

Glueballs in large- N YM by localization on critical points

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By exploiting in large- N YM the change of variables from the gauge connection to the ASD part of its curvature by a non- $SUSY$ version of the Nicolai map, we show that certain twistor Wilson loops supported on a Lagrangian submanifold of twistor space are localized on lattices of surface operators of Z_N holonomy that form translational invariant sectors labelled by the magnetic charge $k = 1, 2, \dots, N - 1$ at a point. The localization is obtained reducing the loop equation in the ASD variables in the holomorphic gauge, regularized by analytic continuation to Minkowski space-time, to a critical equation, by exploiting the invariance of the v.e.v. of twistor Wilson loops by deformations for the addition of backtracking arcs ending with cusps on the singular divisor of surface operators. Alternatively localization is obtained contracting the YM measure in the ASD variables on the fixed points of a semigroup that acts on the fiber of the Lagrangian twistor fibration which twistor Wilson loops are supported on and leaves invariant their v.e.v.. The renormalized effective action induced by the localized YM measure in the ASD variables scales according to a large- N beta function of $NSVZ$ type that reproduces the first two universal perturbative coefficients. Because of a non-trivial Jacobian due to the lack of supersymmetry a multiplicative renormalization by a Z factor of the ASD field occurs. The masses squared of the fluctuations of surface operators in the sectors labelled by k , supported on the Lagrangian submanifold analytically continued to Minkowski space-time, form a trajectory linear in k that does not include any massless state. The glueballs propagators in the holomorphic/antiholomorphic sector defined by correlations of a complex combination of the ASD curvature and its adjoint saturate at short distances the logarithms of perturbation theory by a sum of pure poles. The anomalous dimensions of long gauge invariant operators belonging to the holomorphic/antiholomorphic sector that are implied by the Z factor coincide with the anomalous dimensions of the scalar operators that occur as the antiferromagnetic ground state of the Hamiltonian spin chain in the thermodynamic limit, that it is known to provide the anomalous dimensions in the ASD one-loop integrable sector of large- N YM . In this framework Regge trajectories of higher spins are related to fluctuations of surface operators with pole singularities of any order.

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1. Introduction

A technical presentation of the main ideas and results of this paper is in the synopsis of the next section and in a very sketchy way in the list of contents.

This section is more of introductory nature, in order to convey in a simpler way part of the meaning of the technical ideas.

The problem of the YM mass gap as reported in [1] has an infrared and an ultraviolet nature at the same time.

Indeed the renormalization group (RG) requires that every mass scale of the YM theory must depend on the canonical coupling constant, g_{YM} , only through the RG invariant scale, Λ_{YM} :

$$\Lambda_{YM} = \Lambda \exp\left(-\frac{1}{2\beta_0 g_{YM}^2}\right) (\beta_0 g_{YM}^2)^{-\frac{\beta_1}{2\beta_0^2}} (1 + \dots) \quad (1.1)$$

whose dependence on the coupling constant is equivalent to the knowledge of the exact beta function of the theory in some scheme. This in turn implies that an amazing asymptotic accuracy, as g_{YM} vanishes¹ when the cutoff, Λ , diverges, is needed to solve the mass gap problem and that the mass gap is zero to every order of perturbation theory.

One possibility is that such finest asymptotic accuracy may be achieved only by an exact solution both on the ultraviolet side, for the beta function, and on the infrared side, for the mass gap.

While an exact solution of the YM theory, to use just an euphemism, seems completely outside the reach of the present techniques, in this paper we propose an exact solution in the large- N limit of the $SU(N)$ YM theory for the beta function and for the glueballs spectrum restricted to a special sector of the theory.

From a purely computational point of view we may present our results in terms of the glueballs propagators in a certain sector of the large- N YM theory. The glueballs propagators that we refer to are initially defined in Euclidean signature, constructed by means of fluctuations, $\delta\mu$ ², of certain surface operators³ supported on a Lagrangian submanifold of four-dimensional space-time⁴:

$$\Lambda_W^6 \int \langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}}(\mu\bar{\mu})(z, \bar{z}, z, \bar{z}) \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}}(\mu\bar{\mu})(0, 0, 0, 0) \rangle_{conn} e^{i(p_z \bar{z} + p_{\bar{z}} z)} d^2 z \quad (1.2)$$

where $(z = x_0 + ix_1, \bar{z} = x_0 - ix_1, u = x_2 + ix_3, \bar{u} = x_2 - ix_3)$ are complex coordinates in Euclidean $R^2 \times R^2$, $d^2 z = dx_0 dx_1$ and

$$\mu = \frac{1}{2}(\mu_{01}^- - i\mu_{03}^-) \quad (1.3)$$

¹Therefore the mass gap problem is not a strong coupling problem.

²The precise definition is in Eq.(12.39).

³The idea of integrating in YM theory on local systems associated to an arbitrary parabolic divisor appeared for the first time long ago in some papers by us [2, 3], by embedding the Hitchin fibration [4, 5, 6, 7] in the YM functional integral, and physically corresponds to integrating over surface operators ante litteram [8]. Explaining how surface operators arise in large- N YM is in fact the subject of this paper.

⁴The plane (z, \bar{z}, z, \bar{z}) is Lagrangian for the symplectic form $dz \wedge d\bar{z} - du \wedge d\bar{u}$ in four dimensions.

with $\bar{\mu}$ the Hermitian conjugate. The following identification holds up to (infinite) factors ⁵:

$$\int \mu(z, \bar{z}, z, \bar{z}) e^{i(p_z \bar{z} + p_{\bar{z}} z)} d^2 z \sim \frac{1}{2} (F_{01}^- - iF_{03}^-)(p_z, p_{\bar{z}}, p_z, p_{\bar{z}}) \quad (1.4)$$

where

$$(F_{01}^- - iF_{03}^-)(p_z, p_{\bar{z}}, p_u, p_{\bar{u}}) = \int (F_{01}^- - iF_{03}^-)(z, \bar{z}, u, \bar{u}) e^{i(p_z \bar{z} + p_{\bar{z}} z + p_u \bar{u} + p_{\bar{u}} u)} d^2 z d^2 u \quad (1.5)$$

is the Fourier transform with

$$\begin{aligned} F_{\alpha\beta}^- &= F_{\alpha\beta} - \tilde{F}_{\alpha\beta} \\ \tilde{F}_{\alpha\beta} &= \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta} \end{aligned} \quad (1.6)$$

the anti-selfdual (ASD) part of the curvature of the gauge connection. The identification holds because the fluctuating field of surface operators, $\delta\mu$, has nonvanishing momenta dual in the Fourier sense to the Lagrangian support of surface operators and it occurs in the effective action with zero momenta dual in the Fourier sense to the manifold normal to the support. $\mathcal{N} = N\hat{N}$ is the total rank of the gauge group. The first factor of N is the rank of an $SU(N)$ gauge bundle that is embedded by non-commutative Morita equivalence into $U(N \times \hat{N})$. The construction is explained in sect.(7) and sect.(12). After analytically continuing to Minkowski space-time in its simplest form of our result reads ⁶:

$$\begin{aligned} &\Lambda_W^6 \int \langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}}(\mu\bar{\mu})(x_+, x_-, x_+, x_-) \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}}(\mu\bar{\mu})(0, 0, 0, 0) \rangle_{\text{conn}} e^{i(p_+ x_- + p_- x_+)} dx_+ dx_- \\ &\sim \frac{1}{\mathcal{N}^2} \sum_{k=1}^{\infty} \frac{k \rho_k^{-2} \Lambda_W^6}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ &\sim \frac{1}{\mathcal{N}^2} \sum_{k=1}^{\infty} \frac{k^2 \Lambda_W^6}{-\alpha' p_+ p_- + k \Lambda_W^2} \end{aligned} \quad (1.7)$$

where, $(x_+ = x_4 + x_1, x_- = x_4 - x_1)$ are light-cone coordinates, $(p_+ = p_4 + p_-, p_- = p_4 - p_1)$ are light-cone momenta, and Λ_W is the renormalization group invariant scale in the Wilsonian scheme.

ρ_k is the density, in units of Λ_W^2 ,

$$\rho = \sum_p \delta^{(2)}(z - z_p) \quad (1.8)$$

of surface operators carrying at each lattice point, p , magnetic charge k and holonomy valued in the center, Z_N , of the gauge group, i.e. such that:

$$e^{2i\mu_p} = e^{i\frac{2\pi k}{N}} \quad (1.9)$$

⁵The identification extends to composite operators in a certain asymptotic sense, see below and sect.(12). The infinite factors are actually defined and regularized in sect.(12).

⁶The symbol \sim stays for "equal up to constant irrelevant numerical factors" or "equal up to irrelevant additive terms" depending on the framework.

with:

$$\mu \sim \frac{1}{2}(F_{01}^- - iF_{03}^-) = \sum_p \mu_p \delta^{(2)}(z - z_p(u, \bar{u})) \quad (1.10)$$

and $z_p(u, \bar{u}) = z_p$. ρ_k scales with k as:

$$\rho_k^2 \sim \frac{1}{k} \quad (1.11)$$

and the dimensionless inverse "string tension" in units of Λ_W^{-2} is:

$$\alpha' = \frac{10}{3\pi} \quad (1.12)$$

The peculiar support, (x_+, x_-, x_+, x_-) , of the correlator arises as the projection with Minkowski signature on the base of a Lagrangian submanifold of the twistor space of (complexified) Euclidean space-time that occurs in our approach. The field μ is dimensionless and normalized in such a way that the correlator in Eq.(1.7) be renormalization group invariant⁷. The mass spectrum, $-\alpha' p_+ p_- + k\Lambda_W^2 = 0$, and the multiplicity, k , that occurs in the numerator of Eq.(1.7) in the second line, are in fact exact⁸ in a certain asymptotic expansion in powers of $N^{-\frac{1}{2}}$ of the gauge connection of the surface operators that occur in our approach⁹.

At large N there is a Wilsonian scheme in which the Wilsonian beta function is one-loop exact [10] and a canonical scheme in which the beta function has a Novikov-Shifman-Vainstein-Zacharov form (NSVZ) that reproduces the first two universal perturbative coefficients [10]¹⁰:

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3 \quad (1.13)$$

and

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{1}{(4\pi)^2} g^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2} \quad (1.14)$$

⁷In the canonical normalization also an extra factor of the anomalous dimension occurs. This is reported in the formulae below and discussed in sect.(12).

⁸Large- N exact linearity of the spectrum, rather than only asymptotic linearity, may look surprising. However, the ratio between the masses of the two lowest scalar states in pure $SU(8)$ YM has been found numerically to be compatible with the value $\sqrt{2}$ [9]. We would like to thank Michael Teper for a clarifying discussion about this point at the Galileo Galilei Institute workshop on "Large- N Gauge Theories" (2011), hereafter referred to as the GGI workshop. Indeed the best fits in [9] for the continuum limit ratios of the masses (in units of the lattice RG -invariant scale, Table (7.14)) of the J^{PC} glueballs, $m_{0^{++}} = 4.71(29)$ and $m_{0^{-+}} = 4.72(32)$, to the mass of the lowest scalar, $m_{0^{++}} = 3.32(15)$ agree with very good accuracy with $\sqrt{2}$. Yet, there are larger statistical errors than this agreement may suggest. There is also a $m_{2^{++}} = 4.65(19)$ glueball, essentially degenerate in mass with the two aforementioned scalar states, that may in principle couple to our operators. Nevertheless, for reasons explained in the paper, we suggest that the spectrum in Eq.(1.7) is all made by scalars.

⁹The precise meaning of this statement is clarified in sect.(12).

¹⁰It has been known for some time that the Wilsonian, g_W , and the canonical, g , coupling constant have different beta functions in general [11].

with:

$$\beta_0 = \frac{1}{(4\pi)^2} \frac{11}{3} \quad (1.15)$$

where $g = g_{YM}^2 N$ is the 't Hooft canonical coupling constant and $\frac{\partial \log Z}{\partial \log \Lambda}$ is computed to all orders in the 't Hooft Wilsonian coupling constant, g_W , by:

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{\frac{1}{(4\pi)^2} \frac{10}{3} g_W^2}{1 + c g_W^2} \quad (1.16)$$

with c a scheme dependent arbitrary constant. Indeed since $\frac{\partial \log Z}{\partial \log \Lambda}$ to the lowest order in the canonical coupling is:

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2} \frac{10}{3} g^2 + \dots \quad (1.17)$$

the correct value of the first and second perturbative coefficients of the beta function [12, 13, 14, 15] arise:

$$\begin{aligned} \frac{\partial g}{\partial \log \Lambda} &= -\beta_0 g^3 + \left(\frac{1}{(4\pi)^2} \frac{1}{(4\pi)^2} \frac{10}{3} - \beta_0 \frac{4}{(4\pi)^2} \right) g^5 + \dots \\ &= -\frac{1}{(4\pi)^2} \frac{11}{3} g^3 + \frac{1}{(4\pi)^4} \left(\frac{10}{3} - \frac{44}{3} \right) g^5 + \dots \\ &= -\frac{1}{(4\pi)^2} \frac{11}{3} g^3 - \frac{1}{(4\pi)^4} \frac{34}{3} g^5 + \dots \end{aligned} \quad (1.18)$$

In fact a whole family of correlators of the Fourier transform of composite operators of naive dimension $4L$, $O^L(p_+, p_-)$, constructed by surface operators supported on the aforementioned Lagrangian submanifold, are computed in sect.(12). In the Wilsonian scheme the result reads:

$$\begin{aligned} &\langle Tr_{\mathcal{N}} O^L(p_+, p_-) Tr_{\mathcal{N}} O^L(-p_+, -p_-) \rangle_{conn}^{(W)} \\ &\sim \sum_{k=1}^{\infty} \frac{\Lambda_W^2 k^{2(2L-1)} \Lambda_W^{4(2L-1)}}{-\alpha' p_+ p_- + k \Lambda_W^2} + \dots \\ &\sim (-p_+ p_-)^{4L-2} \sum_{k=1}^{\infty} \frac{\Lambda_W^2}{-\alpha' p_+ p_- + k \Lambda_W^2} + \dots \\ &\sim (-p_+ p_-)^{4L-2} \log \frac{-p_+ p_-}{\Lambda_W^2} + \dots \end{aligned} \quad (1.19)$$

where the dots stand for contact terms, i.e. distributions whose inverse Fourier transform is supported at coinciding points.

In the canonical scheme for the same objects anomalous dimensions arise:

$$\begin{aligned} &\langle Tr_{\mathcal{N}} O^L(p_+, p_-) Tr_{\mathcal{N}} O^L(-p_+, -p_-) \rangle_{conn}^{(C)} \\ &= g^4 (-p_+ p_-) Z^{-\frac{8L-8}{2}} (-p_+ p_-) \langle Tr_{\mathcal{N}} O^L(p_+, p_-) Tr_{\mathcal{N}} O^L(-p_+, -p_-) \rangle_{conn}^{(W)} \\ &\sim g^4 (-p_+ p_-) Z^{-\frac{8L-8}{2}} (-p_+ p_-) (-p_+ p_-)^{4L-2} \log \frac{-p_+ p_-}{\Lambda_W^2} \end{aligned} \quad (1.20)$$

where $g(-p_+p_-)$ and $Z(-p_+p_-)$ are the (RG improved) momentum dependent canonical coupling in Eq.(1.14) and renormalization factor in Eq.(1.16).

For large L the anomalous dimensions agree¹¹ with the anomalous dimensions of the ground state [16, 17] of the Hamiltonian spin chain in the thermodynamic limit, furnishing an identification, that is at least asymptotic for large L , between composite surface operators¹² and composite local operators of YM in some regularization scheme. Indeed the Hamiltonian spin chain is an integrable model by which the anomalous dimensions of composite operators of large- N YM in the ASD and SD sector can be computed exactly at one-loop.

The exactness of our formula for the spectrum is not affected by the possibly only large- L asymptotic identification of the operators, since the spectrum depends only on the occurrence of poles in any correlator of gauge invariant operators. However, the aforementioned asymptotic identification suggests that all the glueballs in our spectrum are in fact scalar, since this is so for the operators that correspond to the ground state of the Hamiltonian spin chain in the thermodynamic limit [17].

Yet, there is an extension of our approach to fluctuations of surface operators defined by connections with wild singularities (i.e. pole singularities of any order), that are naturally associated to Regge trajectories of higher spins (see sect.(13)). Their contribution is not computed in this paper.

We explain the basic ideas underlying our computations as follows.

In the last thirty years we witnessed the geometrization of theoretical high energy physics. This geometrization has several faces but the one that we refer to consists in computing exactly functional integrals by geometrical methods. The key idea in solving the analytical problem of performing an integral by geometrical methods lies in the work of Duistermaat and Heckman [18] on exact localization of the integral of the exponential of the Hamiltonian of a torus action on a compact symplectic manifold on the fixed points of the torus action¹³:

$$\frac{1}{n!} \int \omega^n \exp(-\varepsilon H) = \sum_P \left(\frac{2\pi}{\varepsilon}\right)^n Pf^{-1}(\omega_P^{-1} \partial^2 H_P) \quad (1.21)$$

where Pf is the Pfaffian of the skew matrix, $\omega_P^{-1} \partial^2 H_P$, at the fixed points, P , and $\partial^2 H_P$ the Hessian of the Hamiltonian at P . Such localization has the following cohomological nature according to Atiyah and Bott [20] and Bismut [21, 22]. In finite dimension the integral of the exponential of a closed form ω , $d\omega = 0$, on a compact manifold without boundary, M , defines a cohomology class invariant for the addition to the closed form of an exact differential, $d\alpha$, (i.e. of a coboundary), since $d\alpha$ is trivially closed because $d^2 = 0$. Rescaling the exact differential by a large factor, under suitable positivity assumptions, the integral of the cohomology class gets localized on the critical points of the exact differential and the saddle-point approximation turns out to be exact:

$$\int_M \exp(-\omega - t d\alpha) = \int_M \exp(-\omega) \quad (1.22)$$

¹¹The agreement is at one loop since anomalous dimensions are universal, i.e. scheme independent, only at one loop. Actually they agree also for $L = 1$, since in this case the anomalous dimension is determined by the beta function via the factor of g^4 .

¹²We actually mean that we identify the Fourier transform of our composite surface operators with the Fourier transform of composite local operators in the same fashion as in Eq.(1.4). For example, $O^L(p_+, p_-) \sim |(F_{01}^- + iF_{03}^-)|^{2L}(p_+, p_-, p_+, p_-)$.

¹³For a comprehensive review see [19].

Indeed in finite dimension the integral is a polynomial in t and all the t derivatives in $t = 0$ vanish because they are the integral of a coboundary and $\int_M d\alpha = 0$. Bismut [21, 22] was the first one to extend rigorously this kind of argument to infinite dimensions, actually to a functional integral in one dimension, i.e. to quantum mechanics. In the quantum mechanical setting there are (essentially) no existence problems for functional integrals and the localization argument is in fact a mathematical proof.

However, the subject blossomed in quantum field theory only after Witten paper on localization in two-dimensional Yang-Mills theory [23] and Witten work on Donaldson invariants [24] that introduced localization in four-dimensional supersymmetric gauge theories, by identifying the differential needed to define the cohomology with a twisted super-charge, Q , satisfying $Q^2 = 0$. In turn Witten twist of supersymmetry requires to start with at least an $\mathcal{N} = 2$ $SUSY$ YM theory.

Thus the infinite-dimensional field theoretical analog of Eq.(1.22) is:

$$\int \mathcal{O} \exp(-S_{SUSY} - tQ\alpha) \quad (1.23)$$

with:

$$\begin{aligned} Q\mathcal{O} &= 0 \\ QS_{SUSY} &= 0 \end{aligned} \quad (1.24)$$

There have been a number of applications of the localization idea in four-dimensional $SUSY$ gauge theories, among which we mention the Nekrasov computation [25] of the prepotential in $\mathcal{N} = 2$ $SUSY$ gauge theories, that reproduces by localization methods the Seiberg-Witten solution [26] for the same object, and Pestun [27] computation of certain twist- $SUSY$ invariant Wilson loops in $\mathcal{N} = 4, 2, 2^*$ $SUSY$ gauge theories.

From a purely mathematical point of view these exact results state the equality between a mathematically not well defined object, the original functional integral, and a mathematically well defined and explicit answer, the result of the localization.

However, from the point of view of theoretical physics, these results are in fact satisfactory since, waiting for a realization of the constructive program of quantum field theory in four dimensions [28], the explicit answer that is found by localization defines the functional integral by the rules by which it is computed and contributes to fix the properties that the yet-to-come mathematical construction of the functional integral has to satisfy: the localization property indeed.

The aim of this paper is to add, rather surprisingly, a non-supersymmetric chapter to the aforementioned exact results.

The simplest way to present our basic result is to compare it with Nekrasov computation of the prepotential. In the first part of Nekrasov computation the functional integral that evaluates the cohomology of 1 (i.e. the partition function) is reduced by cohomological localization to a sum of finite dimensional integrals over the instantons moduli spaces:

$$Z = \lim_{t \rightarrow \infty} \int 1 \exp(-S_{SUSY} - tQ\alpha) = \sum_k \exp\left(-\frac{16\pi^2 k N}{2g_W^2}\right) \Lambda^{2kN} \int_{\mathcal{M}_k} \wedge \omega \quad (1.25)$$

This depends on the supersymmetry and has no analog in the pure YM case. On the contrary, in the second part of Nekrasov computation, the finite dimensional integrals over instantons moduli are reduced to a sum over the fixed points for the action of the torus $U(1)^{N-1} \times U(1) \times U(1)$

in $SU(N) \times O(4)$ by applying the Duistermaat-Heckman formula, after a suitable ultraviolet and infrared regularization of the moduli space, by means of a non-commutative deformation parameterized by ε :

$$\frac{1}{(2kN)!} \int_{\mathcal{M}_k^\varepsilon} \wedge \omega \exp(-\varepsilon H) = \left(\frac{2\pi}{\varepsilon}\right)^{kN} \sum_P Pf^{-1}(\omega_P^{-1} \partial^2 H_P) \quad (1.26)$$

Here $SU(N)$ is the (global) gauge group at infinity and $O(4)$ the group of Euclidean rotations. These groups are symmetry groups also of the pure YM theory. The result of the localization can be resummed into an exact formula for the prepotential [26], \mathcal{F} , that is a function of the quantum moduli of the theory, that are related to the v.e.v. of the eigenvalues of the complex scalar field in the adjoint representation of the $\mathcal{N} = 2$ YM theory:

$$Z = \exp\left(\frac{1}{\varepsilon^2} \mathcal{F}\right) \quad (1.27)$$

Despite the prepotential is obtained by the localization of a trivial observable, the cohomology of 1, from a physical point of view it contains the interesting information of the localization. Indeed, according to Seiberg-Witten [26], the prepotential contains exact highly non-trivial quantum information. It determines an exact beta function and the low energy effective action in the Coulomb branch of the theory as a function of the translational invariant condensate of the eigenvalues of the scalar field¹⁴. Thus the prepotential is used to reach conclusions about the physical theory [26] that by far exceed the very limited framework of its derivation by localization.

A general feature of cohomological localization is that the saddle-point computation can be employed only for the specific observables that satisfy Eq.(1.24). A fortiori in pure YM , that has no $SUSY$, there is no hope that localization may hold, if any, but for very special observables.

To say it in a nutshell, our basic idea for pure YM is to construct special trivial observables, called twistor Wilson loops for geometrical reasons, since they are supported on a Lagrangian submanifold of twistor space of complexified Euclidean space-time. In a technical sense the trivial twistor Wilson loops are in the homology of 1, rather than in the cohomology of 1, since in pure YM there is no $SUSY$ and thus no interesting cohomology¹⁵. The loop equation for twistor Wilson loops can be solved, since they are trivial, in the sense that it can be reduced to a critical equation for an effective action, i.e. it can be localized. Despite the effective action is obtained by trivial observables, it carries highly non-trivial quantum information, that exceeds by far the framework of localization of 1.

The effective action determines an exact large- N beta function and turns out to be a function of the density, ρ_k , in units of Λ_W^2 , of the condensate of surface operators of magnetic charge k that occur in the localization of the twistor Wilson loops.

In addition the effective action restricted to fluctuations of surface operators supported on a Lagrangian submanifold with Minkowski signature, obtained by a certain Wick rotation from the Lagrangian submanifold which the twistor Wilson loops are supported on, determines the glueballs spectrum.

¹⁴In the Coulomb branch the eigenvalues are generically all different in such a way that the unbroken gauge group is $U(1)^{N-1}$.

¹⁵There is in fact the Becchi-Rouet-Stora (BRS) cohomology associated to gauge-fixing, that leads to localization on gauge-fixed slices of gauge orbits, but it is not relevant for our purposes.

The analytic continuation to Minkowski space-time is the only way to regularize gauge invariantly the holomorphic loop equation for the twistor Wilson loops, that in turn leads to localization on the critical points of the effective action.

We describe now in more detail what the twistor Wilson loops are.

They compute the holonomies along loops of a modified non-Hermitian YM connection, the twistor connection. Its curvature is a non-Hermitian linear combination of the ASD part of the curvature of the ordinary gauge connection. These loops are supported on a Lagrangian submanifold in twistor space of complexified Euclidean space-time, locally the product of a two-dimensional surface immersed in complexified space-time with local (complex) coordinates (z, \bar{z}) and of a one-dimensional curve immersed in the fiber of the twistor fibration with (not necessarily real) coordinate λ .

The twistor Wilson loops are chosen in the adjoint representation. The operator definition of twistor Wilson loops involves the parameter λ , but their vacuum expectation value (v.e.v.) is λ -independent, in fact trivially 1 at large- N . Hence there is a non-compact real version $U(1)_R$, of the complexification $U(1)_C$, of one of the aforementioned $U(1)$ ¹⁶ that acts by rescaling λ in such a way that the v.e.v. of twistor Wilson loops is invariant under the aforementioned action.

As a consequence we show that twistor Wilson loops in pure YM are localized on the sheaves, defined by the change of variables from the gauge connection to the ASD part of its curvature in the functional integral, fixed by the action of $U(1)_R$. In addition we show that there is a dense set in function space in a neighborhood of the fixed sheaves¹⁷, that at large- N is classified by local systems on a sphere with a very large number of punctures and with fixed conjugacy class of the holonomy of the twistor connection around the lattice of punctures, with values in the complexification, $SU(N)_C$, of the gauge group, modulo the global action of $SU(N)_C$.

In the physics terminology the local systems are lattices of surface operators satisfying the self-duality (SD) equations with singularities:

$$F_{\alpha\beta}^- = \sum_p \mu_{\alpha\beta}^-(p) \delta^{(2)}(z - z_p) \quad (1.28)$$

In addition the v.e.v. of the aforementioned twistor Wilson loops in the adjoint representation factorizes in the large- N limit in the product of the v.e.v. in the fundamental and conjugate representation. Then to each factor the following argument applies.

On the dense set described by local systems, by translational invariance we can assume that all the conjugacy classes of the holonomies are a copy of the same adjoint orbit, and that the orbit for a holonomy around one arbitrarily chosen point can be put by the global action of $SU(N)_C$ in canonical form, i.e. either in diagonal or in Jordan form. Now the global compact $SU(N)$ gauge group acts on such diagonal or Jordan holonomy by conjugation.

If the global gauge group is unbroken, as it is believed to be the case for pure YM , only the holonomies that are fixed by the entire $SU(N)$ may occur at large- N . Thus these holonomies at a

¹⁶It is a $U(1)$ in the Cartan subgroup of $O(4)$.

