as in N=4 sYM).

• At $P = 0$ we have the linear dilaton vacuum. The theory has a mass gap but continuous spectrum. The order of the deconfining transition depends on the subleading terms of the potential and can be of any order larger than two.

Gurdogan+Gursoy+E.K.


T
λ_min
α>1
T_min
α=1 T_min
α<1
λh


F
Big black holes
Small black Holes
r_h
r_{min}

Classification of zero temperature solutions

For any positive+monotonic potential $V(\lambda)$, $\lambda \equiv e^{\phi}$ with the asymptotics :

$$V(\lambda) = V_0 + V_1\lambda + V_2\lambda^2 + \ldots \quad V_0 > 0, \qquad \lambda \to 0$$

$$V(\lambda) = V_\infty \lambda^{2Q} (\log\lambda)^P, \quad V_\infty > 0, \qquad \lambda \to \infty$$

the zero-temperature superpotential equation has three types of solutions, that we name the *Generic*, the *Special*, and the *Bouncing* types:

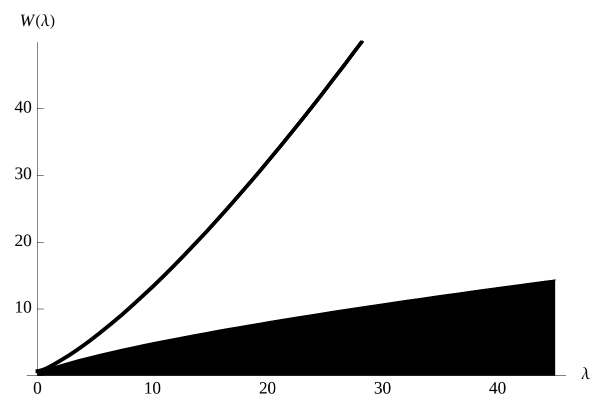
• A continuous one-parameter family that has a fixed power-law expansion near $\lambda = 0$, and reaches the asymptotic large-λ region where it grows as

$$W \simeq C_b\ \lambda^{4/3} \qquad \lambda \to \infty \quad , \quad C_b > 0$$

These solutions lead to backgrounds with “bad” (i.e. non-screened) singularities at finite r_0,

$$b(r) \sim (r_0 - r)^{1/3}, \qquad \lambda(r) \sim (r_0 - r)^{-1/2}$$

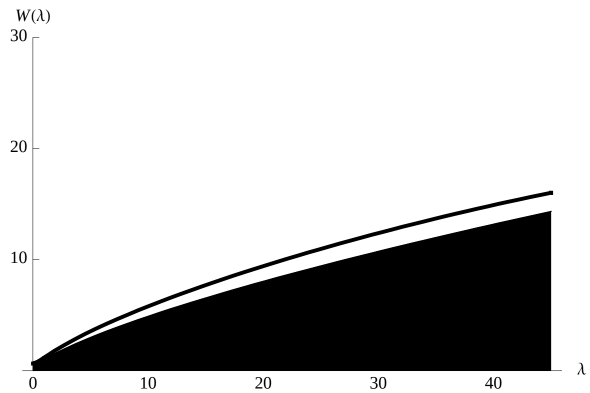
We call this solution *generic*.



• A unique solution, which also reaches the large-λ region, but slower:

$$W(\lambda) \sim W_\infty \lambda^Q (\log \lambda)^{P/2}, \qquad W_\infty = \sqrt{\frac{27 V_\infty}{4(16 - 9Q^2)}}$$

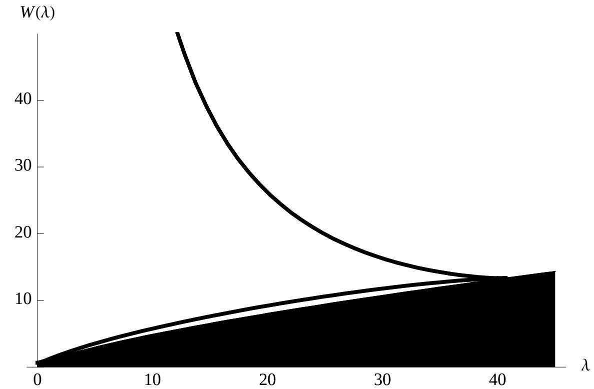
This leads to a repulsive singularity, provided $Q < 2\sqrt{2}/3$ [**?**]. We call this the *special* solution.