¹⁷We refer to the support of the fixed measure as fixed sheaves to imply not any manifold structure for such a locus. However, for a dense set in function space the fixed sheaves at large- N are in fact a manifold that is parameterized by the disjoint union of moduli of local systems with fixed conjugacy class of the holonomy of the twistor connection. We refer to fixed points instead when the fixed locus has no moduli and it is a set of disconnected points.

preferred point are in fact valued in the center of the gauge group and their orbits reduce to points. But then by translational invariance all the orbits reduce to the center ¹⁸.

Besides we show that there is a homological explanation for this localization on fixed points based on a new localization theory of the loop equation for twistor Wilson loops, such that the actual fixed points that contribute to the twistor Wilson loops are the critical points of a certain effective action determined by the loop equation [10]. The localization by homology of the loop equation, i.e. its reduction to a critical equation, is obtained deforming the loop by adding vanishing boundaries that are backtracking arcs ending with the cusps of the local system [10], an operation allowed by the large- N triviality of twistor Wilson loops, by dualizing the idea of deforming a closed form by a coboundary in the cohomological interpretation of the Duistermaat-Heckman localization.

In order to get localization, the first main technical innovation of our approach is a reformulation of the YM theory in terms of a change of variables that in the $\mathcal{N} = 1$ $SUSY$ YM theory has been known as the Nicolai map [29, 30]. The Nicolai map in the $\mathcal{N} = 1$ $SUSY$ YM theory was worked out by De Alfaro, Fubini, Furlan and Veneziano [31, 32] as a change of variables from the gauge connection to the anti-selfdual (ASD) part of the gauge curvature, that needs a gauge fixing to be locally invertible, with the property that the Jacobian of the map cancels precisely the fermion determinant in the light-cone gauge.

As a preparatory exercise, the Nicolai map allows us to introduce localization also in the pure $\mathcal{N} = 1$ $SUSY$ YM theory ¹⁹ by means of the tautological Parisi-Sourlas supersymmetry associated to the cancellation of the Jacobian with the fermion determinant [35, 36]. While it has been known for some time that the Nicolai map can be associated to cohomological localization ²⁰, the localization by the Nicolai map has never been worked out in asymptotically free gauge theories because of the following difficulty. Naively the Nicolai map maps $\mathcal{N} = 1, 2$ $SUSY$ YM into a free theory, that cannot hold true literally. The question arises for example how to reproduce the $NSVZ$ beta function [38, 39] by means of the Nicolai map. Our simple but key observation is that the cancellation of determinants occurs only up to zero modes. Therefore the divergences associated to the Pauli-Villars regulator of the zero modes occur.

In fact understanding how the $NSVZ$ beta function occurs by cohomological localization via the Nicolai map in this paper is only an exercise for understanding localization of the aforementioned twistor Wilson loops in large- N pure YM in the ASD variables. The crucial point is that the localization on the fixed points of the $U(1)_R$ action can be obtained only in the ASD variables.

The second main technical innovation consists in interpreting our non- $SUSY$ version of the

¹⁸The same conclusion is reached by an inductive argument on the holonomies around each point, without assuming translational invariance, since once the holonomy around a point is shown to be in the center by the assumption of unbroken gauge group, the global $SU(N)_C$ still acts on the holonomies around each of the remaining points.

¹⁹Witten already observed in his paper [24] on Donaldson invariants that although most naturally formulated in the $\mathcal{N} = 2$ $SUSY$ theory localization could be extended to certain theories with only $\mathcal{N} = 1$ $SUSY$, called $\mathcal{N} = 2^*$ theories that involve anyway the occurrence of a scalar field. In pure $\mathcal{N} = 1$ $SUSY$ gauge theory Witten localization does not apply directly since there are no scalars. It is always possible to give the scalars of the $\mathcal{N} = 2^*$ $SUSY$ theories large masses in order to obtain at low energy pure $\mathcal{N} = 1$ $SUSY$ YM theory. This leads to the modern "weak coupling" approach to the computation of the gluino condensate [33, 34].

²⁰In the lectures [37] it is worked out the zero dimensional case of the Nicolai map and it is shown indeed that coincides with localization.

Nicolai map in the pure YM theory [10, 2, 3] as hyper-Kähler reduction [40, 41] on a dense set in function space, that corresponds to a lattice of surface operators in the physics terminology. This is an analytical and differential geometric construction that does not need any supersymmetry. It reduces the YM functional integral to a finite dimensional integral with respect to a product measure on a lattice and it is the analog of the first part of Nekrasov computation in the supersymmetric case.

The physics interpretation is that the localization of the twistor Wilson loops in the large- N YM theory is described in terms of variables that are of purely magnetic type, realizing, in the technical sense of localization of twistor Wilson loops, a new version of 't Hooft long-standing ideas ²¹ on the YM vacuum as a dual superconductor [42, 43, 44].

In particular 't Hooft duality in YM theories with fields in the adjoint representation requires that, if the theory has a mass gap, then either the Z_N magnetic charges condense (confining phase) or the Z_N electric charges condense (Higgs phase). Localization by homology of twistor Wilson loops in pure YM realizes the first alternative, in which the electric charge is unbroken ²² and the magnetic charge is broken in superselection sectors labelled by k , the magnetic charge at a (lattice) point, that are degenerate ²³ for the large- N renormalized effective action that occurs in the holomorphic loop equation. From the localized renormalized effective action restricted to fluctuations supported on the aforementioned Lagrangian submanifold it follows also that in each sector there is a mass gap proportional to \sqrt{k} in units of the common RG -invariant scale.

Yet, we should stress that localization is by no means a universal concept, but it applies only to special observables. Therefore, it would be completely wrong to employ surface operators of Z_N holonomy to compute general observables of the YM theory, as it would be completely wrong to employ instantons to compute anything but the gluino condensate in $\mathcal{N} = 1$ $SUSY$ YM .

The mathematics interpretation is that we are in fact representing the YM functional integral as an adelic integral over (the moduli space of) local systems.

The two aforementioned technical innovations are crucial for our twofold approach to localization in pure YM .

Firstly, as we just explained, following the spirit of the Duistermaat-Heckman idea our new kind of localization in the large- N pure YM theory involves the action of a semigroup fixing the v.e.v. of twistor Wilson loops and contracting the support of the functional YM measure, resolved into ASD orbits by our non-supersymmetric version of the Nicolai map. At technical level the hyper-Kähler reduction to surface operators furnishes a structure theory of the locus of the fixed-points.

Secondly, as well as the Duistermaat-Heckman localization on fixed points has a cohomological explanation, so the new localization on fixed points in pure YM theory has a homological explanation. Indeed there exists a new holomorphic loop equation for twistor Wilson loops that can be localized, i.e. reduced to a critical equation, by deformations of the loop that are vanishing boundaries (backtracking arcs) in homology, in the dual sense to which a cohomology class represented by an integral of the exponential of a closed form can be localized by deformations that are coboundaries in cohomology. At technical level the localization of the holomorphic loop

²¹In addition to 't Hooft original papers [42, 43, 44] see also [45] for a very neat account of 't Hooft duality.

²²The adjoint action of the global gauge group leaves invariant the center, Z_N .

²³The classical action scales as k times the square density of surface operators, ρ^2 , and the renormalized square density scales as $\frac{1}{k}$.

equation for twistor Wilson loops requires that the backtracking arcs end with cusps supported on the singular divisor of the surface operators.

Our new holomorphic loop equation for twistor Wilson loops is derived using the standard technique of the celebrated loop equation of Makeenko and Migdal [46, 47] invented long ago, but the crucial difference is that the integration variable that gives origin to the loop equation in our case is not the gauge connection but instead the ASD field of our non- $SUSY$ version of the Nicolai map, in a holomorphic gauge defined by a further change of variables. The resulting loop equation resembles for the cognoscenti the holomorphic loop equation of Dijkgraaf and Vafa [48, 49, 50] for the holomorphic chiral ring of $\mathcal{N} = 1$ $SUSY$ gauge theories [51].

The homological localization of the holomorphic loop equation completes the analogy with Nekrasov computation. As well as the prepotential, i.e. the effective action in the low energy sector as a function of the condensates of the $\mathcal{N} = 2$ $SUSY$ theory, is computed by cohomological localization of 1, so the large- N effective action of the YM theory in the twistor sector, as a function of the condensates of surface operators of Z_N holonomy, is computed by homological localization of 1, the trivial twistor Wilson loops.

The twistor sector is defined by correlation functions obtained by holomorphic/antiholomorphic fusion a la Cecotti-Vafa [52, 53] of the holomorphic/antiholomorphic ASD curvature of surface operators restricted to the aforementioned Lagrangian submanifold of twistor space of complexified space-time with Minkowski signature.

On the ultraviolet side, a striking result that follows from the localization on fixed points is that the large- N beta function for the Wilsonian coupling constant in the ASD variables is one-loop exact, because for twistor Wilson loops, precisely because of the localization, a certain kind of saddle-point approximation turns out to be exact. Thus the quantum corrections for these observables are completely accounted by functional determinants whose diagrammatic expansions contains only one-loop Feynman graphs and possibly the logarithm of the powers of the Pauli-Villars regulator of zero modes. Thus in the large- N pure YM theory Eq.(1.1) can be replaced by the much simpler:

$$\Lambda_W = \Lambda \exp\left(-\frac{1}{2\beta_0 g_W^2}\right) \quad (1.29)$$

where Λ_W is the RG invariant scale in the Wilsonian scheme in the ASD variables.

At the same time we show that, in the regularization scheme of the homological localization of the holomorphic loop equation, the one-loop exactness for the Wilsonian beta function implies a large- N exact beta function for the canonical coupling of $NSVZ$ type, that reproduces the one- and two-loop perturbative universal coefficients. In this scheme the large- N canonical beta function of the pure YM theory in the ASD variables is given by Eq.(1.14).

We should stress that the computation of the canonical beta function depends crucially on exploiting the gluing rules for functional integrals in the specific case of the localization on local systems.

In particular we show that these gluing rules coincide with the ones of topological strings via the gluing of an associated arc complex [54, 55, 56, 57, 58]. Indeed the homological localization is based on "the most local part" of the homology of the essential arc complex of a punctured sphere.

The requirement that the homology be essential, i.e. the exclusion of arcs that can be deformed to a puncture, rules out the local relative homology of compact support around a puncture. In fact "the most local part" of the essential homology that is relevant for us is the essential homology of the arc complex with no polygons, that involves only links ending with two different cusps, one in the divisor at the ultraviolet and one in the divisor at the infrared. Naively the Wilsonian beta function in our scheme is a purely ultraviolet concept, and therefore does not distinguish between the local homology and the essential homology of links. Yet, the canonical beta function, that in our scheme involves infrared physics too, does.

In addition, while fixed point arguments suffice to display localization directly in large- N YM theory by the change to the ASD variables in the functional integral and the dense hyper-Kahler resolution on local systems, the corresponding effective action, i.e. the logarithm of the density of the localized measure, has intrinsic finite holomorphic ambiguities due to the freedom of making holomorphic changes of variables and possibly holomorphic anomalies at loci where the holomorphic change of variables may develop singularities. We are able to fix these holomorphic ambiguities only using the localization by homology of the loop equation, via the choice of the holomorphic gauge that is necessary to write down the holomorphic loop equation.

On the infrared side, the holomorphic gauge in the loop equation is essential, because the mass gap and the glueballs spectrum occur precisely because of a non-trivial Jacobian from the unitary to the holomorphic gauge in the effective action.

We end this introduction with some loose heuristic considerations as to why the line of thought of this paper may be able to overcome the main difficulties of the ultraviolet and infrared problem of YM in the restricted sense specified above.

While the mass gap problem as formulated in full generality for every correlation function and for every compact gauge group in [1] appears presently almost hopeless in our opinion, the program of solving the $SU(N)$ YM theory in the large- N limit has attracted considerable attention and efforts.

A promising avenue is to find an equivalent string theory [59, 60] by effectively resumming 't Hooft perturbative double expansion in powers of g and N^{-1} [61]. In this string theory the v.e.v. of any Wilson loop of the YM theory in the large- N limit would be computed by a string diagram that is a disk with the loop as boundary. No other interesting observables, but the RG -invariant condensates, exist at the leading large- N order because of the factorization of the v.e.v. of local normalized gauge invariant operators.

Now, on the field theory side, the knowledge of all the Wilson loops that would be implied by the string solution contains a vast information in a mathematical sense. Indeed it has been known for some time that the ambient algebra of the master field [62] that solves the large- N Makeenko-Migdal loop equation for ordinary Wilson loops [46, 47] is the Cuntz algebra with four generators [63, 64, 65, 66, 67, 68, 69] whose Fock space representation is known to be of type II_1 but not hyperfinite [70], i.e. not the weak limit of matrix algebras. Indeed such Fock representation is isomorphic to a free group factor with the same number of generators, which is the main explicit example of the "elusive" type II_1 non-hyperfinite factors [70]. It is clear [3] that obtaining the relevant non-hyperfinite information would be extremely difficult in case the von Neumann algebra generated by the actual solution shares with the ambient algebra the non-hyperfinite character.

On the contrary, to the next to leading $\frac{1}{N}$ order, the connected two-points correlation functions

of local gauge invariant operators are conjectured to be the most simple as possible: a sum of an infinite number of propagators of free fields [71], saturating the logarithms of short distance perturbation theory [72]:

$$\int \langle \frac{1}{N} \sum_{\alpha\beta} Tr F_{\alpha\beta}^2(x) \sum_{\alpha\beta} \frac{1}{N} Tr F_{\alpha\beta}^2(0) \rangle_{conn} e^{ipx} d^4x = \sum_r \frac{Z_r}{p^2 + M_r^2} \sim g^4(p) p^4 \log\left(\frac{p^2}{\mu^2}\right) \quad (1.30)$$

Now any string solution, as it is usually meant, cannot avoid to solve the leading order problem for the Wilson loops, in order to solve the much simpler looking subleading problem for the free glueballs spectrum. This makes such a general, large- N exact, string solution very difficult in our opinion. We may wonder as to whether we can solve the easy looking subleading problem for the free glueballs spectrum avoiding and thus loosing the information about the hard looking problem for the Wilson loops. Our answer is positive to a certain extent: we construct trivial Wilson loops, the twistor Wilson loops indeed, whose v.e.v. is 1 in the leading large- N limit. However, they admit non-trivial $\frac{1}{N}$ corrections and thus morally they couple to a certain non-trivial sector of the large- N theory.

By the way, in this restricted sense we believe that there is also an explicitly solvable string theory, that captures the sector of YM accessible to the twistor Wilson loops defined in this paper. The outlook for this twistor string is described in the conclusions.

Coming back to the field theoretical framework, we use our localization theory to localize the twistor Wilson loops, that are in the "homology of 1", precisely in the dual sense to which Nekrasov localized the "cohomology of 1" to get the Seiberg-Witten prepotential. Indeed, although the prepotential is obtained by a "trivial" cohomology, it allows one to reconstruct the low energy effective theory. Precisely in the same sense, since twistor Wilson loops live in a "trivial" homology, they can be localized by suitable deformations. Yet, the interesting information is contained in the effective action, i.e. in the localized measure. The quadratic small fluctuations of the effective action around the localized loci of the measure furnish the glueballs spectrum in the twistor sector.

2. Synopsis

We summarize here the main technical arguments in a logic order and the main results.

In the prologue we describe in some detail the computation of the beta function in the following cases. The one-loop beta function in YM by the usual background field method. The $NSVZ$ beta function in $\mathcal{N} = 1$ $SUSY$ YM by cohomological localization in the Nicolai variables. The one-loop YM beta function for the Wilsonian coupling in the ASD variables by the usual background field method.

These concrete examples are used to furnish a comparison with the computation of the YM beta function by our new localization. This section contains many definitions and computational technicalities that are referred to throughout the whole paper.

We define also the change of variables from the gauge connection to the ASD curvature in the pure YM case. In particular we show that, since in pure YM the Jacobian to the ASD variables is not cancelled, as opposed to the $\mathcal{N} = 1$ $SUSY$ YM case in the light-cone gauge, a multiplicative Z renormalization of the ASD field occurs. In sect.(12) this Z factor is related to the anomalous

dimensions of a large class of composite operators that occur as scalar polynomials in the ASD curvature in the one-loop integrable sector of large- N YM .

In sect.(4) we define twistor Wilson loops in non-commutative gauge theories. The twistor Wilson loops are defined on a non-commutative deformation of space-time, that is used as a tool to define the large- N limit much in the way Nekrasov used a non-commutative deformation as a tool to regularize the instantons moduli space. We recall some features of non-commutative gauge theories that we employ in the following sections.

We display the following properties of twistor Wilson loops. The v.e.v. twistor Wilson loops is fiber independent and trivially 1 in the large- N limit. In addition twistor Wilson loops are supported on Lagrangian submanifolds of twistor space.

In sect.(5) we show that the curvature of the twistor connection is of purely ASD type and we describe the localization in large- N pure YM theory of twistor Wilson loops on the fixed sheaves of a semigroup acting on the fiber of the Lagrangian fibration which twistor loops are supported on and contracting the support of the functional measure in the ASD variables.

We refer to this kind of localization as the quasi-localization lemma, since the resulting localized measure is still represented by a residual functional integration on a certain complex path, supported on distribution valued sheaves in fact, rather than by a sum over fixed points.

The quasi-localization lemma is a purely formal computation that involves a quite disputable formal exchange of the order of limit and integration. For twistor Wilson loops the exchange of order of limit and integration is justified in sect.(8) by explicit computation.

We write down the corresponding effective action, i.e. the logarithm of the density of the localized YM measure. Because of the residual complex integration the effective action has an ambiguity by holomorphic change of variables that we can solve only through the loop equation of sect.(6). The fixed sheaves in the quasi-localization lemma are characterized by the vanishing of two of the three ASD fields of the non- $SUSY$ Nicolai map.

In sect.(6) we write the holomorphic loop equation for twistor Wilson loops. The holomorphic ambiguity of the effective action of sect.(5) is fixed by a change of variables to the holomorphic gauge, necessary to write down the holomorphic loop equation. It is precisely the Jacobian to the holomorphic gauge that generates the glueballs potential.

This implies that the glueballs potential in the holomorphic/antiholomorphic sector defined by twistor loops in the fundamental and conjugate representation must be singular at the fixed points, as it is indeed, for the theory to have a mass gap, since the contribution of the Jacobian to the effective action is formally the logarithm of the square of a holomorphic function.

In sect.(7) we introduce a regularization of the large- N functional integral by integrating on "infinite-dimensional local systems" ²⁴ on non-commutative space-time.

The idea of integrating on local systems associated to an arbitrary parabolic divisor appeared for the first time long ago in a paper by us [2, 3], by embedding the Hitchin fibration [4, 7] in the YM functional integral, and physically corresponds to integrating over surface operators ante litteram [8]. We employ Morita equivalence [73] to reduce to the case of ordinary space-time for

²⁴We write it in quotes because these infinite dimensional objects admit unstable finite dimensional subbundles, thus violating a fundamental property of finite dimensional local systems.

finite rank bundles. Thereafter we reconstruct the large- N limit of YM as an inductive limit on the finite rank local systems.

Following the mathematical literature [40, 4, 74, 75, 77, 78, 7, 79, 80, 82] we discuss the topological, holomorphic and differential geometric features of the finite rank local systems [80, 81]. As topological objects local systems are representations of the fundamental group of a punctured Riemann surface. As holomorphic objects they are holomorphic connections with regular singularities. As differential geometric objects they are parabolic harmonic bundles, i.e. parabolic Hitchin bundles [40] equipped with a harmonic metric by a Hitchin-Kobayashi correspondence [79]. Remarkably in our setting the harmonic bundles arise as the hyper-Kähler reduction [40] induced by our version of the non- $SUSY$ Nicolai map.

Physically the hyper-Kähler reduction [40] is a resolution dense in function space [2], in a neighborhood of the fixed sheaves, of the ASD field [3] as a linear combination of two-dimensional delta distributions supported on a lattice of surface operators [8]. These are local systems that occur in the mathematics and physics literature for completely different reasons, among which we mention the Hitchin-Kobayashi correspondence [79], non-abelian Hodge theory [74, 75], twistor D -modules [82], and last but not least the physics version [8] of the geometric Langlands correspondence [83, 84, 85, 86, 87, 88, 89, 90].

In sect.(8) we get our localization on fixed points. Indeed we combine the quasi-localization lemma of sect.(5) with the idea of sect.(7) of integrating on local systems to get localization on fixed points. Reducing to finite dimensional local systems needs Morita duality and is allowed implicitly by the triviality of twistor Wilson loops.

In fact the quasi-localization lemma depends on the aforementioned disputable formal exchange of the order of limit and integration. This is justified a posteriori by explicit computation by showing that the v.e.v. of the trivial twistor Wilson loops is independent on the order of limit and integration. Of course the integral of the limit is the localized measure in the ASD variables, while the limit of the integral is the original YM measure on the gauge connections.

In particular the choice of the approximating sequence by finite dimensional local systems (i.e. stable bundles) is essential for localization on fixed points. We show that the fixed manifold restricted to the dense hyper-Kähler locus of local systems is a Lagrangian submanifold of the moduli space of surface operators. In addition we show that, assuming that the gauge group is unbroken, the fixed manifold is in fact a collection of fixed points represented by surface operators with Z_N holonomy. This localization is our analog of the Duistermaat-Heckman localization.

In sect.(9) we use it to compute the Wilsonian beta function of the large- N YM theory.

In sect.(10) we get homological localization of the holomorphic loop equation for the twistor Wilson loops by means of the lattice version of the holomorphic loop equation obtained integrating over the local systems.

The triviality of twistor Wilson loops plays a key role here, since it allows arbitrary deformations of the twistor Wilson loops without changing their expectation value. In this section the localization on fixed points is not a consequence of the assumption that the gauge group is unbroken, but a consequence of the reduction of the loop equation to a critical equation.

The localization is obtained deforming the twistor Wilson loops by backtracking arcs ending with the cusps of the local system. This is the localization by homology that combines the holo-

morphic loop equation of sect.(6) with the idea of integrating over local systems of sect.(7). This is our analog of cohomological localization by deforming by a coboundary.

In particular we derive the glueballs potential by computing the Jacobian to the holomorphic gauge of sect.(6) for the local systems of sect.(7).

At mathematical level homological localization involves the essential arc complex of a punctured sphere and a combinatorial model of the gluing of the arcs developed in the mathematical literature [54, 55, 56], in turn inspired by (topological) strings [57, 58].

The version of localization via the loop equation is the one that has, in our opinion, more chances to hold in a strictly mathematical sense. The loop equation for twistor Wilson loops occurs as a formal Schwinger-Dyson or Ward identity derived imposing that the integral of a functional derivative vanishes in function space in the ASD variables in the holomorphic gauge.

The left hand side of the loop equation contains the effective action of the theory that implies a one-loop exact Wilsonian beta function. The right hand side is still divergent, but it can be regularized in a gauge invariant way by analytic continuation to Minkowski space-time and, by deforming the loop, it can be made to vanish. Thus, despite the loop equation is obtained only as a formal identity (as the Makeenko-Migdal equation is), its solution is defined via its would-be properties, essentially the fact that it allows analytic continuation to Minkowski space-time.

The glueballs spectrum occurs only after analytic continuation to Minkowski space-time of the effective action renormalized in Euclidean space for fluctuations of surface operators restricted to the Lagrangian submanifold.

The basic idea is that once the large- N localization is obtained for twistor Wilson loops, that are non-local extended objects, the localized effective action is used to compute physical fluctuations of local operators restricted to certain channels.

In sect.(11) we compute the canonical beta function of large- N YM by means of our holomorphic loop equation restricted to the local systems of sect.(7) and we check agreement with the first two universal perturbative coefficients. The result depends crucially on the gluing rules for local systems.

In sect.(12) we display our main result about the mass gap and the glueballs spectrum using the effective action of sect.(6) together with its extension to the hyper-Kähler locus in a neighborhood of the fixed points of sect.(7).

To do computations we employ the local model of the singular part of the connections with regular singularities around surface operators of sect.(7). We show that at the (renormalized) critical points the local model is in fact asymptotic for large N . We find a trajectory with mass squared exactly linear in k and residues at the poles determined by the multiplicity of the eigenvalues of the ASD curvature at the fixed points and by certain finite counterterms as a function of k in a neighborhood of the fixed points.

We display in some detail how the glueballs spectrum for the trajectory in the twistor sector follows from the effective action. We also use our explicit solution to check the long-standing conjecture that the sum of pure poles in the large- N limit on our trajectory saturates the logarithms that occur in the glueballs propagators in perturbation theory.

In the Wilsonian scheme we reproduce a factor of a logarithm that occurs in perturbation theory, that arises in our scheme by the RG-invariant spectral sum over the glueballs.

In the perturbative canonical scheme the sum of a logarithm (with a coefficient that can be normalized to 1) and of its square occurs. The coefficient of the square of the logarithm depends on the operator and is related to the anomalous dimension.

We observe that in the canonical scheme the multiplicative Z renormalization, that occurs because of the Jacobian to the ASD variables mentioned in sect.(3.4), implies through the localization of composite surface operators the same anomalous dimensions as for the operators associated to the ground state of the Hamiltonian spin chain in the thermodynamic limit, which is known to furnish the one-loop anomalous dimensions of long local gauge invariant operators in the one-loop integrable sector of large- N YM . This sector is made by SD or ASD fields.

This cannot be the whole story, since we get from surface operators just one trajectory. In sect.(13) using by now standard results in mathematics [80, 91], we extend the hyper-Kähler reduction induced by the non- $SUSY$ Nicolai map to twistor connections with wild singularities, i.e. poles of any order. We suggest that such an extension corresponds physically to the more realistic case of an infinite family of Regge trajectories of increasing spins. We write the basic definition of the functional integral on wild surface operators but explicit computations are left for the future.

In sect.(14) we summarize our conclusions and we outline some features of the twistor string conjectured to be dual to the YM theory restricted to the sector defined by the twistor Wilson loops of this paper.

3. Prologue

3.1 One-loop beta function of YM by the background field method

This computation is now completely standard, but since it is not easily found in textbooks in the form that we will need in the rest of the paper we display it here in some detail²⁵. The basic philosophy is as in [71]. The partition function of pure $SU(N)$ YM is:

$$Z = \int \delta A e^{-S} \quad (3.1)$$

where:

$$S = \frac{N}{2g^2} \int d^4x \operatorname{tr}_f (F_{\alpha\beta})^2 = \frac{N}{4g^2} \int d^4x (F_{\alpha\beta}^a)^2 \quad (3.2)$$

The sum over repeated indices is understood. The action has been rescaled by a factor of N in such a way that the theory admits a non-trivial large- N limit. The coupling constant, g , is the 't Hooft coupling related to the YM coupling, g_{YM} , by $g^2 = g_{YM}^2 N$. The normalization of the action is appropriate for a gauge connection in the fundamental representation of the Lie algebra of $SU(N)$:

$$A_\alpha = A_\alpha^a T^a \quad (3.3)$$

with the Hermitian generators in the fundamental representation normalized as:

$$\operatorname{tr}_f (T^a T^b) = \frac{1}{2} \delta^{ab} \quad (3.4)$$

²⁵A more detailed version of the computation reported here can be found on the SNS web site in "Solutions to problems" by Luca Lopez: www.sns.it/it/scienze/fisiche/pertgauge/.