• A second continuous one-parameter family where $W(\lambda)$ does not reach the asymptotic region. These solutions have two branches that both reach $\lambda = 0$ (one in the UV, the other in the IR) and merge at a point λ_* where $W(\lambda_*) = \sqrt{27V(\lambda_*)/64}$. The IR branch is again a "bad" singularity at a finite value r_0, where $W \sim \lambda^{-4/3}$, and

$$b(r) \sim (r_0 - r)^{1/3}, \qquad \lambda(r) \sim (r_0 - r)^{1/2}.$$

We call this solution *bouncing*.



The special solution marks the boundary between the generic solutions, that reach the asymptotic large-λ region as $\lambda^{4/3}$ and the bouncing ones, that don't reach it.

If $Q > 4/3$, only bouncing solutions exist.

In all types of solutions the UV corresponds to the region $\lambda \to 0$ on the W_+ branch. There the behavior of W_+ is universal: a power series in λ with *fixed* coefficients, plus a subleading non-analytic piece which depends on an arbitrary integration constant C_w:

$$W = \sum_{i=1}^{\infty} W_i \lambda^i + C_w \lambda^{16/9} e^{-\frac{16W_0}{9W_1}\frac{1}{\lambda}} [1 + O(\lambda)]$$

All the power series coefficients W_i are completely determined by the coefficients in the small λ expansion of $V(\lambda)$, the first few being:

$$W_0 = \frac{\sqrt{27V_0}}{8}, \quad W_1 = \frac{V_1}{16}\sqrt{\frac{27}{V0}}, \quad W_2 = \frac{\sqrt{27}(64V_0V_2 - 7V1^2)}{1024V_0^{3/2}}$$

RETURN

The $\gamma\delta = 1$ solutions

$$ds^2 = -\frac{V(r)dt^2}{\left[1-\left(\frac{r_-}{r}\right)^{3-\delta^2}\right]^{\frac{4(1-\delta^2)}{(3-\delta^2)(1+\delta^2)}}} + e^{\delta\phi}\frac{dr^2}{V(r)} + r^2\left[1-\left(\frac{r_-}{r}\right)^{3-\delta^2}\right]^{\frac{2(\delta^2-1)^2}{(3-\delta^2)(1+\delta^2)}}\left(dx^2+dy^2\right),$$

$$V(r) = \left(\frac{r}{\ell}\right)^2 - 2\frac{m\ell^{-\delta^2}}{r^{1-\delta^2}} + \frac{(1+\delta^2)q^2\ell^{2-2\delta^2}}{4\delta^2(3-\delta^2)^2 r^{4-2\delta^2}} \quad , \quad (r_\pm)^{3-\delta^2} = \ell^{2-\delta^2}\left[m \pm \sqrt{m^2 - \frac{(1+\delta^2)q^2}{4\delta^2(3-\delta^2)^2}}\right]$$

$$e^\phi = \left(\frac{r}{\ell}\right)^{2\delta}\left[1-\left(\frac{r_-}{r}\right)^{3-\delta^2}\right]^{\frac{4\delta(\delta^2-1)}{(3-\delta^2)(1+\delta^2)}} \quad , \quad \mathcal{A} = \left(\Phi - \frac{q\ell^{2-\delta^2}}{(3-\delta^2)r^{3-\delta^2}}\right)dt \quad , \quad \Phi = \frac{q\ell^{2-\delta^2}}{(3-\delta^2)r_+^{3-\delta^2}}$$

where the parameters m and q are integration constants linked to the gravitational mass and the electric charge. There is an overall scale ℓ

$$\ell^2 = \frac{\delta^2-3}{\Lambda}.$$

RETURN

The $\gamma = \delta$ solutions

$$ds^2 = -V(r)dt^2 + e^{\delta\phi}\frac{dr^2}{V(r)} + r^2\left(dx^2 + dy^2\right)$$

$$V(r) = \left(\frac{r}{\ell}\right)^2 - 2m\ell^{-\delta^2}r^{\delta^2-1} + \frac{q^2}{4(1+\delta^2)r^2}$$

$$e^{\phi} = \left(\frac{r}{\ell}\right)^{2\delta} \quad , \quad \mathcal{A} = \left(\Phi - \frac{\ell^{\delta^2}q}{(1+\delta^2)r^{1+\delta^2}}\right)dt \quad , \quad \Phi = \frac{q\ell^{\delta^2}}{(1+\delta^2)r_+^{1+\delta^2}}$$