The curvature of the YM connection is:

$$[D_\alpha, D_\beta] = iF_{\alpha\beta} \quad (3.5)$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta] \quad (3.6)$$

$$F_{\alpha\beta}^a = \partial_\alpha A_\beta^a - \partial_\beta A_\alpha^a - f^{abc} A_\alpha^b A_\beta^c \quad (3.7)$$

where $D_\alpha = \partial_\alpha + iA_\alpha$ is the covariant derivative. To perform the one-loop computation of the effective action it is convenient to split the gauge connection into a classical background field and a fluctuating quantum field $A_\alpha = \bar{A}_\alpha + \delta A_\alpha$. The Fourier transform of the quantum field is supposed to be supported on momenta much larger than the momenta of the classical background field. The gauge-fixing is performed by the Faddeev-Popov procedure. It is convenient to choose the Feynman gauge with respect to the background gauge field \bar{A}_α :

$$\bar{D}_\alpha \delta A_\alpha - C = 0 \quad (3.8)$$

where we denote by the dot the adjoint action in the Lie algebra:

$$D_\alpha(\bar{A}) \cdot \delta A_\beta = \partial_\alpha \delta A_\beta + i[\bar{A}_\alpha, \delta A_\beta] \quad (3.9)$$

$$(D_\alpha(\bar{A}) \cdot \delta A_\beta)^a = \partial_\alpha \delta A_\beta^a - f^{abc} \bar{A}_\alpha^b \delta A_\beta^c = D_\alpha^{ac}(\bar{A}) \delta A_\beta^c \quad (3.10)$$

where

$$\begin{aligned} D_\alpha^{ac}(\bar{A}) &= \partial_\alpha \delta^{ac} - f^{abc} A_\alpha^b \\ &= \partial_\alpha \delta^{ac} + iA_\alpha^{ac} \end{aligned}$$

with $A_\alpha^{ac} = if^{abc} A_\alpha^b$ and $[T^a, T^b] = if^{abc} T^c$. C is an auxiliary gaussian field whose covariance is chosen in such a way to cancel a longitudinal term in the YM action quadratic in the fluctuating field, by adding $\frac{N}{g^2} \int d^4x \text{tr}_f (\bar{D}_\alpha \delta A_\alpha)^2$ to the action. Quantities such as \bar{D}_α are evaluated at the background field \bar{A}_α . The gauge fixed partition function reads:

$$Z = \int \delta A \delta C \exp(-S_{YM}) \text{Det}(-\Delta_{\bar{A}}) \exp\left(-\frac{N}{g^2} \int d^4x \text{tr}_f (C^2)\right) \delta(\bar{D}_\alpha \delta A_\alpha - C) \quad (3.11)$$

where we have inserted the Faddeev-Popov determinant of (minus) the Laplacian in the background field:

$$-\Delta_{\bar{A}} = -\Delta^2 - i\partial_\alpha \bar{A}_\alpha - 2i\bar{A}_\alpha \partial_\alpha + \bar{A}_\alpha \bar{A}_\alpha \quad (3.12)$$

As a consequence the gauge-fixed action is:

$$\int \frac{N}{2g^2} \text{tr}_f F_{\alpha\beta}^2(\bar{A} + \delta A) + \frac{N}{g^2} \text{tr}_f (\bar{D}_\alpha \delta A_\alpha)^2 d^4x \quad (3.13)$$

and the one-loop partition function reads:

$$Z_{1-loop} = e^{-\Gamma_{1-loop}(\bar{A})} = e^{-S_{YM}(\bar{A})} \text{Det}^{-1/2}(-\Delta_{\bar{A}} \delta_{\alpha\beta} - 2i\bar{F}_{\alpha\beta}) \text{Det}(-\Delta_{\bar{A}}) \quad (3.14)$$

where $\Gamma_{1-loop}(\bar{A})$ is the effective action for the background connection, \bar{A}_α , to one-loop order and $\bar{F}_{\alpha\beta}(\dots) = [\bar{F}_{\alpha\beta}, \dots]$.

It is very instructive to understand the origin of the spin term, $-2i\bar{F}_{\alpha\beta}$, in the first functional determinant of Eq.(3.14). By the splitting of the connection into $A_\alpha = \bar{A}_\alpha + \delta A_\alpha$ the curvature decomposes as follows:

$$F_{\alpha\beta}(\bar{A} + \delta A) = F_{\alpha\beta}(\bar{A}) + D_\alpha(\bar{A}) \cdot \delta A_\beta - D_\beta(\bar{A}) \cdot \delta A_\alpha + i[\delta A_\alpha, \delta A_\beta] \quad (3.15)$$

Performing the square we keep only up to the quadratic terms in δA_α in the action, since we are doing a one-loop computation, we understand integration on space-time in the following and we freely integrate by parts. We use the equation of motion, $D_\alpha(\bar{A}) \cdot F_{\alpha\beta}(\bar{A}) = 0$, to eliminate the linear term in the action. Therefore we get:

$$\begin{aligned} F_{\alpha\beta}^2(\bar{A} + \delta A) &= F_{\alpha\beta}^2(\bar{A}) + (D_\alpha(\bar{A}) \cdot \delta A_\beta - D_\beta(\bar{A}) \cdot \delta A_\alpha)^2 + 2iF_{\alpha\beta}(\bar{A})[\delta A_\alpha, \delta A_\beta] \\ &= F_{\alpha\beta}^2(\bar{A}) + 2(D_\alpha(\bar{A}) \cdot \delta A_\beta)^2 - 2D_\alpha(\bar{A}) \cdot \delta A_\beta D_\beta(\bar{A}) \cdot \delta A_\alpha + 2iF_{\alpha\beta}(\bar{A})[\delta A_\alpha, \delta A_\beta] \end{aligned} \quad (3.16)$$

Using

$$D_\alpha D_\beta = D_\beta D_\alpha + iF_{\alpha\beta} \quad (3.17)$$

$$tr_f(\delta A_\beta [F_{\alpha\beta}, \delta A_\alpha]) = tr_f(F_{\alpha\beta} [\delta A_\alpha, \delta A_\beta]) = -tr_f(\delta A_\alpha [F_{\alpha\beta}, \delta A_\beta]) \quad (3.18)$$

the quadratic form in Eq.(3.16) becomes:

$$\begin{aligned} &tr_f((D_\alpha \delta A_\beta - D_\beta \delta A_\alpha)^2 + 2iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta]) \\ &= tr_f(-2\delta A_\alpha \Delta_A \delta_{\alpha\beta} \delta A_\beta + 2\delta A_\beta D_\alpha D_\beta \delta A_\alpha + 2iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta]) \\ &= tr_f(-2\delta A_\alpha \Delta_A \delta_{\alpha\beta} \delta A_\beta + 2\delta A_\beta D_\beta D_\alpha \delta A_\alpha + 2i\delta A_\beta F_{\alpha\beta} \delta A_\alpha + 2iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta]) \\ &= tr_f(-2\delta A_\alpha \Delta_A \delta_{\alpha\beta} \delta A_\beta - 2(D_\alpha \delta A_\alpha)^2 - 4i\delta A_\alpha [F_{\alpha\beta}, \delta A_\alpha]) \end{aligned} \quad (3.19)$$

where we skip the label of the background field since no confusion can arise. In the Feynman gauge the second term in the last line is cancelled by the gauge-fixing. Finally the quadratic form written in components becomes:

$$\begin{aligned} &tr_f(\delta A_\alpha (-2\Delta_A \delta_{\alpha\beta} - 4i adF_{\alpha\beta}) \delta A_\beta) \\ &= tr_f(\delta A_\alpha^a t^a (-2\Delta_A \delta_{\alpha\beta} - 4i adF_{\alpha\beta}) \delta A_\beta^b t^b) \\ &= \delta A_\alpha^a ((-\Delta_A \delta_{\alpha\beta} - 2i adF_{\alpha\beta}) \delta A_\beta)^a \\ &= \delta A_\alpha^a (-\Delta_A \delta_{\alpha\beta} - 2i adF_{\alpha\beta})^{ac} \delta A_\beta^c \\ &= \delta A_\alpha^a (-(\Delta_A)^{ac} \delta_{\alpha\beta} + 2f^{abc} F_{\alpha\beta}^b) \delta A_\beta^c \end{aligned} \quad (3.20)$$

where $adF_{\alpha\beta}(\dots) = [F_{\alpha\beta}, \dots]$, $(adF_{\alpha\beta})^{ac} = if^{abc} F_{\alpha\beta}^b$ and

$$(\Delta_A)^{ac} = (D_\alpha)^{ad} (D_\alpha)^{dc} = \Delta^2 \delta^{ac} + i\partial_\alpha A_\alpha^{ac} + 2iA_\alpha^{ac} \partial_\alpha - A_\alpha^{ad} A_\alpha^{dc} \quad (3.21)$$

QED.

The following identity holds:

$$Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - 2i adF_{\alpha\beta}) = Det^{-1/2}(-\Delta_A \delta_{\alpha\beta}) Det^{-1/2}(1 - 2i(-\Delta_A)^{-1} adF_{\alpha\beta}) \quad (3.22)$$

The first factor gives:

$$Det^{-1/2}(-\Delta_A \delta_{\alpha\beta}) = Det^{-2}(-\Delta_A) \quad (3.23)$$

Therefore the one-loop effective action reads:

$$e^{-\Gamma_{1-loop}(A)} = e^{-S_{YM}(A)} Det^{-1/2}(1 - 2i(-\Delta_A)^{-1} adF_{\alpha\beta}) Det^{-1}(-\Delta_A) \quad (3.24)$$

The first determinant is the spin contribution while the second determinant is the orbital contribution.

We can factorize away a trivial infinite constant from the orbital contribution:

$$\begin{aligned} Det^{-1}(-\Delta_A) &= Det^{-1}(-\Delta^2 - i\partial_\alpha A_\alpha - 2iA_\alpha \partial_\alpha + A_\alpha A_\alpha) \\ &= Det^{-1}(-\Delta^2) Det^{-1}(1 + (-\Delta^2)^{-1}(-i\partial_\alpha A_\alpha - 2iA_\alpha \partial_\alpha + A_\alpha A_\alpha)) \end{aligned} \quad (3.25)$$

where the operators occurring in Eq.(3.25) are now defined by Eq.(3.21). Using

$$Det(1 + M) = e^{Tr \log(1+M)} = e^{Tr M - Tr(M)^2/2 + \dots} \quad (3.26)$$

at the lowest non-trivial order we get:

$$\begin{aligned} Det^{-1}(-\Delta_A) &= Det^{-1}(-\Delta) \exp\left(-Tr\left((-\Delta)^{-1}(-i\partial_\alpha A_\alpha - 2iA_\alpha \partial_\alpha + A_\alpha A_\alpha)\right)\right) \\ &\exp\left(Tr\left((-\Delta)^{-1}(-i\partial_\alpha A_\alpha - 2iA_\alpha \partial_\alpha + A_\alpha A_\alpha)\right)(-\Delta)^{-1}(-i\partial_\alpha A_\alpha - 2iA_\alpha \partial_\alpha + A_\alpha A_\alpha)\right)/2 \end{aligned} \quad (3.27)$$

where the trace is over the space-time, the Lie algebra and the vector indices. The term $Det^{-1}(-\Delta)$ is an irrelevant constant while the Lie algebra trace of the term linear in A_α vanishes. The term $Tr((-\Delta)^{-1}A_\alpha A_\alpha)$ is a quadratically divergent tadpole that cancels in any gauge invariant regularization scheme, since it would give rise to a mass counterterm for the gauge connection. Therefore it can be ignored. There remains an interesting divergence:

$$Det^{-1}(-\Delta_A) \sim \exp\left(Tr\left((-\Delta)^{-1}(i\partial_\alpha A_\alpha + 2iA_\alpha \partial_\alpha)\right)(-\Delta)^{-1}(i\partial_\alpha A_\alpha + 2iA_\alpha \partial_\alpha)\right)/2 \quad (3.28)$$

that evaluated in momentum space leads to:

$$\exp\left(\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 p}{(2\pi)^4} tr(A_\alpha(-k)A_\beta(k)) \frac{(2p_\alpha - k_\alpha)(2p_\beta - k_\beta)}{p^2(p-k)^2}\right) \quad (3.29)$$

where the trace tr on the Lie algebra indices refers to the matrices defined in Eq.(3.21). The logarithmically divergent part of the integral over $d^4 p$ has to be transverse in such a way that

$$\int \frac{d^4 p}{(2\pi)^4} \frac{(2p_\alpha - k_\alpha)(2p_\beta - k_\beta)}{p^2(p-k)^2} = \Pi(k^2)(k^2 \delta_{\alpha\beta} - k_\alpha k_\beta) + \dots \quad (3.30)$$

where the dots stand for the quadratically divergent part that can be ignored because of the aforementioned reasons. Taking the trace over the vector indices one gets:

$$\int \frac{d^4 p}{(2\pi)^4} \frac{4p^2 + k^2 - 4pk}{p^2(p-k)^2} = 3k^2 \Pi(k^2) + \dots \quad (3.31)$$

We are interested in extracting the logarithmic divergencies by expanding the denominator in powers of k/p up to the appropriate order:

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^4} \frac{4p^2 + k^2 - 4pk}{p^2(p-k)^2} &= \int \frac{d^4 p}{(2\pi)^4} \frac{4p^2 + k^2 - 4pk}{p^4(1 + (k^2 - 2kp)/p^2)} \\ &\sim \int \frac{d^4 p}{(2\pi)^4} \frac{-4k^2 + k^2 + 8(pk)^2/p^2}{p^4} = - \int \frac{d^4 p}{(2\pi)^4} \frac{k^2}{p^4} = - \frac{2k^2}{(4\pi)^2} \log \frac{\Lambda}{\mu} = 3k^2 \Pi(k^2) \end{aligned} \quad (3.32)$$

where we have replaced

$$p_\alpha p_\beta \rightarrow \frac{1}{4} p^2 \delta_{\alpha\beta} \quad (3.33)$$

into the integral and similarly $(pk)^2 \rightarrow p^2 k^2/4$ and we have regularized

$$\int \frac{d^4 p}{p^4} = 2\pi^2 \int_0^\infty \frac{dp}{p} \rightarrow 2\pi^2 \log \frac{\Lambda}{\mu} \quad (3.34)$$

Therefore:

$$\Pi(k^2) = - \frac{2}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \quad (3.35)$$

Hence the orbital contribution to the beta function is:

$$\begin{aligned} \text{Det}^{-1}(-\Delta_A) &\sim \exp\left(-\frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \int \frac{d^4 k}{(2\pi)^4} \text{tr}(A_\alpha(k^2 \delta_{\alpha\beta} - k_\alpha k_\beta) A_\beta)\right) \\ &= \exp\left(-\frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \int \frac{d^4 k}{(2\pi)^4} A_\alpha^{ac}(k^2 \delta_{\alpha\beta} - k_\alpha k_\beta) A_\beta^{ca}\right) \\ &= \exp\left(-\frac{N}{3(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right)^2 \frac{1}{4} \int d^4 x (F_{\alpha\beta}^a)^2\right) \end{aligned} \quad (3.36)$$

where in the last step we used:

$$f^{abc} f^{abd} = N \delta^{cd} \quad (3.37)$$

and

$$\frac{1}{4} \int d^4 x F_{\alpha\beta}^a F_{\alpha\beta}^a \sim \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} A_\alpha^a(-k) A_\beta^a(k) (k^2 \delta_{\alpha\beta} - k_\alpha k_\beta) \quad (3.38)$$

at quadratic order.

Now we have to compute the spin contribution to the effective action. Since $\text{tr} F_{\alpha\beta} = 0$ up to quadratic order in $F_{\alpha\beta}$ we get:

$$\text{Det}^{-1/2}(1 - 2i(-\Delta_A)^{-1} adF_{\alpha\beta}) = \exp\left(-\text{Tr}((-\Delta_A)^{-1} adF_{\alpha\beta} (-\Delta_A)^{-1} adF_{\beta\alpha})\right) \quad (3.39)$$

At lowest order $(-\Delta_A) \sim (-\Delta)$, therefore

$$\begin{aligned} \text{Tr}((-\Delta)^{-1} adF_{\alpha\beta} (-\Delta)^{-1} adF_{\beta\alpha}) &= \int d^4 x \int d^4 y \text{tr}(G(x-y) adF_{\alpha\beta}(y) G(y-x) adF_{\beta\alpha}(x)) \\ &= -N \int d^4 x \int d^4 y G(x-y)^2 F_{\alpha\beta}^a(y) F_{\alpha\beta}^a(x) \end{aligned} \quad (3.40)$$

where in coordinate space

$$G(x-y) = \frac{1}{4\pi^2(x-y)^2} \quad (3.41)$$

and

$$\text{tr}(adF_{\alpha\beta}adF_{\beta\alpha}) = (adF_{\alpha\beta})^{ac}(adF_{\beta\alpha})^{ca} = if^{abc}F_{\alpha\beta}^b if^{cda}F_{\beta\alpha}^d = -NF_{\alpha\beta}^c F_{\alpha\beta}^c \quad (3.42)$$

Assuming that the background field carries momenta much smaller than the fluctuating field we can expand $F_{\alpha\beta}(y) = F_{\alpha\beta}(x) + \dots$ by Taylor series and keep the first term since we are interested only in the divergent terms. Thus defining $z = x - y$

$$\begin{aligned} \text{Tr}((-\Delta)^{-1}adF_{\alpha\beta}(-\Delta)^{-1}adF_{\beta\alpha}) &\sim -\frac{N}{(4\pi^2)^2} \int \frac{d^4z}{z^4} \int d^4x (F_{\alpha\beta}^a)^2 \\ &= -\frac{2\pi^2 N}{(4\pi^2)^2} \log \frac{\Lambda}{\mu} \int d^4x (F_{\alpha\beta}^a)^2 \\ &= -\frac{4N}{(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right)^2 \frac{1}{4} \int d^4x (F_{\alpha\beta}^a)^2 \end{aligned} \quad (3.43)$$

Therefore at this order the divergent part reads:

$$\text{Det}^{-1/2}(1 - 2i(-\Delta_{\bar{A}})^{-1} adF_{\alpha\beta}) \sim \exp\left(\frac{4N}{(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right)^2 \frac{1}{4} \int d^4x (F_{\alpha\beta}^a)^2\right) \quad (3.44)$$

Finally the local part of the one-loop effective action reads:

$$\begin{aligned} \Gamma_{1-loop} &= S_{YM} + \left(\frac{N}{3(4\pi)^2} - \frac{4N}{(4\pi)^2}\right) \log\left(\frac{\Lambda}{\mu}\right)^2 \frac{1}{4} \int d^4x (F_{\alpha\beta}^a)^2 \\ &= \left(\frac{1}{g^2(\Lambda)} - \frac{11N}{3(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right)^2\right) \frac{1}{4} \int d^4x (F_{\alpha\beta}^a)^2 \end{aligned} \quad (3.45)$$

Therefore the bare coupling constant, $g(\Lambda)$, renormalizes as:

$$\frac{1}{2g^2(\Lambda)} = \frac{1}{2g^2(\mu)} + \frac{11}{3} \frac{1}{(4\pi)^2} \log \frac{\Lambda}{\mu} \quad (3.46)$$

or

$$g^2(\Lambda) = \frac{g^2(\mu)}{1 + 2\frac{11}{3(4\pi)^2} g^2(\mu) \log \frac{\Lambda}{\mu}} \quad (3.47)$$

that is the solution at one loop of the equation that defines the β function:

$$\begin{aligned} \beta(g) &= \frac{\partial g}{\partial \log \Lambda} = -\beta_0 g^3 + \dots \\ \beta_0 &= \frac{11}{3(4\pi)^2} \end{aligned} \quad (3.48)$$

Eq. (3.46) can be also written as:

$$\Lambda e^{-\frac{1}{2\beta_0 g^2(\Lambda)}} = \mu e^{-\frac{1}{2\beta_0 g^2(\mu)}} \quad (3.49)$$

Thus the combination:

$$\Lambda_{YM} = \Lambda e^{-\frac{1}{2\beta_0 g^2(\Lambda)}} \quad (3.50)$$

is independent on the cutoff Λ and it is a renormalization group invariant at one loop.

3.2 A SUSY interlude: cohomological localization by the Nicolai map in $\mathcal{N} = 1$ SUSY YM

Shortly after Nicolai discovered [29, 30] that the vanishing of the vacuum energy in an unbroken supersymmetric theory implies the existence of a change of variables whose Jacobian formally sets the functional integral in ultralocal form, De Alfaro, Fubini, Furlan and Veneziano²⁶ [31, 32] worked out explicitly the Nicolai map in the case of $\mathcal{N} = 1$ SUSY YM.

They found that in this case the Nicolai map is actually the change of variables from the gauge connection to the ASD part of its curvature in the light-cone gauge, with the property that its Jacobian cancels the gluino determinant.

In this section we reconsider the Nicolai map of $\mathcal{N} = 1$ SUSY YM paying particular attention to the fact that, while generically in function space the aforementioned cancellation occurs exactly, in a renormalizable but not finite supersymmetric quantum field theory such as $\mathcal{N} = 1$ SUSY YM there should exist loci in function space where the cancellation occurs in fact only up to zero modes.

Indeed if it were not so the theory would be in fact mapped into a theory of free fields with zero beta function.

It is quite clear that the Jacobian of the Nicolai map develops zero modes precisely at loci in function space where the Nicolai map fails to be one-to-one. If these loci are characterized by moduli then there is a continuous family of zero modes and the Pauli-Villars regularization of these zero modes in the functional integral furnishes in general some contribution to the beta function of the theory, thus resolving the puzzle that the Nicolai map maps formally the theory into a theory of Gaussian fields with vanishing beta function.

Understanding the distribution of these zero modes as a function of the Gaussian random field which the theory is generically mapped on is in fact a non-perturbative problem seemingly as difficult as performing the functional integral in the original variables.

However, we point out in this section that, thanks to the tautological nilpotent Parisi-Sourlas [35, 36] BRS symmetry²⁷ associated to the cancellation of the Jacobian with the gluino determinant, the partition function with the insertion of certain BRS invariant operators necessary to saturate the zero modes of the gluino determinant is in fact localized by cohomological localization (see sect.(1)) on those (Euclidean) instantons²⁸ that can be analytically continued to ultrahyperbolic signature (this constraint arises because the cancellations due to the Nicolai map actually occur only in the light-cone gauge).

Thus because of the localization the occurrence of the zero modes for these special BRS-invariant observables can be understood semiclassically as they coincide with the moduli of the instantons.

The immediate consequence of this localization is an exact formula for the beta function of $\mathcal{N} = 1$ SUSY YM, that quite obviously turns out to be the NSVZ beta function [38, 39], by an almost verbatim reproduction of their original computation.

²⁶We would like to thank Gabriele Veneziano for several discussions about the Nicolai map over the years and at the GGI.

²⁷We may consider this BRS symmetry as a remnant of the supersymmetry after gauge-fixing.

²⁸We refer here to instantons as configurations satisfying the SD equations, without any implication about being defined on S^4 . In fact in the present framework the instantons are naturally defined on $S^2 \times S^2$, because of the analytic continuation from Euclidean (4,0) to ultra-hyperbolic (2,2) signature, that can be handled by twistor techniques [93] (see also the Appendix in [94]).

We can now start to work out the details. It turns out that to get localization we need only the information that the gluino determinant cancels the Jacobian of the Nicolai map. Let us suppose at first that there are no zero modes. Using the identity:

$$\text{Tr}(F_{\alpha\beta}^2) = \text{Tr}(F_{\alpha\beta}^-)^2/2 + \text{Tr}(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \quad (3.51)$$

the partition function reads:

$$Z = \left[\int \exp\left(-\frac{N16\pi^2 Q}{2g^2} - \frac{1}{8g^2} \int \text{Tr}(F_{\alpha\beta}^- F^{-\alpha\beta}) d^4x\right) \text{Det}\left(\frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma}\right) \delta A \right]_{A_+=0} \quad (3.52)$$

where fields live in the adjoint representation of the Lie algebra of $SU(N)$. In Eq.(3.52) we have just expressed the existence of the Nicolai map by inserting its inverse Jacobian, $\text{Det}\left(\frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma}\right)$, in place of the gluino determinant. Thus the theory is mapped into a theory of free fields:

$$Z = \int \exp\left(-\frac{N16\pi^2 Q}{2g^2} - \frac{1}{8g^2} \int \text{Tr}(F_{\alpha\beta}^- F^{-\alpha\beta}) d^4x\right) \delta F_{\alpha\beta}^- \quad (3.53)$$

that holds generically in function space where zero modes do not occur. In fact, taking into account the zero modes, we get:

$$Z = \int \Lambda^{n_b[F_{\alpha\beta}^-] - \frac{n_f}{2}[F_{\alpha\beta}^-]} \int_{\mathcal{M}} \frac{Pf < \frac{\delta A[F_{\alpha\beta}^-]}{\delta m}, \frac{\delta A[F_{\alpha\beta}^-]}{\delta m} >}{Pf < \eta[F_{\alpha\beta}^-], \eta[F_{\alpha\beta}^-] >} \exp\left(-\frac{N16\pi^2 Q[F_{\alpha\beta}^-]}{2g^2} - \frac{1}{8g^2} \int \text{Tr}(F_{\alpha\beta}^- F^{-\alpha\beta}) d^4x\right) \delta F_{\alpha\beta}^- \quad (3.54)$$

where the extra factor is the contribution of the Pfaffians of the bosonic and fermionic zero modes [92] and of the associated Pauli-Villars regulator, whose origin is explained below. Going back to Eq.(3.52) we can write it in a more suggestive form introducing anticommuting fields, $(\rho_{\alpha\beta}, \eta_\gamma)$:

$$Z = \left[\int \exp\left(-\frac{N16\pi^2 Q}{2g^2} - \frac{1}{8g^2} \int d^4x \text{Tr}(F_{\alpha\beta}^- F^{-\alpha\beta} + \rho_{\alpha\beta} \frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \eta_\gamma)\right) \delta A \delta \rho \delta \eta \right]_{A_+=0} \quad (3.55)$$

The non-topological term can be rewritten as:

$$\left[\exp \int \left(-\frac{1}{8g^2} \int d^4x \text{Tr}(E_{\alpha\beta}^- E^{-\alpha\beta} + iE_{\alpha\beta}^- F^{-\alpha\beta} - i\rho_{\alpha\beta} \frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \eta_\gamma)\right) \delta E^- \delta A \delta \rho \delta \eta \right]_{A_+=0} \quad (3.56)$$

The functional integral here has to be interpreted as either in Minkowski space-time (3, 1) or in ultrahyperbolic signature (2, 2), since otherwise the light-cone gauge does not exist. In Minkowski signature an overall factor of i in front of the action is understood but not explicitly displayed. In ultrahyperbolic signature the Gaussian integral is defined by analytic continuation. In this form the partition function enjoys the following tautological Parisi-Sourlas BRS symmetry²⁹:

$$Q_{BRS} A_\gamma = \eta_\gamma$$

²⁹To the best of our knowledge the existence of this symmetry in $\mathcal{N} = 1$ SUSY YM has never been related to cohomological localization, presumably because it leads to results inconsistent with the NSVZ beta function if zero modes are not taken into account properly. However, the zero dimensional version of the Nicolai map, for which of course no zero modes occur, has been related to cohomological localization in [37].

$$\begin{aligned}
 Q_{BRS}\eta_\gamma &= 0 \\
 Q_{BRS}\rho_{\alpha\beta} &= E_{\alpha\beta}^- \\
 Q_{BRS}E_{\alpha\beta}^- &= 0
 \end{aligned} \tag{3.57}$$

with $Q_{BRS}^2 = 0$. The consequence of the existence of this symmetry is that the term $E_{\alpha\beta}^- E^{-\alpha\beta}$ is a coboundary, since $E_{\alpha\beta}^- E^{-\alpha\beta} = Q(\rho_{\alpha\beta} E^{-\alpha\beta})$. Thus it can be cancelled without changing the cohomology class of the integrand. The resulting functional integral reduces to:

$$\left[\exp \int \left(-\frac{1}{8g^2} \int d^4x \text{Tr} (iE_{\alpha\beta}^- F^{-\alpha\beta} - i\rho_{\alpha\beta} \frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \eta_\gamma) \right) \delta E^- \delta A \delta \rho \delta \eta \right]_{A_+=0} \tag{3.58}$$

Thus the complete partition function reads:

$$Z = \left[\int \exp \left(-\frac{N16\pi^2 Q}{2g^2} \right) \delta(F_{\alpha\beta}^-) \text{Det} \left(\frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \right) \delta A \right]_{A_+=0} \tag{3.59}$$

that expresses the fact that the partition function is localized on those instantons that can be analytically continued to Minkowski³⁰ or ultrahyperbolic signature, thus violating the assumption that there are no zero modes. In case we assume the existence of zero modes from the start we have to insert some fermionic contribution to take into account the fermionic zero modes of the gluino determinant. This can be done by identifying lexicographically the gluino zero modes, say λ_i where i is a spinor index, with η_i where i is a vector index:

$$\begin{aligned}
 & \left[\int \Lambda^{-\frac{n_f}{2}} (\text{Tr} \varepsilon_{ij} \eta_i \eta_j)^{\frac{n_f}{2}} \exp \left(-\frac{N16\pi^2 Q}{2g^2} \right) \right. \\
 & \left. \exp \left[-\frac{1}{8g^2} \int d^4x \text{Tr} (E_{\alpha\beta}^- E^{-\alpha\beta} + iE_{\alpha\beta}^- F^{-\alpha\beta} - i\rho_{\alpha\beta} \frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \eta_\gamma) \right] \delta E^- \delta A \delta \rho \delta \eta \right]_{A_+=0} \tag{3.60}
 \end{aligned}$$

Afterwards everything goes through as before and the localized partition function is:

$$Z = \exp \left(-\frac{N16\pi^2 Q(A)}{2g_W^2} \right) \Lambda^{n_b - \frac{n_f}{2}} \int_{\mathcal{M}_Q} \frac{Pf < \frac{\delta A}{\delta m}, \frac{\delta A}{\delta m} >}{Pf < \eta, \eta >} \tag{3.61}$$

where we have explicitly displayed the Pfaffians (i.e. square root of determinants) that occur evaluating the residual integral on the instantons moduli associated to gauge field and gluino zero modes. This is precisely the *NSVZ* result originally found by evaluating the gluino condensate, but for the fact that there is a constraint of analytic continuation for the instantons, say to ultrahyperbolic signature, that can be handled by twistor techniques [93]. In particular the natural framework for the analytic continuation is to start with twistors on Euclidean $S^2 \times S^2$ and analytically continue to ultrahyperbolic signature. Doing so the conformal compactification of ultrahyperbolic space-time occurs, that is $S^2 \times S^2 / Z_2$, where Z_2 acts by antipodal involution $\sigma(x, y) = (-x, -y)$ [93]. Thus the analytic continuation defines in fact the double cover, $S^2 \times S^2$, of the conformal compactification, $S^2 \times S^2 / Z_2$, of ultrahyperbolic space-time. As a consequence the possible values of the second

³⁰In this case the gauge group must be complexified since *SD* equations exist only in Euclidean or ultrahyperbolic signature for Hermitian connections [93, 94].