- There is a "BPS condition" for the existence of a horizon

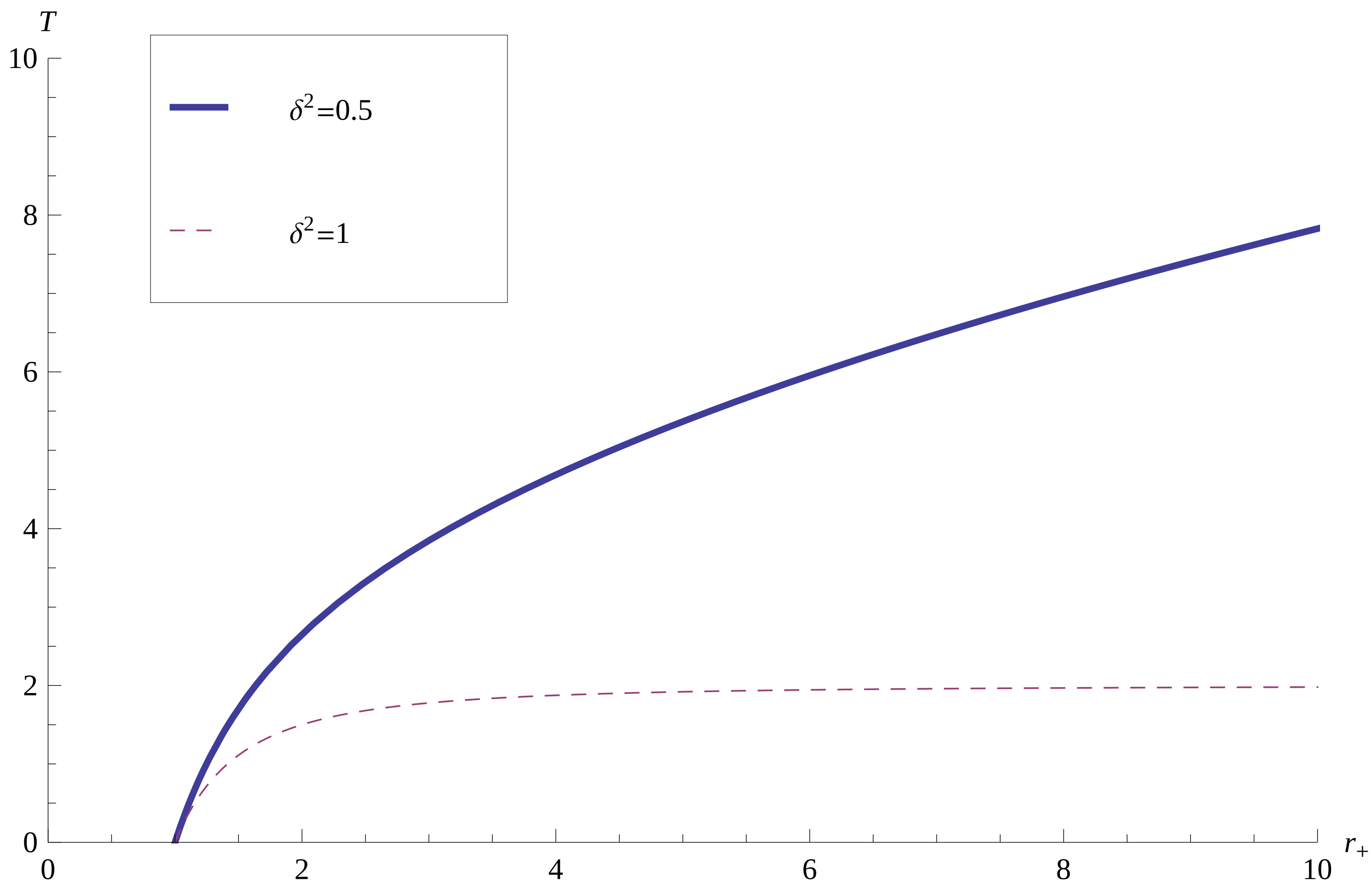
$$m \geq \frac{2q^{\frac{3-\delta^2}{2}}}{1+\delta^2}$$

- U(r) has two roots $0 < r^- < r^+$. The two coincide at the extremality limit, $(1+\delta^2)m = 2q^{\frac{3-\delta^2}{2}}$.