Chern class, Q , (the topological charge) can only be even for ultrahyperbolic instantons [93]. Thus the gluino condensate cannot be saturated by single instantons. This is perhaps related to the old controversy about the strong versus weak coupling evaluation of the condensate (for reviews see [33, 34]), but it is a matter too far away from our main subject to discuss further.

In any case the $NSVZ$ beta function is tautologically reproduced. We show the computation because it is very useful to understand the pure YM case.

A striking consequence of the localization on instantons is that the beta function for the Wilsonian coupling constant, g_W , is one-loop exact since the only sources of divergences are the zero modes via the Pauli-Villars regulator:

$$\frac{16\pi^2 Q}{2g_W^2(\mu)} = \frac{16\pi^2 Q}{2g_W^2(\Lambda)} - (n_b - \frac{n_f}{2}) \log \frac{\Lambda}{\mu} \quad (3.62)$$

Now since $n_b = 4NQ$ and $n_f = 2NQ$ the result for the Wilsonian beta function follows:

$$\frac{1}{2g_W^2(\mu)} = \frac{1}{2g_W^2(\Lambda)} - \frac{3}{(4\pi)^2} \log \frac{\Lambda}{\mu} \quad (3.63)$$

or differentiating with respect to $\log \Lambda$:

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3 \quad (3.64)$$

with

$$\beta_0 = \frac{3}{(4\pi)^2} \quad (3.65)$$

From the one-loop exactness of the Wilsonian beta function it follows the $NSVZ$ formula for the canonical beta function. Indeed the renormalization of the canonical coupling is obtained rescaling the fields in canonical form, i.e. in such a way that the quadratic part of the action is normalized in order to be g independent:

$$\begin{aligned} Z &= \exp\left(-\frac{N16\pi^2 Q(gA_c)}{2g_W^2}\right) \Lambda^{n_b - \frac{n_f}{2}} \int_{\mathcal{M}_Q} \frac{Pf < \frac{\delta g A_c}{\delta m}, \frac{\delta g A_c}{\delta m} >}{Pf < g\eta_c, g\eta_c >} \\ &= \exp\left(-\frac{N16\pi^2 Q(gA_c)}{2g_W^2}\right) \Lambda^{n_b - \frac{n_f}{2}} g^{n_b - n_f} \int_{\mathcal{M}_Q} \frac{Pf < \frac{\delta A_c}{\delta m}, \frac{\delta A_c}{\delta m} >}{Pf < \eta_c, \eta_c >} \\ &= \exp\left(-\frac{N16\pi^2 Q(gA_c)}{2g^2}\right) \Lambda^{n_b - \frac{n_f}{2}} \int_{\mathcal{M}_Q} \frac{Pf < \frac{\delta A_c}{\delta m}, \frac{\delta A_c}{\delta m} >}{Pf < \eta_c, \eta_c >} \end{aligned} \quad (3.66)$$

where we have defined:

$$-\frac{N16\pi^2 Q(gA_c)}{2g^2} = -\frac{N16\pi^2 Q(gA_c)}{2g_W^2} + (4NQ(gA_c) - 2NQ(gA_c)) \log g \quad (3.67)$$

or

$$\frac{1}{2g_W^2} = \frac{1}{2g^2} + \frac{2}{(4\pi)^2} \log g \quad (3.68)$$

Differentiating with respect to $\log \Lambda$ the $NSVZ$ beta function follows:

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\frac{3}{(4\pi)^2} g^3}{1 - \frac{2}{(4\pi)^2} g^2} \quad (3.69)$$

QED

3.3 Non-SUSY Nicolai map in pure YM

Now we have set the stage for the non-SUSY Nicolai map in pure YM.

In the pure YM case we define our change of variables in a more general way than in the SUSY case [31, 32], by allowing arbitrary gauge-fixing. Indeed, while in the SUSY theory we need the light-cone gauge to get the cancellations of the determinants, in the pure YM theory cancellations do not occur at all. Therefore there is no point in choosing a non-covariant gauge.

The YM partition function is (for definitions see sect.(3.1)):

$$Z = \int \exp\left(-\frac{16\pi^2 N Q}{2g^2} - \frac{N}{4g^2} \int \text{tr}_f(F_{\alpha\beta}^-)^2 d^4x\right) \delta A \quad (3.70)$$

where we used the identity $\text{Tr}(F_{\alpha\beta}^2) = \text{Tr}(F_{\alpha\beta}^-)^2/2 + \text{Tr}(F_{\alpha\beta}\tilde{F}_{\alpha\beta})$ as in the SUSY case. We change variables from the connection to the ASD curvature by introducing in the functional integral the appropriate resolution of the identity:

$$1 = \int \delta(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) \delta \mu_{\alpha\beta}^- \quad (3.71)$$

Thus

$$Z = \int \exp\left(-\frac{16\pi^2 N Q}{g^2} - \frac{N}{4g^2} \int \text{tr}_f(\mu_{\alpha\beta}^-)^2 d^4x\right) \delta(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) \delta \mu_{\alpha\beta}^- \delta A \quad (3.72)$$

Exchanging the order of integration we can now perform the integral on the gauge connection because of the delta function. The easiest way to do this is defining the delta function as:

$$\delta(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) = \lim_{\varepsilon \rightarrow 0} N^{-1}(\varepsilon) e^{-\frac{N}{4\varepsilon} \int \text{tr}_f(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-)^2} \quad (3.73)$$

with $N(\varepsilon)$ an irrelevant normalization factor. Thus the Jacobian of the map to the ASD variables can be evaluated as a Gaussian integral for the quadratic form obtained from the expansion to quadratic order of:

$$\text{tr}_f(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-)^2 = \text{tr}_f(P^-(F_{\alpha\beta} - \mu_{\alpha\beta}))^2 \sim \text{tr}_f(P^-(D_{[\alpha} \delta A_{\beta]}))^2 \quad (3.74)$$

where:

$$F_{\alpha\beta} = P^- F_{\alpha\beta} + P^+ F_{\alpha\beta} = \frac{1}{2} F_{\alpha\beta}^- + \frac{1}{2} F_{\alpha\beta}^+ \quad (3.75)$$

and

$$P_{\alpha\beta\gamma\delta}^- = \frac{1}{2} \delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} \quad (3.76)$$

Therefore, integrating by parts freely and using the same identities as in sect.(3.1), we get:

$$\begin{aligned} \text{tr}_f P^-(D_{[\alpha} \delta A_{\beta]})^2 &= \text{tr}_f (D_{[\alpha} \delta A_{\beta]} - \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} D_{[\gamma} \delta A_{\delta]})^2 \\ &= \text{tr}_f (2(D_{[\alpha} \delta A_{\beta]})^2 - D_{[\alpha} \delta A_{\beta]} \varepsilon_{\alpha\beta\gamma\delta} D_{[\gamma} \delta A_{\delta]}) \\ &= \text{tr}_f (-4\delta A_{\alpha} \Delta_{\alpha} \delta A_{\alpha} - 4(D_{\alpha} \delta A_{\alpha})^2 + 4iF_{\alpha\beta} [\delta A_{\alpha}, \delta A_{\beta}] - 4D_{\alpha} \delta A_{\beta} \varepsilon_{\alpha\beta\gamma\delta} D_{\gamma} \delta A_{\delta}) \end{aligned}$$

$$\begin{aligned}
 &= 4tr_f(-\delta A_\alpha \Delta_A \delta A_\alpha - (D_\alpha \delta A_\alpha)^2 + iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta] + D_\gamma D_\alpha \delta A_\beta \varepsilon_{\alpha\beta\gamma\delta} \delta A_\delta) \\
 &= 4tr_f(-\delta A_\alpha \Delta_A \delta A_\alpha - (D_\alpha \delta A_\alpha)^2 + iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta] + \frac{1}{2}([D_\gamma, D_\alpha] \cdot \delta A_\beta \varepsilon_{\gamma\alpha\beta\delta} \delta A_\delta) \\
 &= 4tr_f(-\delta A_\alpha \Delta_A \delta A_\alpha - (D_\alpha \delta A_\alpha)^2 + iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta] + i[\tilde{F}_{\beta\delta}, \delta A_\beta] \delta A_\delta) \\
 &= 4tr_f(-\delta A_\alpha \Delta_A \delta A_\alpha - (D_\alpha \delta A_\alpha)^2 + iF_{\alpha\beta}[\delta A_\alpha, \delta A_\beta] + i\tilde{F}_{\alpha\beta}[\delta A_\alpha, \delta A_\beta]) \\
 &= 4tr_f(\delta A_\alpha (-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta - 2i ad(P^+ F)_{\alpha\beta}) \delta A_\beta) \tag{3.77}
 \end{aligned}$$

The partition function becomes:

$$Z = \int \exp\left(-\frac{16\pi^2 NQ}{g^2} - \frac{N}{4g^2} \int tr_f(\mu_{\alpha\beta}^-)^2 d^4x\right) Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta - iF_{\alpha\beta}^+) \delta\mu^- \tag{3.78}$$

The determinant in Eq.(3.78) does not exist unless the gauge is fixed. This is most conveniently done in a background Feynman gauge:

$$\begin{aligned}
 Z &= \lim_{\varepsilon \rightarrow 0} \int \delta A \delta C \delta\mu^- \exp(-S_{YM}) e^{-\frac{N}{4\varepsilon} \int tr_f(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-)^2 d^4x} \\
 &\quad Det(-\Delta_A) e^{-\frac{N}{\varepsilon} \int d^4x tr_f(C^2)} \delta(D_\alpha \delta A_\alpha - C) \tag{3.79}
 \end{aligned}$$

As a consequence the gauge-fixed partition function in the ASD variables is:

$$\begin{aligned}
 Z &= \int \exp\left(-\frac{16\pi^2 NQ}{g^2} - \frac{N}{4g^2} \int tr_f(\mu_{\alpha\beta}^-)^2 d^4x\right) \\
 &\quad Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - iF_{\alpha\beta}^+) Det(-\Delta_A) \delta\mu^- \tag{3.80}
 \end{aligned}$$

that because of the argument displayed below can be rewritten as:

$$\begin{aligned}
 Z &= \int \exp\left(-\frac{16\pi^2 NQ}{g^2} - \frac{N}{4g^2} \int tr_f(\mu_{\alpha\beta}^-)^2 d^4x\right) \\
 &\quad Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - i\mu_{\alpha\beta}^-) Det(-\Delta_A) \Lambda^{n_b[\mu^-]} \omega^{\frac{n_b[\mu^-]}{2}} \delta\mu^- \tag{3.81}
 \end{aligned}$$

where ω is a Kahler form on the moduli induced by a Kahler form on the connections. A possible choice for ω is:

$$\omega_I = \frac{1}{2\pi} \int d^4x tr_f \left(\frac{\delta A_z}{\delta m_i} \frac{\delta A_{\bar{z}}}{\delta \bar{m}_k} - \frac{\delta A_u}{\delta m_i} \frac{\delta A_{\bar{u}}}{\delta \bar{m}_k} \right) \delta m_i \wedge \delta \bar{m}_k \tag{3.82}$$

As in the *SUSY* case zero modes have to occur, but in the pure *YM* case there are also other contributions to the beta function due to the lack of cancellation of determinants (see sect.(3.4)).

Let us introduce the matrices $\sigma_\alpha = (1, i\tau)$ with τ the three Hermitian Pauli matrices and the self-dual and anti-selfdual matrices (we use the same notation as in [33]):

$$\begin{aligned}
 \sigma_{\alpha\beta} &= \frac{1}{4}(\sigma_\alpha \bar{\sigma}_\beta - \sigma_\beta \bar{\sigma}_\alpha) \\
 \bar{\sigma}_{\alpha\beta} &= \frac{1}{4}(\bar{\sigma}_\alpha \sigma_\beta - \bar{\sigma}_\beta \sigma_\alpha) \tag{3.83}
 \end{aligned}$$

The three variations [33]:

$$P^-(D_{[\alpha}\delta A_{\beta]}) \quad (3.84)$$

can be rewritten as:

$$(\tau_i)_{\dot{b}}^{\dot{a}} \bar{D}^{\dot{b}a} \delta A_{a\dot{a}} = Tr(\tau_i \bar{D} \cdot \delta A) \quad (3.85)$$

and the forth variation:

$$D_{\dot{\alpha}} \delta A_{\alpha} \quad (3.86)$$

as:

$$\bar{D}^{\dot{a}a} \delta A_{a\dot{a}} = Tr(\bar{D} \cdot \delta A) \quad (3.87)$$

in such a way that the four variations can be written together as:

$$\bar{D}^{\dot{a}a} \delta A_{a\dot{b}} \quad (3.88)$$

where we have defined:

$$\begin{aligned} D_{\dot{\alpha}} &= \sigma_{\alpha} D_{\alpha} \\ \bar{D}^{\dot{\alpha}} &= \bar{\sigma}_{\alpha} D_{\alpha} \end{aligned} \quad (3.89)$$

Therefore the Euclidean invariant positive semidefinite quadratic form:

$$\sum_{\alpha>\beta} tr_f(P^-(D_{[\alpha}\delta A_{\beta]}))^2 + tr_f(D_{\dot{\alpha}} \delta A_{\alpha})^2 \quad (3.90)$$

can be written as:

$$\begin{aligned} &\sum_i tr_f |Tr(\tau_i \bar{D} \cdot \delta A)|^2 + tr_f |Tr(\bar{D} \cdot \delta A)|^2 \\ &= \sum_{\dot{a}\dot{b}} tr_f |\bar{D}^{\dot{a}a} \delta A_{a\dot{b}}|^2 \\ &= -tr_f Tr(\delta \bar{A} \bar{D} \cdot \bar{D} \cdot 1 \delta A) \end{aligned} \quad (3.91)$$

where in the second line we have used the completeness of the σ_{α} over the 2×2 matrices. The trace, Tr , refers to the spin indices and we have freely integrated by parts. Therefore the partition function in the Feynman gauge:

$$\begin{aligned} Z &= \lim_{\varepsilon \rightarrow 0} \int \delta A \delta C \delta \mu^- \exp(-S_{YM}) e^{-\frac{N}{4\varepsilon} \int tr_f (F_{\alpha\beta}^- - \mu_{\alpha\beta}^-)^2 d^4x} \\ &\quad Det(-\Delta_A) e^{-\frac{N}{\varepsilon} \int d^4x tr_f (C^2)} \delta(D_{\dot{\alpha}} \delta A_{\alpha} - C) \end{aligned} \quad (3.92)$$

becomes:

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int \delta A \delta \mu^- \exp\left(-\frac{16\pi^2 N Q}{g^2} - \frac{N}{4g^2} \int tr_f (\mu_{\alpha\beta}^-)^2 d^4x\right) Det(-\Delta_A) \\ &\exp\left(-\frac{N}{\varepsilon} \sum_{\alpha>\beta} \int tr_f (P^-(D_{[\alpha}\delta A_{\beta]}))^2 d^4x\right) \exp\left(-\frac{N}{\varepsilon} \int d^4x tr_f (D_{\dot{\alpha}} \delta A_{\alpha})^2\right) \\ &= \int \delta \mu^- \exp\left(-\frac{16\pi^2 N Q}{g^2} - \frac{N}{4g^2} \int tr_f (\mu_{\alpha\beta}^-)^2 d^4x\right) Det(-\Delta_A) Det^{-\frac{1}{2}}(-\bar{D} \cdot \bar{D} \cdot \delta_{\dot{a}\dot{b}}) \end{aligned} \quad (3.93)$$

where:

$$- \mathcal{D} \bar{\mathcal{D}} = -\Delta_A 1 + \sigma_{\alpha\beta} F_{\alpha\beta} \quad (3.94)$$

Hence:

$$\text{Det}^{-\frac{1}{2}}(- \mathcal{D} \bar{\mathcal{D}} \delta_{\dot{a}\dot{b}}) = \text{Det}^{-1/2}(-\Delta_A \delta_{\alpha\beta} - iF_{\alpha\beta}^+) \quad (3.95)$$

In addition:

$$- \bar{\mathcal{D}} \mathcal{D} = -\Delta_A 1 + \bar{\sigma}_{\alpha\beta} F_{\alpha\beta} \quad (3.96)$$

and symmetrically:

$$\text{Det}^{-\frac{1}{2}}(- \bar{\mathcal{D}} \mathcal{D} \delta_{\dot{a}\dot{b}}) = \text{Det}^{-1/2}(-\Delta_A \delta_{\alpha\beta} - iF_{\alpha\beta}^-) \quad (3.97)$$

Now, since $\bar{\mathcal{D}} \mathcal{D}$ and $\mathcal{D} \bar{\mathcal{D}}$ have the same spectrum of non-zero modes, it follows that $-\Delta_A \delta_{\alpha\beta} - iF_{\alpha\beta}^-$ has the same non-zero modes as $-\Delta_A \delta_{\alpha\beta} - iF_{\alpha\beta}^+$, but it has no zero modes precisely when $-\Delta_A \delta_{\alpha\beta} - iF_{\alpha\beta}^+$ has. This is the case when the gauge connection that solves the equation of ASD type, $F_{\alpha\beta}^-(A(m_i)) = \mu_{\alpha\beta}^-$, has moduli, m_i . Indeed taking the derivative of this equation with respect to the moduli one gets:

$$\frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \frac{\delta A_\gamma}{\delta m_i} = 0 \quad (3.98)$$

that implies that the operator:

$$\frac{\delta F_{\alpha\beta}^-}{\delta A_\gamma} \delta A_\gamma = \frac{\delta(P^- F_{\alpha\beta})}{\delta A_\gamma} \delta A_\gamma = P^-(D_{[\alpha} \delta A_{\beta]}) \quad (3.99)$$

has zero modes, and therefore $\bar{\mathcal{D}} \mathcal{D}$ and $(-\Delta_A \delta_{\alpha\beta} - 2i \text{ad}(P^+ F)_{\alpha\beta})$ have zero modes too. QED

3.4 One-loop beta function of pure YM in the ASD variables

We can now use Eq.(3.81) as the definition of the partition function of YM in the ASD variables. We can apply the standard background field method of sect.(3.1) for the computation of the beta function in the ASD variables. The field $\mu_{\alpha\beta} = \bar{\mu}_{\alpha\beta} + \delta\mu_{\alpha\beta}$ can be decomposed in a background, $\bar{\mu}_{\alpha\beta}$, and a fluctuating field, $\delta\mu_{\alpha\beta}$. The correlations of the fluctuating field can contribute only starting from order of g^2 . Therefore the only $O(g^0)$ contributions, relevant for the one-loop beta function, arise from the functional determinants. To evaluate the effective action in the ASD variables it is most convenient to compare it with the standard one-loop effective action of sect.(3.1). In the standard background field method the quadratic form:

$$\frac{N}{2g^2} \int d^4x \text{tr}_f (F_{\alpha\beta})^2 \quad (3.100)$$

is expanded around a solution of the equation of motion, leading to the one-loop effective action:

$$\begin{aligned} Z_{1-loop} &= e^{-\Gamma_{1-loop}(\bar{A})} \\ &= e^{-S_{YM}(\bar{A})} \text{Det}^{-1/2}(-\Delta_{\bar{A}} \delta_{\alpha\beta} - 2i\bar{F}_{\alpha\beta}) \text{Det}(-\Delta_{\bar{A}}) \end{aligned} \quad (3.101)$$

In the ASD variables because of the delta function that defines the resolution of identity, the "action"

$$\lim_{\varepsilon \rightarrow 0} \frac{N}{4\varepsilon} \int \text{tr}_f (F_{\alpha\beta}^- - \mu_{\alpha\beta}^-)^2 \quad (3.102)$$

is expanded around the background $F_{\alpha\beta}^-(\bar{A} + \delta A) = \bar{\mu}_{\alpha\beta}^- + \delta \mu_{\alpha\beta}^-$, leading to the one-loop effective action in the ASD variables:

$$\begin{aligned} Z_{1-loop}^{ASD} &= e^{-\Gamma_{1-loop}^{ASD}(\bar{\mu})} \\ &= e^{-\frac{16\pi^2 N Q}{2g^2} - \frac{N}{g^2} \int \text{tr}_f (\frac{\bar{\mu}_{\alpha\beta}^-}{2})^2 d^4x} \\ &\int \Lambda^{n_b[\bar{\mu}^-]} \omega^{\frac{n_b[\bar{\mu}^-]}{2}} \text{Det}^{-1/2}(-\Delta_{\bar{A}} \delta_{\alpha\beta} - 2i ad \frac{\bar{\mu}_{\alpha\beta}^-}{2}) \text{Det}(-\Delta_{\bar{A}}) \end{aligned} \quad (3.103)$$

Thus the orbital contribution is the same as in the standard background field method. The difference is the spin term, $2iad(P^-F)_{\alpha\beta}$, as opposed to $2iadF_{\alpha\beta}$, and the possible contribution of the zero modes, whose occurrence is not generic but depends on the background. Let us suppose at first that zero modes do not occur. In sect.(3.1) we have seen that the combination of determinants

$$\text{Det}^{-1/2}(-\Delta_{\bar{A}} \delta_{\alpha\beta} - 2i ad F_{\alpha\beta}) \text{Det}(-\Delta_{\bar{A}}) \quad (3.104)$$

leads to the beta function:

$$\frac{1}{2g^2(\mu)} = \frac{1}{2g^2(\Lambda)} + \left(\frac{1}{3(4\pi)^2} - \frac{4}{(4\pi)^2} \right) \log \frac{\Lambda}{\mu} \quad (3.105)$$

where the first term in the brackets is the orbital contribution and the second one is the spin contribution.