- There are two distinct regimes:

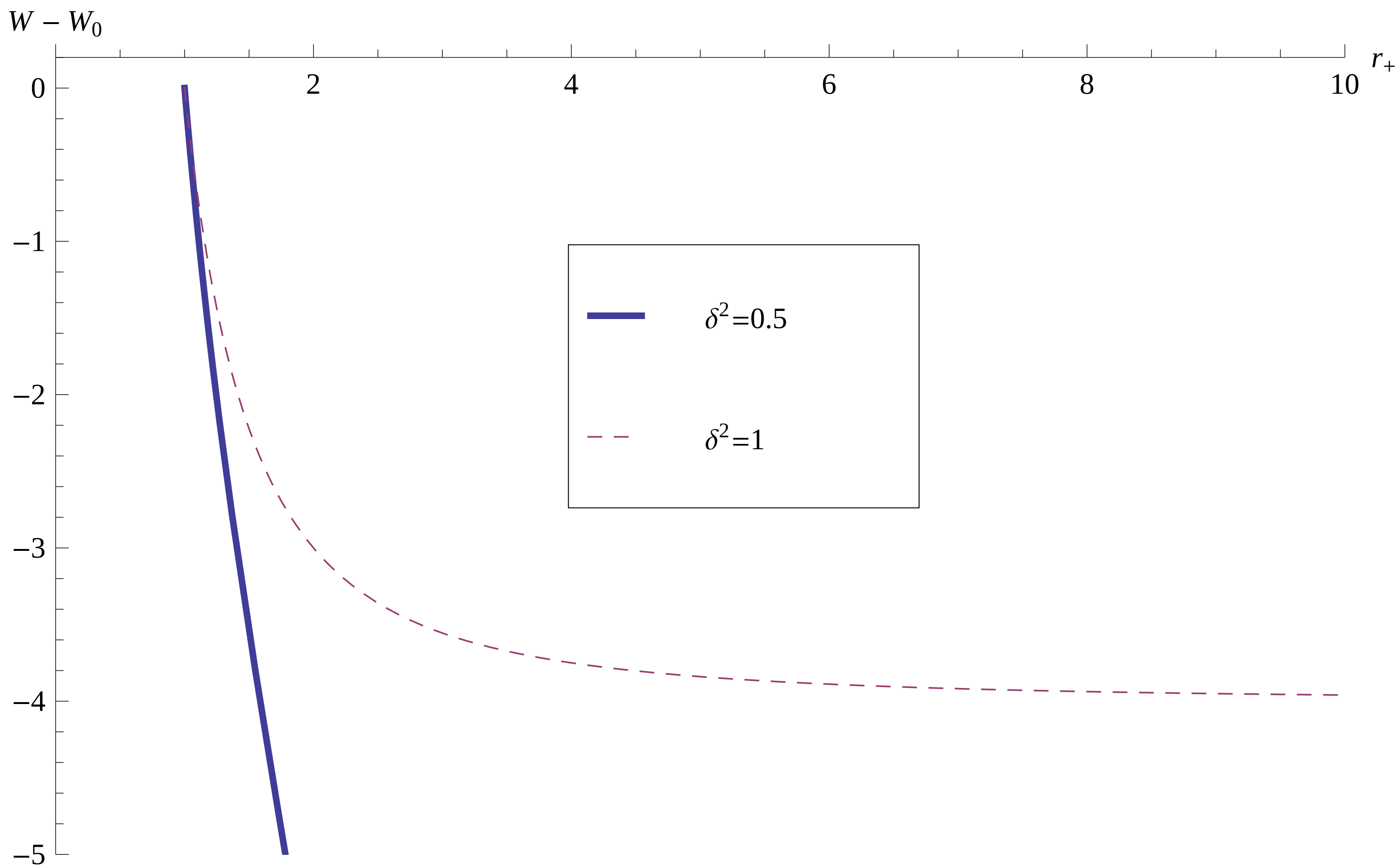
$$0 \leq \delta^2 \leq 1 \qquad\qquad 1 \leq \delta^2 \leq 3$$