In the ASD case the orbital part is the same one, while the spin part differs because of the substitution $F_{\alpha\beta} \rightarrow P^-F_{\alpha\beta} = \frac{\mu_{\alpha\beta}^-}{2}$. Thus both in the standard one-loop effective action and in the ASD variables the orbital contribution is:

$$\begin{aligned} &N \int \text{tr}_f (F_{\alpha\beta})^2 \frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \\ &= N(4\pi)^2 Q \frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} + \frac{N}{2} \int \text{tr}_f (F_{\alpha\beta}^-)^2 \frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \\ &= N(4\pi)^2 Q \frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} + N \int \text{tr}_f (P^-F_{\alpha\beta})^2 \frac{2}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \end{aligned} \quad (3.106)$$

On the contrary, while the spin contribution for the standard one-loop effective action is:

$$-N \int \text{tr}_f (F_{\alpha\beta})^2 \frac{4}{(4\pi)^2} \log \frac{\Lambda}{\mu} \quad (3.107)$$

in the ASD variables, because of the substitution $F_{\alpha\beta} \rightarrow P^-F_{\alpha\beta} = \frac{\mu_{\alpha\beta}^-}{2}$, is:

$$-N \int \text{tr}_f (P^-F_{\alpha\beta})^2 \frac{4}{(4\pi)^2} \log \frac{\Lambda}{\mu} \quad (3.108)$$

Hence

$$\begin{aligned}
 \Gamma_{1-loop}^{ASD} &= \left(\frac{1}{2g^2} + \frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \right) N(4\pi)^2 Q \\
 &+ N \int \text{tr}_f (P^- F_{\alpha\beta})^2 \left(\frac{1}{g^2} + \frac{2}{3(4\pi)^2} \log \frac{\Lambda}{\mu} - \frac{4}{(4\pi)^2} \log \frac{\Lambda}{\mu} \right) \\
 &= \left(\frac{1}{2g^2} + \frac{1}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \right) N(4\pi)^2 Q + 2N \int \text{tr}_f (P^- F_{\alpha\beta})^2 \left(\frac{1}{2g^2} - \frac{5}{3(4\pi)^2} \log \frac{\Lambda}{\mu} \right) \\
 &= \frac{N(4\pi)^2 Q Z_Q^{-1}}{2g^2} + \frac{2NZ^{-1}}{2g^2} \int \text{tr}_f (P^- F_{\alpha\beta})^2
 \end{aligned} \tag{3.109}$$

where:

$$\begin{aligned}
 Z_Q^{-1} &= 1 + \frac{2}{3} \frac{g^2}{(4\pi)^2} \log \frac{\Lambda}{\mu} \\
 Z^{-1} &= 1 - \frac{10}{3} \frac{g^2}{(4\pi)^2} \log \frac{\Lambda}{\mu}
 \end{aligned} \tag{3.110}$$

Thus generically the YM beta function is not reproduced in absence of zero modes as in the $SUSY$ case. Moreover generically the renormalizations of Q and of $\mu_{\alpha\beta}$ are different. However, if the background field satisfies the equation of motion at leading order, $\mu_{\alpha\beta} = 0$, the corresponding YM connection is SD and therefore instantons occur. In this case the zero modes have to be included and the one-loop beta function is reproduced in the ASD variables:

$$\begin{aligned}
 Z_{1-loop}^{ASD} &= e^{-\frac{(4\pi)^2 N Q}{2g^2} \text{Det}^{-1/2} (-\Delta_{\tilde{A}} \delta_{\alpha\beta}) \text{Det} (-\Delta_{\tilde{A}}) \Lambda^{4NQ} \omega^{2NQ}} \\
 &= e^{-\frac{(4\pi)^2 N Q Z_Q^{-1}}{2g^2} \Lambda^{4NQ} \omega^{2NQ}}
 \end{aligned} \tag{3.111}$$

Yet, we may wonder as to whether the YM theory can be exactly localized on instantons as in the $SUSY$ case. It is very instructive to check that it cannot be so, otherwise the two-loop beta function is not reproduced. Rescaling fields in canonical form, as in the $SUSY$ case, we get:

$$\begin{aligned}
 Z_{1-loop}^{ASD} &= e^{-\frac{(4\pi)^2 N Q (g Z_Q^{\frac{1}{2}} A_c) Z_Q^{-1}}{2g_W^2} \Lambda^{4NQ} \omega (g Z_Q^{\frac{1}{2}} A_c)^{2NQ}} \\
 &= e^{-\frac{(4\pi)^2 N Q (g Z_Q^{\frac{1}{2}} A_c) Z_Q^{-1}}{2g_W^2} \Lambda^{4NQ} (g Z_Q^{\frac{1}{2}})^{4NQ} \omega (A_c)^{2NQ}}
 \end{aligned} \tag{3.112}$$

Now we can define as in Eq.(3.67)

$$-\frac{N16\pi^2 Q (g Z_Q^{\frac{1}{2}} A_c) Z_Q^{-1}}{2g^2} = -\frac{N16\pi^2 Q (g Z_Q^{\frac{1}{2}} A_c) Z_Q^{-1}}{2g_W^2} + 4NQ (g Z_Q^{\frac{1}{2}} A_c) \log (g Z_Q^{\frac{1}{2}}) \tag{3.113}$$

that implies:

$$\frac{1}{2g^2} = \frac{1}{2g^2} + \frac{4Z_Q}{(4\pi)^2} \log (g Z_Q^{\frac{1}{2}}) \tag{3.114}$$

and with two-loop accuracy

$$\frac{1}{2g_W^2} \sim \frac{1}{2g^2} + \frac{4}{(4\pi)^2} \log(gZ_Q^{\frac{1}{2}}) \quad (3.115)$$

Taking the derivative with respect to $\log \Lambda$ and assuming by the localization hypothesis that g_W is one-loop exact, we get with two-loop accuracy:

$$\frac{1}{g^3} \frac{\partial g}{\partial \log \Lambda} = -\beta_0 + \frac{4}{(4\pi)^2} \frac{1}{g} \frac{\partial g}{\partial \log \Lambda} + \frac{2}{(4\pi)^2} \frac{\partial \log Z_Q}{\partial \log \Lambda} \quad (3.116)$$

with $\beta_0 = \frac{11}{3} \frac{1}{(4\pi)^2}$. Therefore

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{2g^3}{(4\pi)^2} \frac{\partial \log Z_Q}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2} \quad (3.117)$$

Since

$$\frac{\partial \log Z_Q}{\partial \log \Lambda} \sim -\frac{2}{3} \frac{g_W^2}{(4\pi)^2} \sim -\frac{2}{3} \frac{g^2}{(4\pi)^2} \quad (3.118)$$

it follows that:

$$\frac{\partial g}{\partial \log \Lambda} \sim -\frac{11}{3} \frac{g^3}{(4\pi)^2} - \frac{4}{3} \frac{g^5}{(4\pi)^4} - \frac{4}{(4\pi)^2} \beta_0 g^5 = -\frac{11}{3} \frac{g^3}{(4\pi)^2} - \frac{48}{3} \frac{g^5}{(4\pi)^4} \quad (3.119)$$

Therefore the second coefficient of the beta function, β_1 , differs from the perturbative result:

$$\beta_1 = \frac{48}{3} \frac{1}{(4\pi)^4} \neq \frac{34}{3} \frac{1}{(4\pi)^4} \quad (3.120)$$

Thus it is not possible to localize the YM partition function on instantons. QED

On the contrary, we will see in the following sections that twistor Wilson loops can be localized on surface operators with Z_N holonomy.

4. Twistor loops and non-commutative YM

4.1 Non-commutative Eguchi-Kawai reduction

We recall some fundamental facts about the non-commutative YM theory [73, 95] that will be used throughout the whole paper. These results will allow us to construct the twistor Wilson loops which our approach is entirely based on.

The non-commutative R^d is defined by:

$$[\hat{x}^\alpha, \hat{x}^\beta] = i\theta^{\alpha\beta} 1 \quad (4.1)$$

Let $\hat{\Delta}(x)$ be:

$$\hat{\Delta}(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\hat{x}} e^{-ikx} \quad (4.2)$$

and

$$\hat{f} = \int d^d x f(x) \hat{\Delta}(x) \quad (4.3)$$

for complex functions of rapid decrease in both coordinates and momenta (Schwartz space). This defines an operator/function correspondence such that:

$$\hat{f} \hat{g} = f \star g \quad (4.4)$$

with:

$$(f \star g)(x) = f(x) \exp\left(\frac{i}{2} \partial_x^\alpha \theta^{\alpha\beta} \partial_y^\beta\right) g(y)|_{y=x} \quad (4.5)$$

that can be extended to multiple \star products:

$$f_1(x_1) \star \dots \star f_n(x_n) = \prod_{i < k} \exp\left(\frac{i}{2} \partial_{x_i}^\alpha \theta^{\alpha\beta} \partial_{x_k}^\beta\right) f_1(x_1) \dots f_n(x_n) \quad (4.6)$$

needed in the evaluation of Wilson loops of the non-commutative theory in the function representation: $P_\star \exp i \int_{L_{yz}} A_\alpha(x) dx_\alpha$.

By the operator/function correspondence translations are represented by unitary operators:

$$e^{a\hat{\partial}} \hat{\Delta}(x) e^{-a\hat{\partial}} = \hat{\Delta}(x+a) \quad (4.7)$$

where:

$$\hat{\partial}^i (\hat{x}^j) = \delta^{ij} 1 \quad (4.8)$$

Thus non-commutative derivations can be represented via Eq.(4.1) and satisfy:

$$[\hat{\partial}_\alpha, \hat{\partial}_\beta] = i\theta_{\alpha\beta}^{-1} 1 \quad (4.9)$$

In addition the integration on functions coincides with the operator trace up to a factor:

$$\begin{aligned} (2\pi)^{\frac{d}{2}} Pf(\theta) \hat{T}r \hat{f} &= \int d^d x f(x) \\ (2\pi)^{\frac{d}{2}} Pf(\theta) \hat{T}r(\hat{\Delta}(x) \hat{\Delta}(y)) &= \delta^d(x-y) \\ \int d^d x (f \star g)(x) &= \int d^d x f(x) g(x) \end{aligned} \quad (4.10)$$

The YM action of the $U(N)$ non-commutative gauge theory has the function/

$$\frac{N}{2g^2} \int d^d x tr_N (F_{\alpha\beta} \star F_{\alpha\beta})(x) \quad (4.11)$$

/operator representation:

$$\frac{N}{2g^2} (2\pi)^{\frac{d}{2}} Pf(\theta) tr_N \hat{T}r (-i[\hat{\partial}_\alpha + i\hat{A}_\alpha, \hat{\partial}_\beta + i\hat{A}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \quad (4.12)$$

where the non-commutative gauge connection is valued in the tensor product of the Lie algebra, $u(N)$, of $U(N)$ in the fundamental representation and of the field \star -algebra. This leads to the non-commutative [96, 97]³¹ Eguchi-Kawai reduction [98, 99, 100]

$$\frac{N}{2g^2} \hat{N} \left(\frac{2\pi}{\Lambda} \right)^d \text{tr}_N \text{Tr}_{\hat{N}} (-i[\hat{\partial}_\alpha + i\hat{A}_\alpha, \hat{\partial}_\beta + i\hat{A}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \quad (4.13)$$

where the trace $\text{Tr}_{\hat{N}}$ is taken now over a subspace of dimension \hat{N} , with

$$\hat{N} \left(\frac{2\pi}{\Lambda} \right)^d = (2\pi)^{\frac{d}{2}} Pf(\theta) \quad (4.14)$$

in the large \hat{N}, θ, Λ limit. The simplest way to understand the occurrence of the inverse power of the cutoff³² in the reduced non-commutative action³³ is to study the Makeenko-Migdal [46, 47] loop equation after having reabsorbed the two factors of N, \hat{N} into a unique factor, $\mathcal{N} = N\hat{N}$, that computes the rank of the tensor product. For this we need to write the Wilson loop of the non-commutative theory in the operator notation. In this version the theory is a matrix model of infinite matrices. Thus the Wilson loop must involve a connection constant in space-time:

$$\frac{1}{\mathcal{N}} \text{tr}_N \text{Tr}_{\hat{N}} \Psi(\hat{A}; L_{ww}) = \frac{1}{\mathcal{N}} \text{tr}_N \text{Tr}_{\hat{N}} P \exp \int_{L_{ww}} (\hat{\partial}_\alpha + i\hat{A}_\alpha) dx_\alpha \quad (4.15)$$

Indeed this prescription leads to the correct definition of the Wilson loops of the non-commutative theory in the function representation. The proof is as follows. In the operator representation we can gauge away the non-commutative derivative that occurs in the definition of the Wilson loop by performing a local gauge transformation with values in the infinite-dimensional unitary group acting on the Fock representation of the non-commutative theory:

$$\hat{U}(x) = e^{x_\alpha \hat{\partial}_\alpha} \quad (4.16)$$

where x_α is a commutative space-time coordinate. The operator-valued gauge connection transforms under this gauge transformation in the usual way:

$$\hat{A}_\alpha^{\hat{U}} = \hat{U}(x) \hat{A}_\alpha \hat{U}(x)^{-1} + i\partial_\alpha \hat{U}(x) \hat{U}(x)^{-1} \quad (4.17)$$

where the partial derivative is the usual partial derivation with respect to the commutative parameter x_α . The operator $\hat{\partial}_\alpha$ must instead transform as a Higgs field in order for $-i\hat{\partial}_\alpha + \hat{A}_\alpha$ to be a connection:

$$\hat{\partial}_\alpha^{\hat{U}} = \hat{U}(x) \hat{\partial}_\alpha \hat{U}(x)^{-1} \quad (4.18)$$

Correspondingly the Wilson line:

$$\Psi(\hat{A}; L_{yz}) = P \exp i \int_{L_{yz}} (-i\hat{\partial}_\alpha + \hat{A}_\alpha) dx_\alpha \quad (4.19)$$

³¹We would like to thank Antonio Gonzalez-Arroyo and Chris Korthals-Altes for discussions on non-commutative Eguchi-Kawai reduction and Antonio Gonzalez-Arroyo for a detailed exam of our work at the GGI.

³²We would like to thank Yuri Makeenko for discussing this point with us at the GGI.

³³See [50] for a modern treatment. Another interesting way to understand the same factor in the quenched version of the Eguchi-Kawai reduction is in [101].

transforms as ³⁴:

$$\begin{aligned}
 & \hat{U}(y)\Psi(\hat{A};L_{yz})\hat{U}(z)^{-1} \\
 &= P \exp i \int_{L_{yz}} (-i\hat{\partial}_\alpha^U + \hat{A}_\alpha^U) dx_\alpha \\
 &= P \exp i \int_{L_{yz}} (\hat{U}(x)\hat{A}_\alpha\hat{U}(x)^{-1} - iU(x)\hat{\partial}_\alpha U(x)^{-1} + i\partial_\alpha\hat{U}(x)\hat{U}(x)^{-1}) dx_\alpha \\
 &= P \exp i \int_{L_{yz}} \hat{U}(x)\hat{A}_\alpha\hat{U}(x)^{-1} dx_\alpha \\
 &= P_\star \exp i \int_{L_{yz}} A_\alpha(x) dx_\alpha
 \end{aligned} \tag{4.20}$$

that is the function version of the non-commutative Wilson loop by the operator/function correspondence. Now the Makeenko-Migdal loop equation [46, 47] of the large- N commutative theory is:

$$\begin{aligned}
 & \left\langle \frac{1}{N} \text{tr}_N \left(\frac{\delta S_{YM}}{\delta A_\alpha(x)} \Psi(A;L_{xx}) \right) \right\rangle \\
 &= i \int_{L_{xx}} dy_\alpha \delta^{(d)}(x-y) \left\langle \frac{1}{N} \text{tr}_N \Psi(A;L_{xy}) \right\rangle \left\langle \frac{1}{N} \text{tr}_N \Psi(A;L_{yx}) \right\rangle
 \end{aligned} \tag{4.21}$$

where the normalized commutative YM action is:

$$S_{YM} = \frac{1}{2g^2} \int \text{tr}_N (F_{\alpha\beta})^2 d^d x \tag{4.22}$$

and the v.e.v. is defined with respect of the unnormalized action:

$$\langle \dots \rangle = Z^{-1} \int \dots \exp\left(-\frac{N}{2g^2} \int \text{tr}_N (F_{\alpha\beta})^2 d^d x\right) \delta A \tag{4.23}$$

The loop equation of the non-commutative matrix model is instead [102, 103]:

$$\begin{aligned}
 & \left\langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}} \left(\frac{\delta S_{NC}}{\delta \hat{A}_\alpha} \Psi(\hat{A};L_{xx}) \right) \right\rangle \\
 &= i \int_{L_{xx}} dy_\alpha \left\langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}} \Psi(\hat{A};L_{xy}) \right\rangle \left\langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}} \Psi(\hat{A};L_{yx}) \right\rangle
 \end{aligned} \tag{4.24}$$

where the normalized action, S_{NC} , of the non-commutative theory is:

$$S_{NC} = \frac{1}{2g^2} \left(\frac{2\pi}{\Lambda}\right)^d \text{Tr}_{\mathcal{N}} (-i[\hat{\partial}_\alpha + i\hat{A}_\alpha, \hat{\partial}_\beta + i\hat{A}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \tag{4.25}$$

and the v.e.v. of the non-commutative theory is defined with respect to the unnormalized action:

$$\langle \dots \rangle = Z^{-1} \int \dots \exp\left(-\frac{\mathcal{N}}{2g^2} \left(\frac{2\pi}{\Lambda}\right)^d \text{Tr}_{\mathcal{N}} (-i[\hat{\partial}_\alpha + i\hat{A}_\alpha, \hat{\partial}_\beta + i\hat{A}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2\right) \delta A \tag{4.26}$$

At this point we notice that the factor of $(\frac{2\pi}{\Lambda})^d$ in the normalized non-commutative action is essential to reproduce the loop equation of the commutative gauge theory, since its effect is equivalent

³⁴We ignore central terms that vanish for large θ .

to the insertion of the missing $\delta^{(d)}(0)$ in the right hand side of the non-commutative loop equation. As a consequence the $\delta^{(d)}(x-y)$ of the commutative loop equation is reproduced provided the trace of Wilson lines vanishes for $x \neq y$ [102, 103]:

$$\langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}} \Psi(\hat{A}; L_{xy}) \rangle = 0 \quad (4.27)$$

QED

The occurrence of the inverse power of the cutoff in the matrix model version of the non-commutative theory opens the way to saddle-point computations of new kind in which power-like divergences cancel against the $(\frac{2\pi}{\Lambda})^d$ factor. In particular if the theory is defined on $R^2 \times R_\theta^2$ quadratic divergences cancel. This will turn out to be the case for the surface operators of the theory on $R^2 \times R_\theta^2$ in the limit of large θ introduced in sect.(7) and employed in the whole paper.

4.2 Twistor Wilson loops

We define twistor Wilson loops in the YM theory with gauge group $U(N)$ on $R^2 \times R_\theta^2$ with complex coordinates ($z = x_0 + ix_1, \bar{z} = x_0 - ix_1, \hat{u} = \hat{x}_2 + i\hat{x}_3, \hat{\bar{u}} = \hat{x}_2 - i\hat{x}_3$) and non-commutative parameter θ , satisfying $[\hat{\partial}_u, \hat{\partial}_{\bar{u}}] = \theta^{-1}1$, as follows:

$$\text{Tr}_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) = \text{Tr}_{\mathcal{N}} \text{P exp} i \int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z} \quad (4.28)$$

where $\hat{D}_u = \hat{\partial}_u + i\hat{A}_u$ is the covariant derivative along the non-commutative direction \hat{u} and λ a complex parameter. For many purposes it is not restrictive to choose λ real, although other choices are possible, for example a phase $\lambda = e^{i\delta}$. The plane (z, \bar{z}) is commutative. The loop, L_{ww} , starts and ends at the marked point, w , and lies in the commutative plane. Thus we regard the twistor connection, B_λ , whose holonomy the twistor Wilson loop computes, as a non-Hermitian connection in the commutative plane valued in the tensor product of the $U(N)$ Lie algebra and of the infinite-dimensional operators that generate the Fock representation of the non-commutative plane $(\hat{u}, \hat{\bar{u}})$. B_λ is indeed a connection in the commutative plane since the non-commutative covariant derivative transforms as a Higgs field of the commutative plane. The trace is defined accordingly. The limit of infinite non-commutativity in the plane $(\hat{u}, \hat{\bar{u}})$ is understood, being equivalent to the large- N limit of the commutative gauge theory [96, 73, 104]. The $U(N)$ non-commutative theory for finite θ has tachyon instabilities that occur in non-planar diagrams suppressed by powers of θ^{-1} and N^{-1} [73]. Therefore non-commutativity is for us just a mean to define the large- N limit, as well as it is for Nekrasov just a mean to compactify the moduli space of instantons³⁵.

4.3 Fiber independence of the v.e.v. of twistor Wilson loops

It easy to show that the v.e.v. of the twistor Wilson loops is independent on the parameter λ :

$$\langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) \rangle = \langle \frac{1}{\mathcal{N}} \text{Tr}_{\mathcal{N}} \Psi(\hat{B}_1; L_{ww}) \rangle \quad (4.29)$$

³⁵Once localization is obtained, the glueballs spectrum is computed employing the effective action in the large- N commutative theory, around the localized locus (see sect.(12)).

The proof is obtained changing variables, rescaling covariant derivatives in the usual definition of the functional integral of the non-commutative YM theory:

$$\begin{aligned}
 & \int Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z} \\
 & \exp\left(-\frac{\mathcal{N}}{2g^2} \left(\frac{2\pi}{\Lambda}\right)^2 \int d^2x Tr_{\mathcal{N}} (-i[\hat{D}_\alpha, \hat{D}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2\right) \delta\hat{A} \delta\hat{A} \delta\hat{D} \hat{D} \\
 & = \int Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_z + \hat{D}'_u) dz + (\hat{A}_{\bar{z}} + \hat{D}'_{\bar{u}}) d\bar{z} \\
 & \exp\left(-\frac{\mathcal{N}}{2g^2} \left(\frac{2\pi}{\Lambda}\right)^2 \int d^2x Tr_{\mathcal{N}} (-[\hat{D}'_\alpha, \hat{D}'_\beta]^2 + (\theta_{\alpha\beta}^{-1})^2 1 - 2i[\hat{D}'_\alpha, \hat{D}'_\beta] \theta_{\alpha\beta}^{-1})\right) \delta\hat{A} \delta\hat{A} \delta\hat{D}' \hat{D}' \quad (4.30)
 \end{aligned}$$

where:

$$\begin{aligned}
 \hat{D}'_z &= \hat{D}_z \\
 \hat{D}'_{\bar{z}} &= \hat{D}_{\bar{z}} \\
 \hat{D}'_u &= \lambda \hat{D}_u \\
 \hat{D}'_{\bar{u}} &= \lambda^{-1} \hat{D}_{\bar{u}}
 \end{aligned} \quad (4.31)$$

The formal non-commutative integration measure is invariant under such rescaling because of the pairwise cancellation of the powers of λ and λ^{-1} . The first term in the non-commutative YM action, proportional to $Tr_{\mathcal{N}} [\hat{D}'_\alpha, \hat{D}'_\beta]^2$, is invariant because of rotational invariance in the non-commutative plane.

Indeed every u must be contracted with a \bar{u} by rotational invariance in the non-commutative plane and thus the factors of λ cancel. The only possibly dangerous terms couple the non-commutative parameter to the commutator $Tr_{\mathcal{N}} ([\hat{D}'_\alpha, \hat{D}'_\beta] \theta_{\alpha\beta}^{-1})$ but only $Tr_{\mathcal{N}} ([\hat{D}'_u, \hat{D}'_{\bar{u}}] \theta_{u\bar{u}}^{-1})$ survives, because all the other terms are zero for $R^2 \times R_\theta^2$. But the commutator is invariant under λ -rescaling.

We notice that, after rescaling, the integration variables $(\delta\hat{D}'_u, \delta\hat{D}'_{\bar{u}})$ should be treated as independent. For λ real this is appropriate if we analytically continue the non-commutative plane to Minkowski space-time, after which the λ invariance of the loop is simply invariance under Lorentz boosts³⁶. The analytic continuation is also connected with the large- θ triviality (see below). QED

In fact the twistor Wilson loops are trivially 1 at large- θ to all orders in the 't Hooft coupling constant g :

$$\lim_{\theta \rightarrow \infty} \langle \frac{1}{\mathcal{N}} Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) \rangle = 1 \quad (4.32)$$

Firstly, we show that triviality holds to the lowest non-trivial order in perturbation theory. We have in the Feynman gauge in the large- θ limit³⁷:

$$\langle Tr_{\mathcal{N}} \left(\int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z} \int_{L_{ww}} (\hat{A}_z + \lambda \hat{D}_u) dz + (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) d\bar{z} \right) \rangle$$

³⁶We would like to thank Konstantin Zarembo for discussing with us the λ -independence at the GGI.

³⁷For a very accurate version of this computation see "Solution to problems" by Luca Lopez in the SNS web site: www.sns.it/it/scienze/fisiche/pertgauge/.

$$\begin{aligned}
 &= 2 \int_{L_{w\bar{w}}} dz \int_{L_{w\bar{w}}} d\bar{z} \langle \text{Tr}_{\mathcal{N}}(\hat{A}_z \hat{A}_{\bar{z}}) \rangle + i^2 \langle \text{Tr}_{\mathcal{N}}(\hat{A}_u \hat{A}_{\bar{u}}) \rangle \\
 &= 0
 \end{aligned} \tag{4.33}$$

4.4 Twistor Wilson loops are supported on Lagrangian submanifolds of twistor space

Secondly, we show that triviality holds in the large θ -limit to all orders of perturbation theory. For this aim it is convenient to gauge away the non-commutative derivatives that occur in the definition of twistor Wilson loops. This can be done by performing a local gauge transformation with values in the complexification of the gauge group. Although this is not a symmetry of the theory, the trace of the twistor Wilson loops is left invariant because of the cyclicity property of the trace. Let be

$$\hat{S}(z, \bar{z}) = e^{i\lambda z \hat{\partial}_u + i\lambda^{-1} \bar{z} \hat{\partial}_{\bar{u}}} \tag{4.34}$$

where (z, \bar{z}) are commutative coordinates. The components of the operator-valued gauge connection, \hat{B}_λ , transform under this gauge transformation in the usual way:

$$\begin{aligned}
 \hat{B}_{\lambda, z}^{\hat{S}} &= \hat{S}(z, \bar{z}) \hat{B}_{\lambda, z} \hat{S}(z, \bar{z})^{-1} + i \partial_z \hat{S}(z, \bar{z}) \hat{S}(z, \bar{z})^{-1} \\
 \hat{B}_{\lambda, \bar{z}}^{\hat{S}} &= \hat{S}(z, \bar{z}) \hat{B}_{\lambda, \bar{z}} \hat{S}(z, \bar{z})^{-1} + i \partial_{\bar{z}} \hat{S}(z, \bar{z}) \hat{S}(z, \bar{z})^{-1}
 \end{aligned} \tag{4.35}$$

where the partial derivatives are the usual partial derivations with respect to the commutative parameters (z, \bar{z}) . Correspondingly the twistor Wilson line transforms as³⁸:

$$\begin{aligned}
 &\hat{S}(w, \bar{w}) \Psi(\hat{B}_\lambda; L_{w\bar{w}}) \hat{S}(v, \bar{v})^{-1} \\
 &= P \exp i \int_{L_{w\bar{w}}} \hat{B}_{\lambda, z}^{\hat{S}} dz + \hat{B}_{\lambda, \bar{z}}^{\hat{S}} d\bar{z} \\
 &= P \exp i \int_{L_{w\bar{w}}} (\hat{S}(z, \bar{z}) (\hat{A}_z + \lambda \hat{D}_u) \hat{S}(z, \bar{z})^{-1} - \lambda \hat{\partial}_u) dz + (\hat{S}(z, \bar{z}) (\hat{A}_{\bar{z}} + \lambda^{-1} \hat{D}_{\bar{u}}) \hat{S}(z, \bar{z})^{-1} - \lambda^{-1} \hat{\partial}_{\bar{u}}) d\bar{z} \\
 &= P \exp i \int_{L_{w\bar{w}}} \hat{S}(z, \bar{z}) (\hat{A}_z + i\lambda \hat{A}_u) \hat{S}(z, \bar{z})^{-1} dz + \hat{S}(z, \bar{z}) (\hat{A}_{\bar{z}} + i\lambda^{-1} \hat{A}_{\bar{u}}) \hat{S}(z, \bar{z})^{-1} d\bar{z}
 \end{aligned} \tag{4.36}$$

Therefore the twistor loop lies effectively on the submanifold of four-dimensional commutative space-time defined by:

$$(z, \bar{z}, u, \bar{u}) = (z, \bar{z}, i\lambda z, i\lambda^{-1} \bar{z}) \tag{4.37}$$

with tangent vector:

$$(\dot{z}, \dot{\bar{z}}, \dot{u}, \dot{\bar{u}}) = (\dot{z}, \dot{\bar{z}}, i\lambda \dot{z}, i\lambda^{-1} \dot{\bar{z}}) \tag{4.38}$$

This is a Lagrangian submanifold of the (complexified) Euclidean space with respect to the Kahler form $dz \wedge d\bar{z} + du \wedge d\bar{u}$ that lifts to a Lagrangian submanifold of twistor space provided λ is either real or a unitary phase. The two cases correspond to Lagrangian submanifolds of antipodal and circle type respectively.