- $0 \leq \delta^2 \leq 1$



- Temperature as a function of horizon position

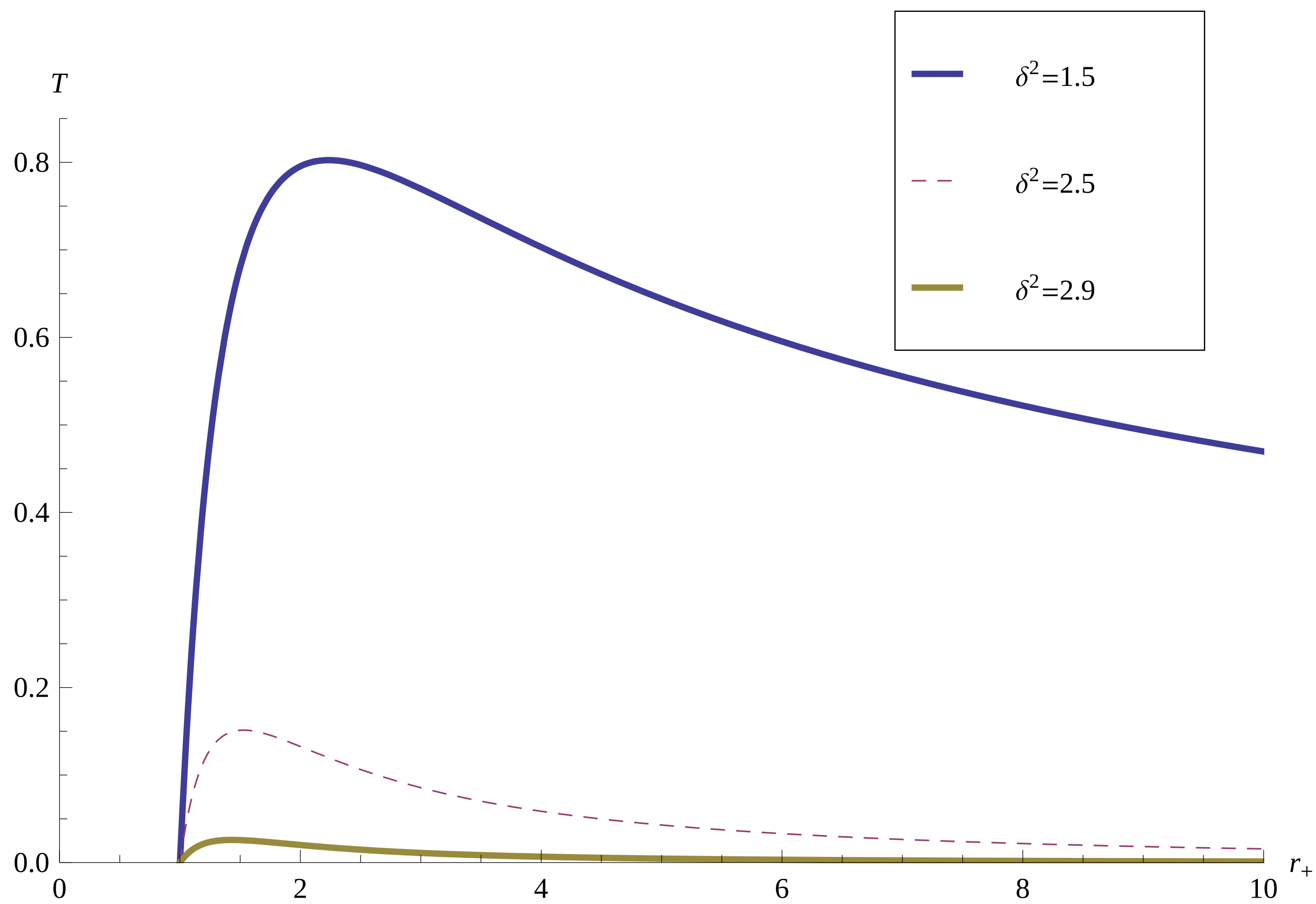
- $0 \leq \delta^2 \leq 1$



- Difference of free energies vs horizon position

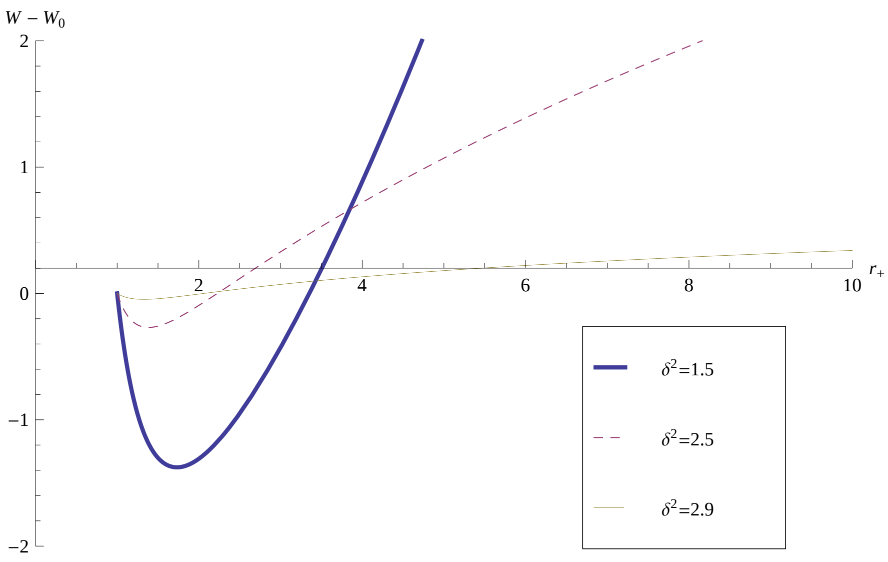
- The BH always dominates

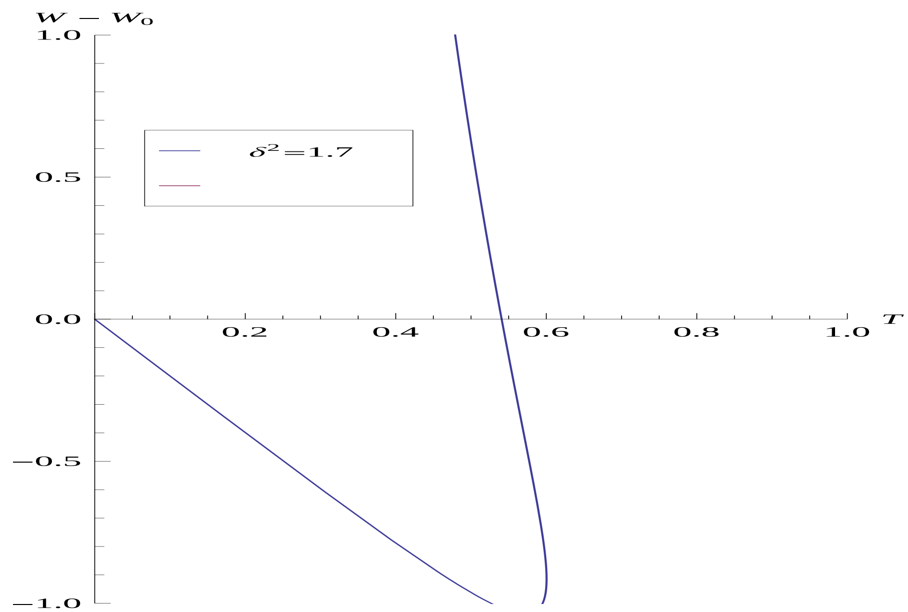
- $1 \leq \delta^2 \leq 3$



- Temperature vs horizon position

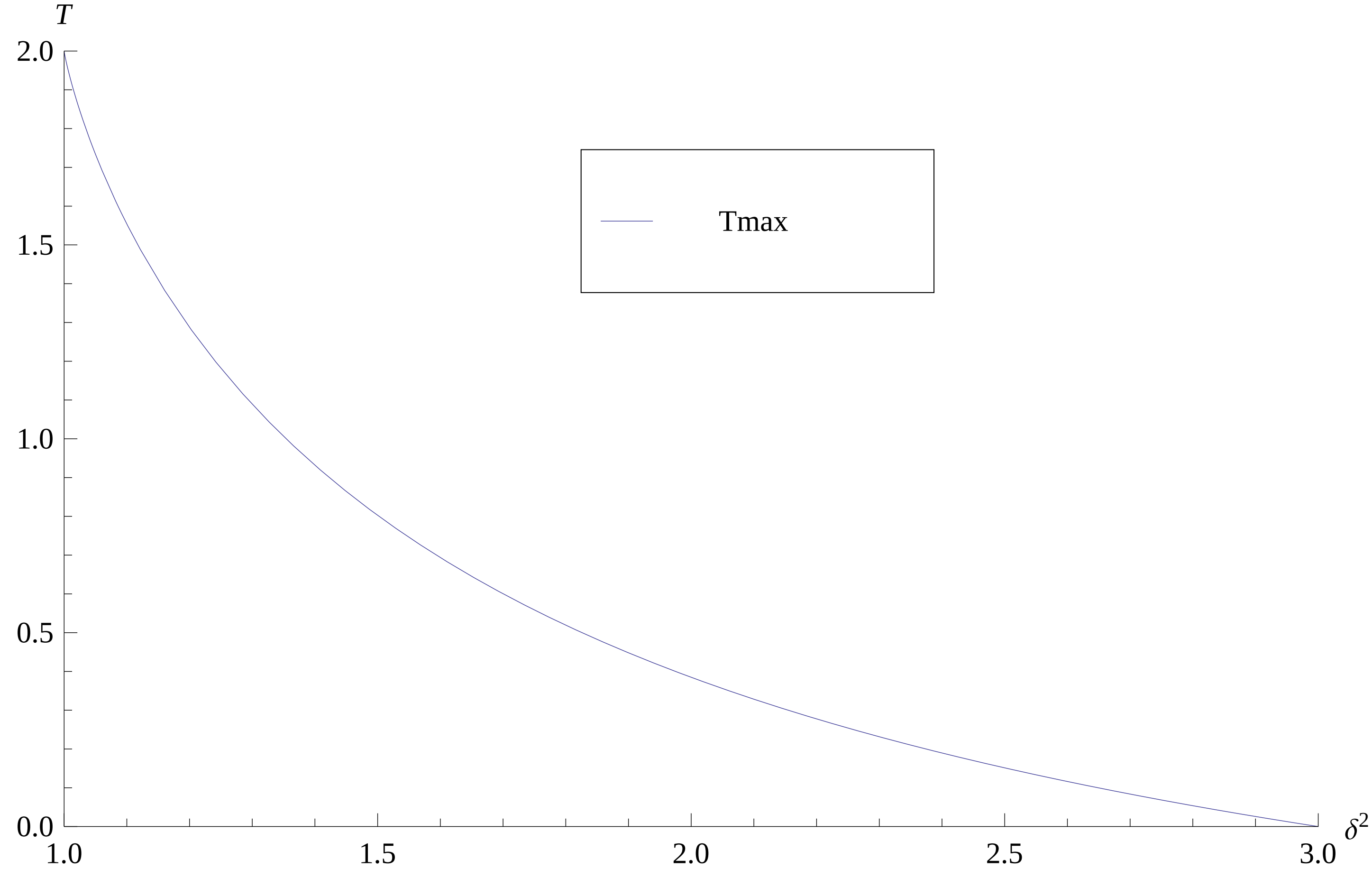
• $1 \leq \delta^2 \leq 3$





• Difference of free energies as a function of horizon position and temperature.

• The BH dominates at low temperatures up to the phase transition



- The maximum temperature as a function of δ^2.

Detailed plan of the presentation

- Title page 0 minutes
- Bibliography 1 minutes
- The plan 2 minutes
- Brief Summary of Results 4 minutes
- The strategy 6 minutes
- Effective Holographic Theories 9 minutes
- Einstein-scalar-U(1) theory 12 minutes
- Naked singularities 15 minutes

- Solutions at Zero Charge Density 25 minutes
- Charged near extremal solutions 32 minutes
- The extremal AC conductivity 35 minutes
- The near-extremal DC conductivity 37 minutes
- Exact Charged solutions 39 minutes
- Solutions with $\gamma\delta = 1$ 51 minutes
- Outlook 53 minutes

- Introduction 57 minutes
- A typical Phase diagram 59 minutes
- Linear Resistivity 60 minutes
- Linear Heat Capacity 61 minutes
- AC conductivity 62 minutes
- Conductivity 63 minutes
- AC Conductivity: Derivation 67 minutes
- Drag calculation of DC conductivity 71 minutes
- Vacuum solutions in the Einstein-Dilaton theory 75 minutes
- Classification of zero temperature solutions 79 minutes
- The $\gamma\delta = 1$ solutions 83 minutes
- Charged solutions $\gamma = \delta$ 90 minutes