³⁸We ignore central terms that vanish for large θ .

4.5 Triviality of twistor Wilson loops in the limit of infinite non-commutativity

The proof of triviality of twistor Wilson loops to all orders of perturbation theory in the limit $\theta \rightarrow \infty$ follows now almost immediately.

Indeed at any order in perturbation theory a generic contribution to an ordinary Wilson loop of a commutative gauge theory contains a correlator of gauge fields, i.e. a Green function, with tensor indices contracted with a product of monomials in $\dot{x}_\alpha(s)$ at generic insertion points on the loop, labeled by s :

$$\int ds_1 ds_2 \dots G_{\alpha_1 \alpha_2 \dots} (x_\beta(s_1) - x_\beta(s_2), \dots) \dot{x}_{\alpha_1}(s_1) \dot{x}_{\alpha_2}(s_2) \dots \quad (4.39)$$

Because of the $O(4)$ invariance of the commutative theory $\dot{x}_{\alpha_1}(s_1)$ is contracted either with another $\dot{x}_{\alpha_2}(s_2)$ or with an $x_{\alpha_2}(s_2)$ to form polynomials in $\dot{x}_\alpha(s) \dot{x}_\alpha(s')$ or in $\dot{x}_\alpha(s) x_\alpha(s')$. Indeed all these monomials necessarily contain at least one factor of \dot{x}_α since the gauge field along the loop has the index contracted with the one of \dot{x}_α . The possible factor of x_α arises from the dependence of the Green functions on the coordinates.

We now specialize to twistor Wilson loops.

In the limit $\theta \rightarrow \infty$ of the non-commutative gauge theory $O(4)$ invariance is recovered, because the theory becomes the large- N limit of the commutative theory, that obviously is $O(4)$ invariant.

Therefore all the monomials just mentioned vanish when evaluated on the Lagrangian submanifold which the twistor Wilson loop lies on, because they are of the form $\dot{z}(s) \dot{\bar{z}}(s') - \dot{z}(s) \dot{\bar{z}}(s') = 0$ or $z(s) \dot{\bar{z}}(s') - z(s) \dot{\bar{z}}(s') = 0$. Thus the "effective propagators" that connect a Feynman graph at any order to the twistor Wilson loop vanish. The only factors that may spoil the triviality occur if singularities due to denominators of Feynman diagrams arise, since $x_\alpha(s) x_\alpha(s')$ vanishes too on the Lagrangian submanifold for the same reasons. To cure this we analytically continue the correlators that occur in the computation of twistor Wilson loops from Euclidean to Minkowski space-time in order to get the $i\epsilon$ prescription, $z_+(s) z_-(s') - z_+(s) z_-(s') + i\epsilon = i\epsilon$ in the denominators.

The gauge invariant prescription of analytic continuation from Euclidean to Minkowski space-time will be used over and over in the paper and it will play a crucial role. QED

We describe now the aforementioned analytic continuation of the twistor Wilson loops at the operator level in the functional integral by means of the following sequence.

Firstly, we analytically continue to Minkowski space-time only the commutative plane. Then at operator level the twistor Wilson loops become:

$$Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) \rightarrow Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_{z_+} + i\lambda \hat{D}_u) dz_+ + (\hat{A}_{z_-} + i\lambda^{-1} \hat{D}_{\bar{u}}) dz_- \quad (4.40)$$

since $x_0 \rightarrow ix_4$, $z \rightarrow iz_+$ and $\bar{z} \rightarrow iz_-$, with $(z_+ = x_4 + x_1, z_- = x_4 - x_1)$ and $A_z \rightarrow -iA_{z_+}$, $A_{\bar{z}} \rightarrow -iA_{z_-}$. The support of the twistor Wilson loops analytically continued in this way becomes:

$$(z, \bar{z}, i\lambda z, i\lambda^{-1} \bar{z}) \rightarrow (z_+, z_-, -\lambda z_+, -\lambda^{-1} z_-) \quad (4.41)$$

that is Lagrangian with respect to $-dz_+ \wedge dz_- + du \wedge d\bar{u}$ for a real section of the complexified Euclidean space-time.

In sect.(6) we write a holomorphic loop equation that the twistor Wilson loops satisfy in Euclidean space-time. The holomorphic loop equation involves in the left hand side an Euclidean

effective action that should be renormalized in Euclidean space-time and in the right hand side a contour integral along the loop that is not well defined in Euclidean space-time and that should be regularized.

We will show that there is an essentially unique way of regularizing by analytical continuation to Minkowski space-time. Thus after renormalization of the effective action the holomorphic loop equation makes sense in Minkowski space-time. In sect.(11) the renormalized effective action in Minkowski space-time restricted to fluctuations of surface operators supported on the Lagrangian submanifold:

$$(z_+, z_-, -\lambda z_+, -\lambda^{-1} z_-) \quad (4.42)$$

is used to compute the glueballs spectrum. In the effective action this Lagrangian submanifold is obtained first restricting to the Lagrangian submanifold in Euclidean space:

$$(z, \bar{z}, -\lambda z, -\lambda^{-1} \bar{z}) \quad (4.43)$$

and then analytically continuing. Despite the renormalized effective action and the holomorphic loop equation make perfect meaning in Minkowski space-time, twistor Wilson loops after the analytic continuation to Minkowski space-time become non-trivial, in such a way that, seemingly, localization does not hold. This looks puzzling and we suggest a way out below.

Hence to get localization we should analytically continue further to $(2, 2)$ signature at operator level. This can be done in two ways. The following choice for the analytic continuation $u \rightarrow iu_+, \bar{u} \rightarrow iu_-$ leads to trivial twistor Wilson loops:

$$Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) \rightarrow Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_{z_+} + \lambda \hat{D}_{u_+}) dz_+ + (\hat{A}_{z_-} + \lambda^{-1} \hat{D}_{u_-}) dz_- \quad (4.44)$$

supported on the Lagrangian submanifold:

$$(z_+, z_-, i\lambda z_+, i\lambda^{-1} z_-) \quad (4.45)$$

for which the correlators that occur in the evaluation of the twistor Wilson loops are vanishing as shown in the triviality proof. The other choice for the analytic continuation $u \rightarrow u_+, \bar{u} \rightarrow u_-$ leads to:

$$Tr_{\mathcal{N}} \Psi(\hat{B}_\lambda; L_{ww}) \rightarrow Tr_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_{z_+} + i\lambda \hat{D}_{u_+}) dz_+ + (\hat{A}_{z_-} + i\lambda^{-1} \hat{D}_{u_-}) dz_- \quad (4.46)$$

supported on the Lagrangian submanifold:

$$(z_+, z_-, -\lambda z_+, -\lambda^{-1} z_-) \quad (4.47)$$

and apparently back to non-trivial twistor loops. Yet, this is may not be the case at global level.

The conformal compactification of Minkowski space-time with $(2, 2)$ signature is $S^2 \times S^2/Z_2$, whose double cover is $S^2 \times S^2$ [57]. Z_2 is the antipodal involution, $\sigma(x, y) = (-x, -y)$, acting on each S^2 . The corresponding Lagrangian submanifold is not orientable, topologically RP^2 , but admits an orientable double cover, topologically S^2 .

Now every small closed loop around say the north pole on the orientable double cover should have a mirror by the antipodal identification around the south pole, in order to project on the non-orientable quotient. Yet, this loop would be the disconnected sum of two loops on the double cover. To cure this we connect the two mirrors by two almost backtracking arcs with opposite orientations in such a way that the resulting loop is simple and has a well defined orientation on the double cover. But then its projection on the non-orientable quotient is the loop around say the north pole followed by the same loop with opposite orientation and thus it is trivial by construction. Hence, because of triviality, localization still holds for this class of twistor loops.

Because of triviality twistor loops are finite at $\theta = \infty$, i.e. they have no cusp and perimeter divergences, in analogy with certain supersymmetric Wilson loops.

Indeed the cognoscenti may have noticed that twistor Wilson loops resemble locally *BPS* Wilson loops of theories with extended supersymmetry. In fact our triviality proof mimics the argument about a certain non-renormalization property [105] of locally *BPS* Wilson loops. Indeed it has been argued in [105] that a locally *BPS* Wilson loop in the four-dimensional $\mathcal{N} = 4$ *SUSY* gauge theory:

$$\text{Tr} P \exp i \int_L A_\alpha \dot{x}_\alpha(s) ds + i \phi_b \dot{y}_b(s) ds \quad (4.48)$$

has no perimeter divergence, to all orders in perturbation theory, because of the local *BPS* constraint:

$$\sum_\alpha \dot{x}_\alpha^2(s) - \sum_b \dot{y}_b^2(s) = 0 \quad (4.49)$$

At lowest order of perturbation theory this constraint assures the cancellation of the contribution to the perimeter divergence of the gauge propagator versus the scalar propagator, because of the factor of i^2 in front of the scalar propagator at that order.

As far as the perimeter divergence is concerned, it is argued in [105] that this cancellation occurs to all orders in perturbation theory, when the locally *BPS* Wilson loop is seen as the dimensional reduction to four dimensions of the ten-dimensional Wilson loop of the ten-dimensional $\mathcal{N} = 1$ *SUSY* YM theory from which the four-dimensional $\mathcal{N} = 4$ *SUSY* theory is obtained.

Remarkably, in the argument of [105] *SUSY* plays no direct role. In fact the argument is based only on $O(10)$ rotational symmetry of the parent $d = 10$ $\mathcal{N} = 1$ *SUSY* gauge theory from which the daughter $d = 4$ $\mathcal{N} = 4$ *SUSY* gauge theory derives by dimensional reduction, as we show momentarily.

In ten dimensions the coefficient of the perimeter divergence of an ordinary unitary Wilson loop, at any order in perturbation theory, must necessarily contain as a factor a polynomial in the $O(10)$ invariant quantity $\sum_M \dot{x}_M^2(s)$. Indeed the perimeter divergence arises when all insertion points coincide, in such a way that all the arguments of the Green function vanish.

In this case the Green function provides a factor that, by $O(10)$ rotational invariance, must be a polynomial in ten-dimensional Kronecker delta, since all the difference vectors in the Green function are zero at coinciding points and thus no other tensorial structure can be produced.

This combines with the factors of $\dot{x}_M(s)$ to produce an invariant polynomial in $\sum_M \dot{x}_M^2(s)$ with no constant term, since the lowest order contribution is zero by direct computation. But $\sum_M \dot{x}_M^2(s) = \sum_\alpha \dot{x}_\alpha^2(s) - \sum_b \dot{y}_b^2(s) = 0$ is zero for a *BPS* Wilson loop because of the *BPS* constraint. A naive

application of this argument to the four-dimensional BPS Wilson loop would imply the absence of the perimeter divergence for this loop on the basis of the $O(10)$ rotational invariance of the theory before the dimensional reduction. QED

5. The quasi-localization lemma for twistor loops in large- N YM

We use the λ -independence to show that the v.e.v. of twistor Wilson loops is localized on the sheaves fixed by the semigroup rescaling λ . This involves a delicate and subtle interchange between limit and integration, that will be justified by direct computation in sect.(8), after introducing a lattice regularization of the functional integral of differential geometric nature in sect.(7).

In this section for simplicity we use a notation that does not distinguish between commutative and non-commutative theories and therefore we do not add hats to operator valued quantities of non-commutative theories. The framework has been set in the previous section, therefore this use should not generate ambiguities.

It is convenient to choose our twistor Wilson loops in the adjoint representation and to use the fact that in the large- N limit their v.e.v. factorizes in the product of the v.e.v. of the fundamental representation and of its conjugate. Then, for the factor in the fundamental representation, localization proceeds as follows. We write the YM partition function by means of the non- $SUSY$ analog [10] of the Nicolai map of $\mathcal{N} = 1$ $SUSY$ YM theory worked out in sect.(3.3), introducing in the functional integral the appropriate resolution of identity:

$$1 = \int \delta(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) \delta\mu_{\alpha\beta}^- \quad (5.1)$$

$$Z = \int \exp\left(-\frac{N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr_f(\mu_{\alpha\beta}^{-2}) d^4x\right) \delta(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) \delta\mu_{\alpha\beta}^- \delta A_\alpha \quad (5.2)$$

Q is the second Chern class (the topological charge) and $\mu_{\alpha\beta}^-$ is a field of ASD type. The equations of ASD type in the resolution of identity, $F_{01} - F_{23} = \mu_{01}^-$, $F_{02} - F_{31} = \mu_{02}^-$, $F_{03} - F_{12} = \mu_{03}^-$, can be rewritten in the form of a Hitchin system (taking into account the central extension that occurs in the non-commutative case):

$$\begin{aligned} -iF_A + [D, \bar{D}] - \theta^{-1} 1 &= \mu^0 = \frac{1}{2} \mu_{01}^- \\ -i\partial_A \bar{D} &= n = \frac{1}{4} (\mu_{02}^- + i\mu_{03}^-) \\ -i\bar{\partial}_A D &= \bar{n} = \frac{1}{4} (\mu_{02}^- - i\mu_{03}^-) \end{aligned} \quad (5.3)$$

or equivalently in terms of the non-Hermitian connection whose holonomy is computed by the twistor Wilson loop with parameter ρ , $B_\rho = A + \rho D + \rho^{-1} \bar{D} = (A_z + \rho D_u) dz + (A_{\bar{z}} + \rho^{-1} D_{\bar{u}}) d\bar{z}$,

$$\begin{aligned} -iF_{B_\rho} - \theta^{-1} 1 &= \mu_\rho = \mu^0 + \rho^{-1} n - \rho \bar{n} \\ -i\partial_A \bar{D} &= n \\ -i\bar{\partial}_A D &= \bar{n} \end{aligned} \quad (5.4)$$

The resolution of identity in the functional integral then reads:

$$1 = \int \delta n \delta \bar{n} \int_{C_\rho} \delta \mu_\rho \delta(-iF_{B_\rho} - \mu_\rho - \theta^{-1}1) \delta(-i\partial_A \bar{D} - n) \delta(-i\bar{\partial}_A D - \bar{n}) \quad (5.5)$$

where the measure, $\delta \mu_\rho$, along the path, C_ρ , is over the non-Hermitian path with fixed n and \bar{n} and varying μ^0 . The resolution of identity is independent, as ρ varies, on the complex path of integration C_ρ . Indeed the constraint implied by the first of Eq.(5.4) on C_ρ is a linear complex combination of the Hermitian constraint in the first of Eq.(5.3) with coefficient 1 and of the remaining two with coefficients depending on ρ .

Let us consider the v.e.v. of twistor Wilson loops:

$$\begin{aligned} & \int \delta n \delta \bar{n} \int_{C_\rho} \delta \mu_\rho \exp\left(-\frac{N8\pi^2}{g^2}Q - \frac{N4}{g^2} \int Tr_f(\mu^0)^2 + 4Tr_f(n\bar{n})d^4x\right) \\ & Tr_f P \exp i \int_{L_{ww}} (A_z + \lambda D_u) dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}}) d\bar{z} \\ & \delta(-iF_{B_\rho} - \mu_\rho - \theta^{-1}1) \delta(-i\partial_A \bar{D} - n) \delta(-i\bar{\partial}_A D - \bar{n}) \delta A \delta \bar{A} \delta D \delta \bar{D} \end{aligned} \quad (5.6)$$

and let us change variables in the functional integral rescaling the non-commutative covariant derivatives:

$$\begin{aligned} & \int \delta n \delta \bar{n} \int_{C_\rho} \delta \mu_\rho \exp\left(-\frac{N8\pi^2}{g^2}Q - \frac{N4}{g^2} \int Tr_f(\mu^0)^2 + 4Tr_f(n\bar{n})d^4x\right) \\ & Tr_f P \exp i \int_{L_{ww}} (A_z + D'_u) dz + (A_{\bar{z}} + D'_{\bar{u}}) d\bar{z} \\ & \delta(-iF_A + [D', \bar{D}'] - \theta^{-1}1 - \mu^0 - i\frac{\lambda}{\rho} \partial_A \bar{D}' + i\frac{\rho}{\lambda} \bar{\partial}_A D' - \rho^{-1}n + \rho \bar{n}) \\ & \delta(-i\lambda \partial_A \bar{D}' - n) \delta(-i\lambda^{-1} \bar{\partial}_A D' - \bar{n}) \delta A \delta \bar{A} \delta D' \delta \bar{D}' \end{aligned} \quad (5.7)$$

Taking the limit $\lambda \rightarrow 0$ inside the functional integral, the last line implies localization on $n = 0$ and $\bar{\partial}_A D' = 0$. The δn integral is performed by means of the delta function. The independence on the path C_ρ in the neighborhood of $\rho = 0$, that we denote, choosing $\rho = \alpha\lambda$, $C_{\alpha\lambda^+}$ with α fixed as $\lambda \rightarrow 0^+$, implies that the $\delta \bar{n}$ integral decouples and that $\partial_A \bar{D}' = 0$ as well.

Indeed on $C_{\alpha\lambda^+}$ the integral over $\delta \bar{n}$ decouples and the integral on the path $C_{\alpha\lambda^+}$ collapses to the integral on the Hermitian field μ^0 . But the argument of the remaining delta function contains the combination $-iF_A + [D', \bar{D}'] - \theta^{-1}1 - \mu^0 + i\alpha^{-1} \partial_A \bar{D}'$, that can be α independent and thus zero for every α only if the two terms are zero separately. Therefore also $\partial_A \bar{D}' = 0$.

Notice that for this argument to hold it is not necessary to consider the variables (n, D') as Hermitian conjugated to (\bar{n}, \bar{D}') . Indeed they are not so because of the λ rescaling and/or the analytic continuation that is necessary to regularize the twistor loops for the triviality property to hold.

We notice that the localized density has a holomorphic ambiguity, since we can represent the same measure using a different density, performing holomorphic transformations without spoiling the quasi-localization lemma : $\delta \mu_{0^+} = \frac{\delta \mu_{0^+}}{\delta \mu'_{0^+}} \delta \mu'_{0^+}$. The final result for the localized effective measure is:

$$\left[\int_{C_{0^+}} \delta \mu'_{0^+} \frac{\delta \mu_{0^+}}{\delta \mu'_{0^+}} \exp\left(-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr_f(\mu_{\alpha\beta}^-) d^4x\right) \delta(F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) \right]_{n=\bar{n}=0} \delta A_\alpha \quad (5.8)$$

where we have reintroduced the covariant notation.

Thus the twistor loops are localized on the fixed sheaves for which two of the ASD fields vanish: $\mu_{02}^- = \mu_{03}^- = 0$. QED

The integration on the gauge connection in Eq.(5.8) can be explicitly performed in the Feynman gauge in the way explained in sect.(3.3), to obtain:

$$\begin{aligned}
 & \left| \int_{C_1} \delta\mu' e^{-\Gamma} \right|^2 \\
 &= \left[\int_{C_1} \delta\mu' \exp\left(-\frac{N8\pi^2}{g_W^2} Q - \frac{N}{4g_W^2} \sum_{\alpha \neq \beta} \int Tr_f(\mu_{\alpha\beta}^{-2}) d^4x\right) \right. \\
 & \quad \left. Det^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} - iad_{\mu_{\alpha\beta}^-}) Det(-\Delta_A) \left(\frac{\Lambda}{2\pi}\right)^{n_b} Det^{\frac{1}{2}} \omega \frac{\delta\mu}{\delta\mu'} \times c.c. \right]_{n=\bar{n}=0} \quad (5.9)
 \end{aligned}$$

The complex conjugate factor arises by the conjugate representation, $Det^{\frac{1}{2}} \omega$ is the contribution of the n_b zero modes due to the moduli, and Λ the corresponding Pauli-Villars regulator.

The volume form on the connections admits several different representations as the Liouville measure associated to a symplectic form, since different symplectic forms may lead to the same volume form. One possible choice for the symplectic form, ω , is ω_I , displayed in Eq.(3.82). However, in sect.(7) and sect.(12) we will see that there is a different choice of ω , compatible with holomorphic/antiholomorphic fusion, that is most convenient in the computation of the glueballs spectrum.

6. Holomorphic loop equation for twistor Wilson loops

6.1 Holomorphic loop equation

We now specialize to the case $\rho = 1$ for the twistor connection $B_1 = B$. The partition function reads:

$$\begin{aligned}
 Z &= \int \delta n \delta \bar{n} \int_{C_1} \delta\mu' \frac{\delta\mu}{\delta\mu'} \exp\left(-\frac{N8\pi^2}{g^2} Q - \frac{N4}{g^2} \int Tr_f(\mu^0)^2 + 4Tr_f(n\bar{n}) d^4x\right) \\
 & \quad \delta(-iF_B - \mu - \theta^{-1}1) \delta(-i\partial_A \bar{D} - n) \delta(-i\bar{\partial}_A D - \bar{n}) \delta A \delta \bar{A} \delta D \delta \bar{D} \quad (6.1)
 \end{aligned}$$

Writing the holomorphic loop equation for twistor Wilson loops requires that μ' be chosen in the holomorphic gauge, $B_{\bar{z}} = 0$. The further change of variables to the holomorphic gauge is a new key feature of our approach to the large- N YM theory. It is based on the idea that twistor Wilson loops, being holomorphic functionals of μ' , behave as the chiral (i.e. holomorphic) super-fields of an $\mathcal{N} = 1$ SUSY gauge theory. In fact the new holomorphic loop equation resembles for the cognoscenti the holomorphic loop equation that occurs in the Dijkgraaf-Vafa theory [48, 49, 50, 51] of the glueball superpotential³⁹ in $\mathcal{N} = 1$ SUSY gauge theories. Thus the v.e.v. is taken with respect to the measure:

$$\langle \dots \rangle = Z^{-1} \int \delta n \delta \bar{n} \int_{C_1} \delta\mu' \dots \exp\left(-\frac{N8\pi^2}{g^2} Q - \frac{N4}{g^2} \int Tr_f(\mu\bar{\mu}) + Tr_f(n+\bar{n})^2 d^4x\right)$$

³⁹The name is perhaps misleading because the glueball superpotential cannot be used to compute the mass of any glueball state of the $\mathcal{N} = 1$ SUSY theories.

$$\delta(-iF_B - \mu - \theta^{-1}1)\delta(-i\partial_A\bar{D} - n)\delta(-i\bar{\partial}_A D - \bar{n})\frac{\delta\mu}{\delta\mu'}\delta A\delta\bar{A}\delta D\delta\bar{D} \quad (6.2)$$

The holomorphic loop equation is obtained following the Makeenko-Migdal technique, as an identity that expresses the fact that the functional integral of a functional derivative vanishes:

$$\int (Tr \frac{\delta}{\delta\mu'(z, \bar{z})} e^{-\Gamma\Psi(B'; L_{zz})}) \delta\mu' = 0 \quad (6.3)$$

The new holomorphic loop equation for twistor loops follows:

$$\langle Tr(\frac{\delta\Gamma}{\delta\mu'(z, \bar{z})}\Psi(B'; L_{zz})) \rangle = \frac{1}{\pi} \int_{L_{zz}} \frac{dw}{z-w} \langle Tr\Psi(B'; L_{zw}) \rangle \langle Tr\Psi(B'; L_{wz}) \rangle \quad (6.4)$$

where $\Psi(B'; L_{zz})$ is the holonomy of B in the gauge $B'_z = 0$. The Cauchy kernel arises as the kernel of the operator $\bar{\partial}^{-1}$ that occurs by functionally differentiating $\Psi(B'; L_{zz})$ with respect to μ' via $i\bar{\partial}B'_z = \mu'$. Assuming that the loop L_{zz} is simple, i.e. it has no self-intersections, the holomorphic loop equation linearizes:

$$\langle Tr(\frac{\delta\Gamma}{\delta\mu'(z, \bar{z})}\Psi(B'; L_{zz})) \rangle = \frac{1}{\pi} \int_{L_{zz}} \frac{dw}{z-w} \langle Tr\Psi(B'; L_{zw}) \rangle \langle Tr 1 \rangle \quad (6.5)$$

6.2 Regularization by analytic continuation to Minkowski space-time

The contour integration in the right hand side (i.e in the term that accounts for quantum fluctuations) of the loop equation includes the pole of the Cauchy kernel. We need therefore a regularization.

The natural choice consists in analytically continuing the loop equation from Euclidean to Minkowski space-time, $z \rightarrow i(z_+ + i\varepsilon)$. It is at the heart of the Euclidean approach to quantum field theory that this analytic continuation be in fact possible.

In the approach to localization by the holomorphic loop equation the analytic continuation is performed only after functional integration and renormalization, that are performed in Euclidean space. Thus we think that this procedure has chances to work also from the point of view of the constructive quantum field theory.

In fact the approach to localization via the holomorphic loop equation, when combined with the integration on local systems of the next section, leads to more complete and satisfactory results than the localization on fixed points.

The result of the $i\varepsilon$ regularization of the Cauchy kernel is the sum of two distributions, the principal part of the real Cauchy kernel and a one-dimensional delta function:

$$\frac{1}{z_+ - w_+ + i\varepsilon} = P \frac{1}{z_+ - w_+} - i\pi\delta(z_+ - w_+) \quad (6.6)$$

The loop equation thus regularized is:

$$\begin{aligned} & \langle Tr(\frac{\delta\Gamma_M}{\delta\mu'(z_+, z_-)}\Psi(B'; L_{z_+z_+})) \rangle \\ &= \frac{1}{\pi} \int_{L_{z_+z_+}} (P \frac{dw_+}{z_+ - w_+} + dw_+ \pi\delta(z_+ - w_+)) \langle Tr\Psi(B'; L_{z_+w_+}) \rangle \langle Tr\Psi(B'; L_{w_+z_+}) \rangle \end{aligned} \quad (6.7)$$

where now both the distributions on the right hand side are integrable along the loop. This regularization has the great virtue of being manifestly gauge invariant, an unusual feature for loop equations. In addition this regularization is not loop dependent.

The right hand side of the loop equation contains now two contributions. A delta-like one dimensional contact term, that is supported on closed loops and a principal part distribution that is supported on open loops. Since by gauge invariance it is consistent to assume that the expectation value of open loops vanishes, as in Eq.(4.27), the principal part does not contribute and the loop equation in the holomorphic gauge reduces to:

$$\begin{aligned} & \langle \text{Tr} \left(\frac{\delta \Gamma_M}{\delta \mu'(z_+, z_-)} \Psi(B'; L_{z_+ z_+}) \right) \rangle \\ &= \int_{L_{z_+ z_+}} dw_+ \delta(z_+ - w_+) \langle \text{Tr} \Psi(B'; L_{z_+ w_+}) \rangle \langle \text{Tr} \Psi(B'; L_{w_+ z_+}) \rangle \end{aligned} \quad (6.8)$$

Taking $w_+ = z_+$ and using the transformation properties of μ' and of the holonomy of B' , the preceding equation can be rewritten in terms the curvature, μ , and of the connection, B , in a unitary gauge:

$$\begin{aligned} & \langle \text{Tr} \left(\frac{\delta \Gamma_M}{\delta \mu(z_+, z_-)} \Psi(B; L_{z_+ z_+}) \right) \rangle \\ &= \int_{L_{z_+ z_+}} dw_+ \delta(z_+ - w_+) \langle \text{Tr} \Psi(B; L_{z_+ w_+}) \rangle \langle \text{Tr} \Psi(B; L_{w_+ z_+}) \rangle \end{aligned} \quad (6.9)$$

where we have used the condition that v.e.v. of the trace of open loops vanishes to substitute the holonomy of B' with the holonomy of B .

7. Integrating on surface operators in large- N YM

7.1 Integrating on infinite-dimensional local systems

We would like to give a precise mathematical meaning to the formal manipulations of the functional measure in sect.(5) and sect.(6). One possibility would be introducing a lattice regularization of the functional integral according to Wilson [106]. However, this kind of lattice regularization would spoil completely the geometrical structure, since in the Wilson regularization the gauge connection lives on links and the curvature on plaquettes, a fact that makes exploiting the map from the connection to the ASD curvature problematic, not to mention the understanding of the moduli of the loci at which this map is not one-to-one, for which the zero modes necessary to get the correct beta function occur (see sect.(3.3) and sect.(3.4)).

Therefore we introduce a new regularization of the YM functional integral that allows us to maintain the differential geometric structure. The differential geometric structure is crucial to get a structure theory of the locus of the fixed points of the functional measure and to understand the zero modes of the determinants, that in turn affect the beta function of the theory.

Our new regularization of the YM theory in the large- N limit is performed in two steps. In the first step the resolution of identity in the Nicolai map on $R^2 \times R^2_{\mathfrak{g}}$ is represented in the operator notation of sect.(4.1) as a functional integral on infinite-dimensional parabolic bundles, as suggested

long ago in [2, 3]:

$$1 = \int \delta(-i[\hat{D}_\alpha, \hat{D}_\beta]^- - \sum_p \hat{\mu}_{\alpha\beta}^-(p) \delta^{(2)}(z - z_p) - \theta_{\alpha\beta}^{-1} \hat{1}) \prod_p \delta \hat{\mu}_{\alpha\beta}^-(p) \quad (7.1)$$

In this notation all the dependence on the non-commutative coordinates is absorbed into the infinite dimensional nature of the operators that occur in the non-commutative Eguchi-Kawai reduction. Therefore the base of the infinite-dimensional parabolic bundles is the two-dimensional surface, R^2 , labelled by the commutative coordinates (z, \bar{z}) .

This amounts to substitute the continuous field, $\hat{\mu}_{\alpha\beta}^-(z, \bar{z})$, of the Nicolai map with the lattice field, $\hat{\mu}_{\alpha\beta}^-(p)$, by the resolution:

$$\hat{\mu}_{\alpha\beta}^-(z, \bar{z}) = \sum_p \hat{\mu}_{\alpha\beta}^-(p) \delta^{(2)}(z - z_p) \quad (7.2)$$

This resolution is dense in the sense of distributions, since for any smooth test function of compact support:

$$N_D^{-1} \sum_p f(z_p, \bar{z}_p) \hat{\mu}_{\alpha\beta}^-(p) \rightarrow \int f(z, \bar{z}) \hat{\mu}_{\alpha\beta}^-(z, \bar{z}) d^2 z \quad (7.3)$$

On this dense set in function space⁴⁰ the resolution of identity of the Nicolai map can be interpreted as hyper-Kähler reduction [2, 3]. Indeed the three constraints of ASD type are the Hermitian and the complex moment maps for the Hamiltonian action of the infinite-dimensional unitary gauge group on the commutative plane R^2 :

$$\begin{aligned} -iF_{\hat{A}} + [\hat{D}, \hat{D}] - \theta^{-1} 1 &= \sum_p \hat{\mu}_p^0 \delta^{(2)}(z - z_p) \\ -i\partial_{\hat{A}} \hat{D} &= \sum_p \hat{n}_p \delta^{(2)}(z - z_p) \\ -i\bar{\partial}_{\hat{A}} \hat{D} &= \sum_p \hat{\bar{n}}_p \delta^{(2)}(z - z_p) \end{aligned} \quad (7.4)$$

with respect to the three symplectic forms [2, 3, 10]⁴¹:

$$\begin{aligned} \omega_I &= \frac{1}{2\pi} \int d^2 z \text{tr}_f \hat{T} r (\delta \hat{A}_z \wedge \delta \hat{A}_{\bar{z}} + \delta \hat{D}_u \wedge \delta \hat{D}_{\bar{u}}) \\ \omega_J - i\omega_K &= \frac{1}{2\pi i} \int d^2 z \text{tr}_f \hat{T} r (\delta \hat{A}_z \wedge \delta \hat{D}_{\bar{u}}) \\ \omega_J + i\omega_K &= \frac{1}{2\pi i} \int d^2 z \text{tr}_f \hat{T} r (\delta \hat{A}_{\bar{z}} \wedge \delta \hat{D}_u) \end{aligned} \quad (7.5)$$

⁴⁰The three operators $\hat{\mu}_{\alpha\beta}^-(p)$ are reduced in fact to large finite dimensional matrices by Morita equivalence as explained momentarily. In the finite dimensional case these matrices all commute as a consequence of the local model of the Hitchin equations [80, 81]. Therefore the resolution of Eq.(7.2) turns out to be dense in function space in the sense of the distributions only in a certain neighborhood of the fixed points, $\hat{\mu}_{02}^-(p) = \hat{\mu}_{03}^-(p) = 0$. The missing degrees of freedom occur as moduli (see below). Nevertheless the local degrees of freedom and the moduli in a neighborhood of the fixed points are enough to reproduce the correct universal one- and two- loop contributions to the beta function (see sect.(9) and sect.(11)). Hence the mentioned degrees of freedom are in fact dense in function space in a neighborhood of the fixed points in the large- N limit (see below).

⁴¹We use the same labels (I, J, K) of the symplectic forms as in [8] for the finite dimensional case.

as it follows immediately from the interpretation as (infinite-dimensional) Hitchin systems [40, 41].

Thus we come to the remarkable conclusion that, on a dense set associated to a lattice divisor, the Nicolai map can be interpreted, up to gauge equivalence, as a resolution of the gauge connection into orbits parametrized formally by hyper-Kähler moduli spaces, that arises as the quotient of the manifold defined by Eq.(7.4) for the action of the unitary gauge group [40, 41]. Yet, for the moment, this construction is somehow formal, because of the infinite-dimensional nature of the bundles involved. Thus it is unclear what the moduli theory of these infinite-dimensional bundles is.

7.2 Reducing to finite dimension by Morita duality and inductive structure

In order to reduce to finite dimensional bundles a possible way out is to compactify the non-commutative plane, R_θ^2 , on a non-commutative torus of large area, L^2 . The corresponding non-commutative $U(N)$ gauge theory enjoys, for rational values of the dimensionless non-commutative parameter, $2\pi\theta L^{-2} = \frac{\hat{M}}{\hat{N}}$, Morita duality [73, 104, 107, 108] to a theory on a commutative torus of area $L^2\hat{N}^{-2}$, with gauge group $U(N \times \hat{N})$, with the same 't Hooft coupling constant g , and with twisted boundary conditions corresponding to a 't Hooft flux. In addition the Wilson loops of the $U(N)$ non-commutative gauge theory and of the corresponding $U(N \times \hat{N})$ commutative Morita equivalent theory are related by the relation [109]:

$$\begin{aligned} & \hat{N} \frac{2\pi\theta}{L^2} \text{tr}_N \hat{T} r P \exp\left(\int_{L_{xx}} \hat{D}_\alpha dx_\alpha\right) \\ &= \hat{N} \frac{1}{L^2} \int d^2x \text{tr}_N \hat{T} r P_\star \exp\left(i \int_{L_{xx}} A_\alpha dx_\alpha\right) \\ &= \text{tr}_N T r_{\hat{N}} P \exp\left(i \int_{L_{xx}} A_\alpha dx_\alpha\right) \end{aligned} \quad (7.6)$$

The first equality is just an identity of the function/operator correspondence of sect.(4.1). The second equality is the actual content of the Morita correspondence.

We can understand this relation by noticing that it assures that the naturally normalized Wilson loops of the two theories coincide in the large- (\hat{N}, N) limit.

We choose in this paper $\frac{2\pi\theta}{L^2} = \frac{\hat{N}-1}{\hat{N}}$, because it allows us to perform the large θ limit, necessary to reproduce the large- N' limit of the commutative $SU(N')$ theory, uniformly for large \hat{N} .

From a physical point of view we apply 't Hooft duality ideas to the $U(N)$ non-commutative theory. The structure group of this theory is the tensor product of $U(N)$ and of the group of \star -gauge transformations. We assume Z_N magnetic condensation for the $U(N)$ factor. The situation is less clear for the \star -group factor, since it is not a Lie group.

To clarify this issue we proceed as follows. Starting from the infinite-dimensional non-commutative case we can perform an infinite-dimensional unitary gauge transformation, $\hat{U}(u, \bar{u})$, depending on the commutative parameters, (u, \bar{u}) , in such a way that:

$$-i[\hat{D}_\alpha, \hat{D}_\beta]^- - \sum_p \hat{\mu}_{\alpha\beta}^-(p) \delta^{(2)}(z - z_p) - \phi_{\alpha\beta} \hat{1} = 0 \quad (7.7)$$

becomes:

$$[\partial_\alpha \hat{A}_\beta(z, \bar{z}, u, \bar{u}) - \partial_\beta \hat{A}_\alpha(z, \bar{z}, u, \bar{u}) + i[\hat{A}_\alpha(z, \bar{z}, u, \bar{u}), \hat{A}_\beta(z, \bar{z}, u, \bar{u})]]^-$$

$$= \sum_p \hat{U}(u, \bar{u}) \hat{\mu}_{\alpha\beta}^-(p) \hat{U}^{-1}(u, \bar{u}) \delta^{(2)}(z - z_p) + \phi_{\alpha\beta} \hat{1} \quad (7.8)$$

where:

$$\hat{A}_\beta(z, \bar{z}, u, \bar{u}) = \hat{U}(u, \bar{u}) \hat{A}_\beta(z, \bar{z}) \hat{U}^{-1}(u, \bar{u}) + i \partial_\beta \hat{U}(u, \bar{u}) \hat{U}^{-1}(u, \bar{u}) \quad (7.9)$$

and $\phi_{\alpha\beta}$ is an antisymmetric field that takes into account a possible $U(1)$ background flux in addition to the central term that arises by the inverse of the non-commutativity.

The content of Morita duality is that these seemingly infinite-dimensional equations admit a finite dimensional solution [73] on a commutative torus with coordinates (u, \bar{u}) , gauge group $U(N \times \hat{N})$, twisted boundary conditions corresponding to a 't Hooft flux and a $U(1)$ background flux, $\phi'_{\alpha\beta}$. Despite the structure group is $U(N \times \hat{N})$, the Morita equivalent theory does not describe the most general $U(N \times \hat{N})$ gauge theory, but only the one that satisfies the twisted boundary conditions. In particular the twisted boundary conditions require solutions of the kind:

$$\begin{aligned} & [\partial_\alpha A_\beta(z, \bar{z}, u, \bar{u}) - \partial_\beta A_\alpha(z, \bar{z}, u, \bar{u}) + i[A_\alpha(z, \bar{z}, u, \bar{u}), A_\beta(z, \bar{z}, u, \bar{u})]]^- \\ &= \sum_p U(u, \bar{u}) \mu_{\alpha\beta}^-(p) U^{-1}(u, \bar{u}) \delta^{(2)}(z - z_p) + \phi'_{\alpha\beta} 1_{N \times \hat{N}} \end{aligned} \quad (7.10)$$

where now $U(u, \bar{u})$ are $U(\hat{N})$ matrices and $\mu_{\alpha\beta}^-(p)$ lives in the tensor product of Lie algebras, $u(N) \times u(\hat{N})$, [73, 104, 107, 108]. These are the equations that define surface operators in finite dimension but for the fact that the gauge connection satisfies twisted boundary conditions on the torus. Because the Morita equivalent twisted connections live in the tensor product, $u(N) \times u(\hat{N})$, there is a natural embedding in the Morita equivalent theory of $SU(N)$ connections with Z_N holonomy in $Z_N \times 1_{\hat{N}}$ and of $SU(\hat{N})$ connections with $Z_{\hat{N}}$ holonomy in $1_N \times Z_{\hat{N}}$. Thus we apply 't Hooft arguments about the condensation of the magnetic charge to the tensor product $Z_N \times 1_{\hat{N}}$ on the locus of Eq.(7.10). Alternatively we assume the condensation of the magnetic $1_N \times Z_{\hat{N}}$ charges. In fact we find that our formulae for the glueballs spectrum are symmetric, mutatis mutandis, for the exchange $N \rightarrow \hat{N}$. Therefore from now on we consider only the $Z_N \times 1_{\hat{N}}$ case.

For computational technical reasons it may be convenient to get rid of the twist by means of the thermodynamic limit, to get back (commutative) R^2 . Going back to the original gauge (that would be singular on the commutative torus but not on R^2) we obtain:

$$\begin{aligned} & [\partial_\alpha A_\beta(z, \bar{z}, u, \bar{u}) - \partial_\beta A_\alpha(z, \bar{z}, u, \bar{u}) + i[A_\alpha(z, \bar{z}, u, \bar{u}), A_\beta(z, \bar{z}, u, \bar{u})]]^- \\ &= \sum_p \mu_{\alpha\beta}^-(p) \delta^{(2)}(z - z_p) + \phi'_{\alpha\beta} 1_{N \times \hat{N}} \end{aligned} \quad (7.11)$$

that is the standard defining equation of a lattice of surface operators in presence of a central magnetic field. The central term is now irrelevant in the large- N limit ⁴² since without the non-commutativity it does split, at difference of Eq.(7.7).

The twisted and untwisted theories describe the same thermodynamic and large- N, \hat{N} limits, but for the fact that in the twisted theory the traces of Wilson loops scale exactly by a factor of \hat{N} times the normalized v.e.v. of the non-commutative theory, according to Eq.(7.6).

⁴²In sect.(12.7) it still plays a role.

For our purposes it suffices that this property is satisfied for localized twistor Wilson loops. In the large- θ limit, because of their triviality, this can be mimicked in the untwisted commutative $U(N \times \hat{N})$ theory by restricting the ansatz for the localized locus to holonomies valued in the complexification of $u(N) \times 1_{\hat{N}}$ in the large- N, \hat{N} limit. Indeed in this sector the \hat{N} -scaling is automatic and the large- N limit of the commutative untwisted theory coincides with the non-commutative one in the thermodynamic limit.

The solutions of Eq.(7.11) describe surface operators with singularities supported on the (u, \bar{u}) plane and, because of translational invariance of the vacuum ⁴³ in the (u, \bar{u}) plane, reduce to the standard two-dimensional Hitchin equations in the (z, \bar{z}) plane. The large- N and large- θ limit of the non-commutative theory is therefore recovered as a double large- N and large- \hat{N} limit.

For further use we need to know that the second Chern class has an extension to surface operators as a parabolic Chern class. In the notation of [8]:

$$\frac{1}{16\pi^2} \int d^4x F_{\alpha\beta} \tilde{F}_{\alpha\beta} = Q + \sum_p \text{tr}_f(\alpha_p m_p) + \frac{1}{2} \sum_p D_p \cap D_p \text{tr}_f(\alpha_p^2) \quad (7.12)$$

where Q is the usual second Chern class of the $U(N)$ bundle without the parabolic structure, α_p is the vector of the parabolic weights at the point p , i.e. the vector of the eigenvalues of F_{01}^- modulo 2π in the fundamental representation, m_p the magnetic flux through the surface D_p of the singular divisor $p \times D_p$ of the surface operator and $D_p \cap D_p$ the index of self-intersection of the surface D_p .

There is one more symplectic form that plays an important role in this paper. It occurs as the symplectic form associated to the twistor connection:

$$\begin{aligned} \omega_\rho &= \frac{1}{2\pi} \int d^2z \text{tr}_f \hat{T}r(\delta \hat{B}_{\rho z} \wedge \delta \hat{B}_{\rho \bar{z}}) \\ &= \omega_I - i\rho(\omega_J + i\omega_K) - i\rho^{-1}(\omega_J - i\omega_K) \end{aligned} \quad (7.13)$$

For $\rho = -1$ it will be employed as an ingredient of the holomorphic/antiholomorphic fusion in sect.(12). Indeed it follows from Eq.(5.4) that $\omega_{-1} = \omega$ depends holomorphically on μ_{-1} .

In sect.(12) we employ a modification of ω , ω' , defined integrating over a punctured sphere rather than over its compactification obtained adding the singular divisor. The relation between the two is (see Eq.(3.30) of [118]):

$$\omega = \omega' + \sum_p \text{Tr}(\mu_p (\delta g_p g_p^{-1})^2) \quad (7.14)$$

where the terms in the sum over p represent the Kirillov forms on the adjoints orbits at p . ω' depends only on the holonomy of the connection (see Eq.(3.13) of [118]).

To summarize we have started from the point-like parabolic singularities of the non-commutative Eguchi-Kawai reduced theory and we have ended with surface-like singularities in the Morita equivalent commutative theory. We can turn the argument around and say that the aforementioned point-like parabolic singularities of the non-commutative partial large- N Eguchi-Kawai reduction are daughters of codimension-two singularities of the four-dimensional parent gauge theory. Codimension-two singularities of this kind were introduced in [2, 3] in the pure YM theory as

⁴³We mean in fact the vacuum on which twistor Wilson loops can be localized on.

an "elliptic fibration of parabolic bundles" for the purpose of getting control over the large- N limit of the pure YM theory exploiting the integrability of the Hitchin fibration. In [8] they were introduced in the $\mathcal{N} = 4$ SUSY YM theory for the study of the geometric Langlands correspondence, under the name of "surface operators", and this is now the name universally used in the physical literature.

7.3 Moduli of surface operators

To study the moduli space of surface operators in Eq.(7.11) it is convenient to compactify the (z, \bar{z}) plane on a sphere. The moduli space has three different equivalent descriptions that are all employed in this paper. There is a vast mathematics [74, 75, 76, 77, 78, 79, 80, 81, 82] and physics literature [8, 86] on parabolic Hitchin bundles⁴⁴. Thus we summarize briefly the essential results [74, 75, 80, 81].

The first description of the moduli space is of differential geometric nature as a Hitchin system and hyper-Kahler quotient, that in our approach follows by the the non-SUSY non-commutative Nicolai map on a dense set, as we just discussed. This is the description that occurs combining the quasi-localization lemma with the idea of integrating on surface operators. In the hyper-Kahler description the structure group of the bundles involved is compact. In our case $U(N)$ or $SU(N)$. Thus we refer to the gauge fixing in this framework as the unitary gauge. It is convenient to write Eq.(7.11) in non-covariant notation exactly as Hitchin equations (we disregard for the moment the central $U(1)$ extension that splits):

$$\begin{aligned} -iF_A - [A_u, A_{\bar{u}}] &= \sum_p \mu_p^0 \delta^{(2)}(z - z_p) \\ -i\partial_{\bar{A}} A_{\bar{u}} &= \sum_p n_p \delta^{(2)}(z - z_p) \\ -i\bar{\partial}_{\bar{A}} A_u &= \sum_p \bar{n}_p \delta^{(2)}(z - z_p) \end{aligned} \quad (7.15)$$

Because of the delta function at p in general the gauge connection has a pole singularity. The triple $(\mu_p^0, n_p, \bar{n}_p)$ determines the coefficients of the leading behavior of the gauge connection around the pole. The local model arises by restricting to such leading behavior [80, 81]. Since μ_p^0 is Hermitian it is always diagonalizable. Let us consider first the semisimple case for which by definition all the eigenvalues of μ_p^0 ⁴⁵ are different modulo π .

A study of the local model implies that in this case also n_p and \bar{n}_p can be diagonalized simultaneously with μ_p^0 by a compact gauge transformation, g_p [80]. Thus the matrix, $\mu_p = \mu_p^0 + n_p - \bar{n}_p$, commutes with its adjoint, $\bar{\mu}_p$, i.e. it is normal. Hence the compact adjoint orbits at a point, $g_p \lambda_p g_p^{-1}$, where λ_p are the (in general complex) eigenvalues of μ_p , label some moduli of the solution of Eq.(7.15). There are other moduli that are not immediately manifest in the unitary gauge. They arise as moduli of the metric of the Hitchin bundle that is implicit in the definition of the unitary structure [81].

When some eigenvalues of μ_p^0 are degenerate modulo π the asymptotic behavior of the connection changes. This is recalled below after describing the other representations of the moduli space.

⁴⁴These references are by no means a complete list.

⁴⁵We have normalized μ_p in such a way that the holonomy around p is $e^{2i\mu_p}$.

The second description of the moduli space is of holomorphic nature. It arises by a meromorphic connection in a holomorphic gauge. Indeed the Hitchin equations imply the flatness equation:

$$\begin{aligned} -iF(B) &= \sum \mu_p \delta^{(2)}(z - z_p) \\ F(B) &= \partial_z B_{\bar{z}} - \partial_{\bar{z}} B_z + i[B_z, B_{\bar{z}}] \end{aligned} \quad (7.16)$$

for the non-Hermitian connection:

$$\begin{aligned} B_z &= A_z + iA_u \\ B_{\bar{z}} &= A_{\bar{z}} + iA_{\bar{u}} \end{aligned} \quad (7.17)$$

The moduli space arises as the Kahler quotient of the space of solutions of the flatness equation, Eq.(7.16), with respect to the action of the complexification of the gauge group. Because of a well known result [40, 41] it coincides with the hyper-Kahler quotient of the three equations, Eq.(7.15), with respect to the action of the compact gauge group [80]. The structure of the moduli space is particularly transparent in a holomorphic gauge:

$$B_{\bar{z}} = 0 \quad (7.18)$$

In this gauge B_z is a meromorphic connection:

$$i\partial_{\bar{z}} B_z = \sum_p \mu'_p \delta^{(2)}(z - z_p) \quad (7.19)$$

with residue at p determined by μ'_p , that is conjugate to μ_p by a gauge transformation in the complexification of the gauge group. This description is the most transparent to understand the moduli space because all the local moduli are labelled by the adjoint orbit in the complexification of the gauge group, $\mu'_p = G_p \lambda_p G_p^{-1}$.

The holomorphic description arises in the holomorphic loop equation.

It implies also that Hitchin equations are associated to local systems, i.e. to fiber bundles with locally constant transition functions [74, 110]. A local system on a complex curve is the same as a representation of the fundamental group of a Riemann surface with punctures [74, 110]. This is the topological description of the moduli, and it is also the easiest to understand globally. Indeed the residues of the meromorphic connection, B_z , determine its holonomy around p :

$$\begin{aligned} M_p &= P e^{i \int_{L_p} B_z dz} \\ &= e^{2i\mu'_p} \end{aligned} \quad (7.20)$$

The global moduli space on a punctured sphere is therefore the quotient of the algebraic variety:

$$\prod_p M_p = 1 \quad (7.21)$$

modulo the adjoint action of the complexification of the global gauge group (this description has been employed in sect.(1)).

We come now to the non-semisimple case [74, 80, 81].

If the eigenvalues of the holonomy around p , $e^{2i\lambda_p}$, are not all different, the holonomy cannot be diagonalized in general but it can be put in Jordan form. In this case in the unitary gauge some eigenvalues of n_p and of its Hermitian conjugate \bar{n}_p are degenerate as well and the Higgs field, A_u , in some directions in color space has not anymore a pole singularity but only a milder one, a pole divided by powers of a logarithm. The power of the logarithm and the coefficient of the pole are determined by the off-diagonal parameters in the Jordan form of the holonomy [81].

Perhaps the most important property of the hyper-Kähler construction from a physical point of view in the semisimple case is the fact that in a unitary gauge the coefficients of the delta function at a point $(\mu_p^0, n_p, \bar{n}_p)$ commute and thus can be diagonalized at the same time by a unitary gauge transformation.

On the opposite in a holomorphic gauge the residues of the meromorphic connection and the local holonomies of the twistor connection cannot be diagonalized in general by a unitary transformation but generically only by a transformation in the complexification of the gauge group. Thus there is mismatch in the number of local degrees of freedom between the holomorphic and unitary descriptions.

This is explained by the fact that in the unitary description the missing degrees of freedom arise as moduli of the Hermitian metric associated to the Higgs field [81]. Physically this means that the dimension of local fluctuations of the ASD curvature, μ_p , of semisimple type that occur by the hyper-Kähler construction in a unitary gauge is just one-half of the dimension that occurs in the holomorphic description of local fluctuations, μ'_p . This leads to a non-trivial Jacobian from the unitary to the holomorphic gauge whose logarithm turns out to be the glueballs potential.

In the next section we consider solutions of the Hitchin equations for connections with Z_N holonomy. These connections have no (local) moduli since the adjoint orbit of the center is the center. They turn out to be the Hitchin bundles that occur at the fixed points of the quasi-localization lemma. They satisfy the Hitchin equations:

$$\begin{aligned} -iF_A - [A_u, A_{\bar{u}}] &= \sum_p \lambda_p \delta^{(2)}(z - z_p) \\ \partial_{\bar{z}} A_{\bar{u}} &= 0 \\ \bar{\partial}_{\bar{z}} A_u &= 0 \end{aligned} \tag{7.22}$$

with $e^{2i\lambda_p} \in Z_N$. Therefore:

$$2\lambda_p = \text{diag}(\underbrace{2\pi(k-N)/N}_k, \underbrace{2\pi k/N}_{N-k}) \tag{7.23}$$

These equations are invariant for the following $U(1)$ action:

$$\begin{aligned} A_u &\rightarrow e^{i\theta} A_u \\ A_{\bar{u}} &\rightarrow e^{-i\theta} A_{\bar{u}} \end{aligned} \tag{7.24}$$

Since there are no moduli this $U(1)$ must act by gauge transformations:

$$\begin{aligned} g_{\theta} A_u g_{\theta}^{-1} &= e^{i\theta} A_u \\ g_{\theta} A_{\bar{u}} g_{\theta}^{-1} &= e^{-i\theta} A_{\bar{u}} \\ g_{\theta} A_z g_{\theta}^{-1} &= A_z \\ g_{\theta} A_{\bar{z}} g_{\theta}^{-1} &= A_{\bar{z}} \end{aligned} \tag{7.25}$$

More generally the set of moduli fixed by this $U(1)$ action is the Lagrangian submanifold of the hyper-Kähler moduli space for which the Higgs field, A_u , is nilpotent [111, 78, 83].

There are fundamentally two interesting types of orbits in the Lagrangian cone of the hyper-Kähler moduli space [111]: the orbits with unitary holonomies, for which the Higgs field vanishes identically; the orbits that correspond to Hodge bundles, for which the holonomies are valued in a real version of the complexification of the gauge group [111].

In the first case the holonomies can always be diagonalized, despite the eigenvalues may not be all different. In the second case the holonomies cannot be diagonalized, but can be set in Jordan form. Both the orbits play a role in the computation of the Wilsonian beta function (see sect.(9)).

8. Localization on fixed points in large- N YM

We use the description of surface operators as local systems (i.e. as representations of the fundamental group) to obtain localization on fixed points.

We assume that the quasi-localization lemma holds for twistor Wilson loops in the adjoint representation and we justify the exchange of the order of integration and limits a posteriori by direct computation.

Physically the basic idea is that twistor Wilson loops in the adjoint representation, of both the non-commutative theory with gauge group $U(N)$ and of its Morita equivalent counterpart, satisfy the general criteria of 't Hooft duality. Thus, since a localization theory should hold for these loops, we expect that they may be localized, by large- N factorization, on a condensate of the tensor product of Z_N magnetic vortices with the conjugate representation. We have noticed in the previous section that the correct ansatz for the localized locus is in fact $Z_N \times 1_{\hat{N}}$, i.e. Z_N occurs with degeneracy \hat{N} .

The complexification of the global gauge group acts on the holonomy at one point, p_1 , by the adjoint action, in such a way that M_{p_1} can be put in canonical form. M_{p_1} can be diagonalized if it has distinct eigenvalues, while in general it can be put in Jordan form.

In the large- N limit it is possible to restrict the integration measure, $d\mu_p$, to orbits whose holonomies have fixed eigenvalues, since this restriction implies an error of subleading order in $\frac{1}{N}$. In addition by translational invariance the conjugacy class of the orbits at all the points p must be the same. Finally the global $SU(N)$ gauge group (and not only the $U(1)^{N-1}$ torus as in the Nekrasov case) must fix M_{p_1} , i.e. $gM_{p_1}g^{-1} = M_{p_1}$, since otherwise M_{p_1} would break spontaneously the global gauge symmetry. Therefore M_{p_1} must be central and thus must be in Z_N . But then all the orbits collapse to a point and there are no moduli at the fixed points ⁴⁶.

However, in any neighborhood of the fixed points of the global gauge group, the orbits are non-trivial and moduli there exist. There are essentially two different ways to describe these neighborhoods of the fixed points. One possibility is deforming infinitesimally the unitary part of the eigenvalues of the holonomy, the other possibility is deforming the holonomy along nilpotent directions. Both possibilities are discussed in the next section.

Thus if we first compute the effective measure in a neighborhood of the fixed points and then we sit on the fixed points the induced measure will contain the powers of the Pauli-Villars regulator

⁴⁶As we have seen in the introduction the requirement of translational invariance can be relaxed.

due to the moduli. This has an analog in the localization of the $\mathcal{N} = 2$ $SUSY$ YM partition function, where generically instantons have moduli (this is essential to get the correct beta function in that case too), but the instantons at the fixed points of the torus action have not.

Thus at the fixed points the contour integral over μ_{0+} in Eq.(5.8) collapses to a discrete sum over sectors with Z_N holonomy. The reduced Eguchi-Kawai effective action of the localized theory is now:

$$\sum_{Z_N} \left[\exp\left(-\frac{N8\pi^2}{N_2 g_W^2} Q - \frac{N}{N_2 4g_W^2} \sum_{\alpha \neq \beta} \int \text{tr}_f(\mu_{\alpha\beta}^{-2}) d^4x\right) \right. \\ \left. \text{Det}^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} - iad_{\mu_{\alpha\beta}^-}) \text{Det}(-\Delta_A) \left(\frac{\Lambda}{2\pi}\right)^{n_b} \text{Det}^{\frac{1}{2}} \omega \frac{\delta \mu_{0+}}{\delta \mu'_{0+}} \times c.c. \right]_{n=\bar{n}=0} \quad (8.1)$$

The connection, A , denotes the solution of the equation $[F_{\alpha\beta}^- - \sum_p \mu_{\alpha\beta}^-(p) \delta^2(z - z_{p(u,\bar{u})}) = 0]_{n=\bar{n}=0}$ in each Z_N sector. $\text{Det}^{\frac{1}{2}} \omega$ is the contribution of the n_b zero modes due to the moduli and Λ the corresponding Pauli-Villars regulator. The complex conjugate factor arises by the conjugate representation.

Thus the holonomy of adjoint twistor Wilson loops at the fixed points is trivial, because of the cancellation of Z_N factors between the fundamental and the conjugate representations. Hence for twistor Wilson loops the same result is obtained performing the limit $\lambda \rightarrow 0$ outside the functional integral, leading to triviality via λ -independence and the all order argument of sect.(4.5), and inside the functional integral, leading to localization on the tensor product of Z_N surface operators and the conjugate representation, and to triviality as well.

9. Wilsonian beta function

9.1 Beta function by restricting the non- $SUSY$ Nicolai map to surface operators

We now proceed to the computation of the beta function. In order to define the renormalization of the coupling constant it is necessary to compute the classical YM action of surface operators. As we have seen in sect.(7) surface operators are defined by connections on parabolic bundles. In a mathematical sense we can think of parabolic bundles in two different ways. Either parabolic bundles occur on space-time with no boundary and with a divisor and a parabolic structure that belong to the space-time. This is the point of view in this paper and in some mathematical literature.

Or they arise on space-time with boundary, where the boundary is the parabolic divisor. This is the point of view of [8]. In the latter case the insertion of a surface operator keeps the finiteness of the action, since the singular parabolic locus is not included in the space-time integral that computes the action. This justifies also the term operators, since their occurrence is the analog of operator insertions⁴⁷.

However, our point of view is that the surface operators are dynamical objects and therefore their singular divisor is included in the path integral in space-time.

Despite the parabolic singularity, the topological term in the action has a well defined mathematical extension to parabolic bundles, as parabolic Chern class (see Eq.(7.13)).

⁴⁷We would like to thank Edward Witten for a discussion about this point.

This is not the case for the term involving the ASD field. As a consequence the classical YM action is quadratically divergent on each singular divisor, $p \times D_p$, of a surface operator, with a divergence proportional to the area of the singular locus of each surface operator. Therefore we need a way to handle this classical divergence.

We have already recalled in sect.(4.1) that when the codimension-two surface is non-commutative, as in our case, the YM action of the corresponding non-commutative reduced Eguchi-Kawai (EK) model is rescaled by a power of the inverse cutoff, that cancels precisely [10] the quadratic divergence that occurs evaluating the classical YM action on surface operators. This allows us to define a new kind of semi-classical computation for which the classical YM action is finite on parabolic bundles. In our case the EK reduction is only partial in such a way that the action is:

$$\frac{N}{2g^2} \hat{N} \left(\frac{2\pi}{\Lambda} \right)^2 \int d^2x \text{tr}_N \text{Tr}_{\hat{N}} (-i[\hat{\partial}_\alpha + i\hat{A}_\alpha, \hat{\partial}_\beta + i\hat{A}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \quad (9.1)$$

where the trace $\text{Tr}_{\hat{N}}$ is taken over a subspace of dimension \hat{N} , with

$$\hat{N} \left(\frac{2\pi}{\Lambda} \right)^2 = 2\pi\theta \quad (9.2)$$

in the large \hat{N} , θ , Λ limit. The contribution of each parabolic singularity in the action reads:

$$\begin{aligned} & \int d^2x \delta^{(2)}(x - x_p)^2 \\ &= \delta^{(2)}(0) \int d^2x \delta^{(2)}(x - x_p) \\ &= \left(\frac{\Lambda}{2\pi} \right)^2 \end{aligned} \quad (9.3)$$

and it is cancelled by its inverse in the EK reduced action. It is convenient to describe the same result in terms of the action of a commutative gauge theory. In this case the action reads:

$$\frac{N}{2g^2} \hat{N} \int d^4x \text{tr}_N \text{Tr}_{\hat{N}} (-i[\partial_\alpha + iA_\alpha, \partial_\beta + iA_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \quad (9.4)$$

and it is divergent on a surface operator as:

$$\begin{aligned} & \int d^4x \delta^{(2)}(x - x_p)^2 \\ &= \delta^{(2)}(0) V_2 \int d^2x \delta^{(2)}(x - x_p) \\ &= \left(\frac{\Lambda}{2\pi} \right)^2 V_2 \\ &= N_2 \end{aligned} \quad (9.5)$$

where V_2 is the area of the singular divisor of the surface operator and the last equality is the definition of N_2 . Thus the reduced action is related to the one of a commutative gauge theory by the factor of N_2^{-1} ⁴⁸:

$$\begin{aligned} & \frac{N}{2g^2} \hat{N} \left(\frac{2\pi}{\Lambda} \right)^2 \int d^2x \text{tr}_N \text{Tr}_{\hat{N}} (-i[\hat{\partial}_\alpha + i\hat{A}_\alpha, \hat{\partial}_\beta + i\hat{A}_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \\ &= \frac{N}{2g^2 N_2} \hat{N} \int d^4x \text{tr}_N \text{Tr}_{\hat{N}} (-i[\partial_\alpha + iA_\alpha, \partial_\beta + iA_\beta] + \theta_{\alpha\beta}^{-1} 1)^2 \end{aligned} \quad (9.6)$$

⁴⁸Precisely the same factor arises in the quenched version of the EK reduction [101].

We can interpret this result by saying that the normalization of the trace in the non-commutative EK reduced theory differs by a factor of N_2^{-1} from the one of a commutative theory. Thus it may be computationally convenient to perform the calculations in the commutative theory and at the end normalize appropriately the traces.

Let us evaluate firstly the action of the surface operators of Z_N holonomy that occur at the fixed points. The Morita equivalent commutative ⁴⁹ theory has gauge group $U(N \times \hat{N})$ and the center of $SU(N)$ is embedded diagonally in $U(N \times \hat{N})$ as $e^{\frac{i2\pi k}{N}} 1_{\hat{N}}$. For a surface operator of Z_N holonomy around the point p of charge k , i.e. such that $M_p = e^{\frac{i2\pi k}{N}}$, $N - k$ eigenvalues, $2\lambda_p$, of the ASD curvature at p , $F_{01}^- = 2\lambda_p \delta^{(2)}(z - z_p)$, are equal to $\frac{2\pi k}{N}$ and k eigenvalues are equal to $\frac{2\pi(k-N)}{N}$, for the curvature to be traceless and to give rise to the holonomy $M_p = e^{\frac{i2\pi k}{N}}$ ⁵⁰. The trace of the square of the eigenvalues of the ASD curvature in the fundamental representation is thus:

$$tr_N(4\lambda^2) = (N - k) \left(\frac{2\pi k}{N}\right)^2 + k \left(\frac{2\pi(k - N)}{N}\right)^2 = (2\pi)^2 \frac{k(N - k)}{N} \quad (9.7)$$

in such a way that the reduced action is:

$$S_{EK} = \frac{N\hat{N}(4\pi)^2}{2g^2} \frac{Q}{N_2} + \frac{\hat{N}^2}{2g^2} \sum_p 2(2\pi)^2 k(N - k) \quad (9.8)$$

and the Morita equivalent one is:

$$S_{YM} = \frac{N\hat{N}(4\pi)^2}{2g^2} Q + \frac{\hat{N}^2}{2g^2} N_2 \sum_p 2(2\pi)^2 k(N - k) \quad (9.9)$$

From these equations it follows that, if Q is finite, its contribution to the action is irrelevant with respect to the one of the ASD curvature at the parabolic singularities.

Once the classical quadratic divergence has been tamed by the EK reduction we need to understand the logarithmic divergences that lead to a non-trivial beta function in the Jacobian for the change of variables from the connection to the ASD curvature. These divergences have been already computed in sect.(3.4), and we can just adapt our previous calculation to the case of a lattice of surface operators. We have already observed that, as in Nekrasov localization, we should evaluate and renormalize the functional measure in a neighborhood of the fixed points and thereafter we should sit on the fixed points. Since the fixed points have no moduli and thus no zero modes, the inverse order, first sitting on the fixed points and then renormalizing the functional measure, would not lead to the correct result.

But let start ignoring for the moment the zero modes and sitting on the fixed points. The observation that the contribution of Q to the classical and the quantum effective action is irrelevant with respect to the one of the ASD curvature at the parabolic singularities resolves the issue about the different factors of Z_Q^{-1} and Z^{-1} that occur in sect.(3.4) as counterterms in the Jacobian of the non- $SUSY$ Nicolai map.

⁴⁹We rescale the area of the Morita equivalent theory from $\frac{L^2}{N_2}$ to L^2 in order to perform the thermodynamic limit uniformly in \hat{N} .

⁵⁰The curvature is not uniquely determined by the holonomy, since parabolic bundles admit extensions over the punctures such that the eigenvalues of the ASD curvature differ by shifts of 2π . Our choice is in some sense minimal. This feature together with many others, known in the mathematical literature, is reviewed in [8].

This observation allows us to ignore also issues related to the possible global non-triviality of the bundles and to the counting of the global moduli, and to concentrate only on the local counterterms associated to the sum over the points of the parabolic divisor.

As in every computation involving an effective action the background field should live on a scale much larger than the quantum fluctuating field. This may look awkward to realize for surface operators that carry delta-like singularities, that involve any momentum scale. In fact it is impossible to realize for an isolated surface operator, but it is possible for a lattice with uniform lattice spacing⁵¹. A typical example is the following one-loop contribution to Z^{-1} in Euclidean configuration space, that arises from the spin term of sect.(3.4) evaluated on surface operators:

$$\frac{1}{(4\pi^2)^2} \sum_{p,p'} \int d^2u d^2v \frac{NTr(\mu_p \bar{\mu}_{p'})}{(|z_p - z_{p'}|^2 + |u - v|^2)^2} \quad (9.10)$$

where the sum over p, p' runs over the planar lattice of the parabolic divisors of the surface operators. There is also an orbital contribution that has the same structure. Indeed the orbital logarithmic contribution to the beta function arises from terms of the kind⁵²:

$$\begin{aligned} & \int d^4x d^4y Tr(A_z(x) \partial_{\bar{z}} \frac{1}{(x-y)^2} A_{\bar{z}}(y) \partial_z \frac{1}{(x-y)^2}) \\ &= \int d^4x d^4y Tr(A_z(x) \frac{2(z-w)}{(x-y)^4} A_{\bar{z}}(y) \frac{2(\bar{z}-\bar{w})}{(x-y)^4}) \\ &= \int d^2z d^2w d^2u d^2v Tr(A_z(x) A_{\bar{z}}(y) \frac{4|z-w|^2}{(|z-w|^2 + |u-v|^2)^4}) \\ &\sim - \int d^2z d^2w d^2u d^2v Tr(A_z(x) A_{\bar{z}}(y) \partial_{\bar{z}} \partial_w \frac{4}{6(|z-w|^2 + |u-v|^2)^2}) \\ &= - \int d^2z d^2w d^2u d^2v Tr(\partial_{\bar{z}} \partial_w (A_z(x) A_{\bar{z}}(y)) \frac{4}{6(|z-w|^2 + |u-v|^2)^2}) \\ &= - \int d^2z d^2w d^2u d^2v Tr(\partial_{\bar{z}} A_z(x) \partial_w A_{\bar{z}}(y) \frac{4}{6(|z-w|^2 + |u-v|^2)^2}) \\ &\sim \sum_{p,p'} \int d^2u d^2v \frac{NTr(\mu_p^0 \mu_{p'}^0)}{(|z_p - z_{p'}|^2 + |u - v|^2)^2} \end{aligned} \quad (9.11)$$

where in the last line we used $\partial_{\bar{z}} A_z(x) \sim \sum_p \mu_p^0 \delta^{(2)}(z - z_p)$ and $\partial_w A_{\bar{z}}(y) \sim -\sum_p \mu_p^0 \delta^{(2)}(w - z_p)$ for the surface operators that occur at the fixed points, with $x = (z, \bar{z}, u, \bar{u})$ and $y = (w, \bar{w}, v, \bar{v})$.

We would like to find a regularization for which the loop expansion of the functional determinants evaluated on surface operators satisfies the usual power counting as in the background-field computation of the beta function.

To avoid quadratic divergences we restrict the sum to $p \neq p'$. Although quadratic divergences in the Morita equivalent theory correspond to finite counterterms in the EK reduced action, we

⁵¹In fact it is sufficient that the lattice spacing be uniform on a scale much larger than the typical scale of the physics involved.

⁵²The term involving $\partial_\alpha A_\alpha$ vanishes identically around the local singularity, while the term involving A_α^2 is quadratically divergent and does not contribute because of cancellations due to gauge invariance in any gauge invariant regularization of the theory. Since in our computations only functional determinants occur, the Pauli-Villars regularization suffices.

would like to avoid divergences at coinciding points in higher orders of the loop expansion for the conventional power counting to hold. We set further $\mu_p = \lambda_p = \lambda$ in such a way that :

$$\frac{1}{(4\pi^2)^2} \sum_{p \neq p'} \int d^2 u d^2 v \frac{NTr(\lambda^2)}{(|z_p - z_{p'}|^2 + |u - v|^2)^2} \quad (9.12)$$

is logarithmically divergent (both in the ultraviolet and the infrared). Indeed introducing a lattice scale a :

$$\begin{aligned} & \frac{1}{(4\pi^2)^2} \sum_{p \neq p'} \int d^2 u d^2 v \frac{NTr(\lambda^2)}{(|z_p - z_{p'}|^2 + |u - v|^2)^2} \\ & \rightarrow \frac{1}{(4\pi^2)^2} \int d^2 u a^{-2} d^2 z a^{-2} \int d^2 w d^2 v \frac{NTr(\lambda^2)}{(|z - w|^2 + |u - v|^2)^2} \\ & = \frac{1}{(4\pi^2)^2} N_2^2 \int d^2 w d^2 v \frac{NTr(\lambda^2)}{(|z - w|^2 + |u - v|^2)^2} \\ & \sim N_2^2 NTr(\lambda^2) \log \frac{\Lambda}{\mu} \\ & = N_2 \sum_p NTr(\lambda^2) \log \frac{\Lambda}{\mu} \end{aligned} \quad (9.13)$$

One factor of N_2 is just the sum on lattice points, the other factor is the "phase space area" $N_2 = (\frac{\Lambda}{2\pi})^2 V_2$ of one surface operator. From the preceding equation we read also that:

$$a^{-1} = \frac{\Lambda}{2\pi} \quad (9.14)$$

To summarize, there exist a point-splitting regularization of the effective action in the background of the lattice of surface operators, that together with the implicit use of the Pauli-Villars regularization, needed to handle quadratic tadpoles, leads to the same power counting as the usual dimensional regularization, with the logarithmic divergences occurring as in Eq.(9.12). Had the contributions with $p = p'$ been included, there would appear quadratic divergences, thus spoiling the usual power counting in higher order terms of the loop expansion. This lattice point-splitting regularization⁵³, followed by Epstein-Glaser renormalization in Euclidean configuration space (see [112] for references) is a possible starting point of a new constructive approach for the large- N YM theory.

We should understand now the contribution of the zero modes.

At first we count the moduli of surface operators and then we give an argument to identify the zero modes with the moduli, as for the instantons.

From the computation of the Z^{-1} factor that we already performed in sect.(3.4) it follows that the renormalization of the action in absence of zero modes would be:

$$\frac{8\pi^2 k(N-k)}{2g_W^2(\mu)} = 8\pi^2 k(N-k) \left(\frac{1}{2g_W^2(\Lambda)} - \frac{1}{(4\pi)^2} \frac{5}{3} \log\left(\frac{\Lambda}{\mu}\right) \right) \quad (9.15)$$

Adding to the action the contribution of the complex conjugate representation we get:

$$\frac{16\pi^2 k(N-k)}{2g_W^2(\mu)} = 16\pi^2 k(N-k) \left(\frac{1}{2g_W^2(\Lambda)} - \frac{1}{(4\pi)^2} \frac{5}{3} \log\left(\frac{\Lambda}{\mu}\right) \right) \quad (9.16)$$

⁵³This regularization has been found during joint work with Arthur Jaffe.

9.2 Dimension of the Lagrangian neighborhood of the fixed points

It follows from Eq.(9.16) that, in order to get the correct one-loop beta function, the contribution of the zero modes to the renormalization of the action should be $-2k(N-k)\log(\frac{\Lambda}{\mu})$, including the fundamental and conjugate representation. The sign is consistent with the Pauli-Villars regularization of zero modes, yet the absolute value of the coefficient of the logarithm is in general an even integer but not a multiple of 4, as it would be implied by the hyper-Kahler reduction⁵⁴. Thus the neighborhood of the fixed points cannot be generic. Remarkably we have already seen in sect.(7) that the fixed points sit automatically inside the Lagrangian cone of the moduli space for which A_u is nilpotent. Since in a Lagrangian neighborhood of the fixed points the dimension of the moduli space is generically one half of the dimension of a hyper-Kahler neighborhood, the correct beta function may arise.

We should classify now which are the Lagrangian neighborhoods of the fixed points that lead to the correct beta function.

One component of the Lagrangian cone corresponds to $A_u = 0$ and gives rise to unitary representations of the fundamental group. The complex dimension of an adjoint orbit for a generic parabolic unitary bundle of rank N is given by:

$$\dim_{\mathbb{C}} = \frac{1}{2}(N^2 - \sum_i m_i^2) \quad (9.17)$$

where m_i are the multiplicities of the eigenvalues. Vortices of Z_N holonomy have no moduli, since the holonomy lives in the center, for which the multiplicity of the eigenvalues (modulo 2π) equals the rank. But the following slight deformation of the eigenvalues gives rise to a non-trivial adjoint orbit for the holonomy:

$$2\lambda = \text{diag}(\underbrace{2\pi(k-N)/N + \varepsilon}_k, \underbrace{2\pi k/N - \varepsilon k/(N-k)}_{N-k}) \quad (9.18)$$

Thus the complex dimension of the orbit is:

$$\dim_{\mathbb{C}} = \frac{1}{2}(N^2 - k^2 - (N-k)^2) = k(N-k) \quad (9.19)$$

and the same holds for the complex conjugate orbit, in such a way that the real dimension of the orbit matches the number of zero modes needed for the correct beta function:

$$\frac{16\pi^2 k(N-k)}{2g_W^2(\mu)} = 16\pi^2 k(N-k) \left(\frac{1}{2g_W^2(\Lambda)} - \frac{1}{(4\pi)^2} \left(\frac{5}{3} + 2 \right) \log\left(\frac{\Lambda}{\mu}\right) \right) \quad (9.20)$$

However, there is a more intrinsic characterization of the Lagrangian neighborhood of the fixed points. Instead of deforming slightly the eigenvalues we may deform the moduli along nilpotent directions. Hence we may require that, in a unitary gauge in the Lagrangian neighborhood, exactly the same Hitchin equations are satisfied as at the fixed points. Therefore the eigenvalues of the ASD field are precisely:

$$2\lambda = \text{diag}(\underbrace{2\pi(k-N)/N}_k, \underbrace{2\pi k/N}) \quad (9.21)$$

⁵⁴A hyper-Kahler manifold has necessarily a real dimension that is a multiple of 4.

but the we allow the Higgs field, A_u , to have a nilpotent residue. In the Lagrangian cone not only the Higgs field has a nilpotent residue, but it is nilpotent itself [81]. These are bundles of Hodge type [111] for which the twistor connection has a holonomy that cannot be diagonalized, but it can be set in Jordan canonical form. In the Lagrangian cone the local moduli at a point for these bundles are parametrized by orbits of the Jordan canonical form for a real version of the complexification of the compact gauge group [111]. Thus in our case, for which the diagonal part of the holonomy is in Z_N , the local holonomy is unipotent.

Since the diagonal part of the holonomy is central, to compute the dimension of the orbit we need to consider only the nilpotent part. Any such matrix is conjugate by the Jordan theorem to a direct sum of k blocks of dimension d_i , such that $\sum_{i=1}^k d_i = N$, where N is the total rank. Each block has d_i zero eigenvalues on the diagonal⁵⁵ and it is upper triangular with all 1 on the super-diagonal.

The classical action is not modified at all by a deformation of the moduli along a nilpotent direction, since the nilpotent part of the holonomy is invisible in the ASD curvature in a unitary gauge [81].

Therefore, to get the correct beta function, we need to construct orbits of nilpotent Jordan matrices, such that the real dimension of the orbits for the action of a real version of the complexification of the gauge group is precisely the double of the complex dimension of the unitary orbits, i.e. of the flag manifolds.

Indeed in this case the conjugate representation describes exactly the same moduli, since the orbit is for the action of a real version of the gauge group.

Nilpotent orbits with the features just described, having double the dimension of a flag, are called Richardson orbits [113].

Here are some examples. The principal nilpotent complex orbit, i.e. the orbit of a nilpotent Jordan block of maximal rank, has precisely the same dimension as the complex orbit in the generic semisimple case, i.e. $N^2 - N = N(N - 1)$, that is always even. Thus the real dimension is a multiple of 4, as it should be.

By Eq.(9.17) the complex dimension is the double of the complex dimension of the maximal flag, and thus the principal nilpotent orbit is a Richardson orbit.

We are looking for a nilpotent orbit with double the dimension of the partial flag in Eq.(9.19).

Let us first recall the general formula for the dimension of the adjoint orbit of a nilpotent [114]:

$$\dim O_N = N^2 - N - 2 \sum_{i=1}^k (i-1)d_i \quad (9.22)$$

For example for the principal nilpotent, $k = 1$, $d_1 = N$, and we get the aforementioned result [115]. For the zero nilpotent, $k = N$, $d_i = 1$, and we get 0. The orbit of the direct sum of k nilpotent Jordan blocks of dimension 2 and of one Jordan block of dimension $N - 2k$ has double the dimension of the partial flag in Eq(9.19).

Indeed in this case:

$$\dim O_N = N^2 - N - 2 \sum_{i=1}^k (i-1)2 - 2 \sum_{i=k+1}^{N-k} (i-1)$$

⁵⁵A nilpotent matrix has all the eigenvalues zero.

$$\begin{aligned}
 &= N^2 - N + 2(2k + N - 2k) - 2 \sum_{i=1}^k i - 2 \sum_{i=k+1}^{N-k} i \\
 &= N^2 - N + 2N - (N - k + 1)(N - k) - (k + 1)k \\
 &= N^2 + N - (N - k + 1)(N - k) - (k + 1)k \\
 &= 2k(N - k)
 \end{aligned} \tag{9.23}$$

QED

We should clarify in which sense the moduli that label the orbits with fixed eigenvalues occur as zero modes. Let us consider first the unitary orbits in the Lagrangian cone. In this case in any smooth unitary gauge the ASD curvature, $\mu_{01}^-(p)$, is conjugated to the eigenvalues by the action of the unitary group.

However, the classical action depends on the eigenvalues but not on the unitary matrices. Thus the unitary matrices define flat directions in the classical action, but the holonomy of the twistor connection actually depends on the unitary matrices that label the orbit.

We can associate to these moduli zero modes of the functional determinants choosing the singular gauge in which the gauge curvature is diagonal. This gauge may be singular in the sense that may be reached by gauge transformations that are possibly singular along lines, i.e. semi-infinite strings that start at the punctures and end at infinity or with another puncture. From the point of view of the homological localization of the holomorphic loop equation of sect.(10) these gauge transformations are allowed, provided the associated strings do not intersect the backtracking arcs that connect the loop to a puncture, in such a way that the twistor connection is not discontinuous across the backtracking arcs.

In this unitary singular gauge the equations for the surface operators of unitary holonomy are:

$$F_{\alpha\beta}^- = \sum_p \lambda_{\alpha\beta}^-(p) \delta^{(2)}(z - z_p(u, \bar{u})) \tag{9.24}$$

with $\lambda_{01}^-(p)$ diagonal matrices and the other $\lambda_{\alpha\beta}^-(p)$ vanishing [80]. But the connection still depends implicitly on the moduli, in such a way that zero modes occur by the standard argument at the end of sect.(3.3).

In the case of Richardson orbits for Hodge bundles the holonomies are conjugated to the Jordan form by a real version of the complexification of the gauge group, that factorizes into a compact and a parabolic subgroup, the compact factor being a subgroup of the unitary group. In a unitary gauge only the orbit of the compact subgroup occurs as an adjoint orbit of the ASD field at a point, while the parabolic group parametrizes the moduli of the metric. Thus it there exists a possibly singular unitary gauge in which all the moduli occur as zero modes as for orbits of unitary holonomy.

10. Homological localization of the holomorphic loop equation

It has been known for many years that (twisted)- $SUSY$ observables can be localized in gauge theories with extended $SUSY$ by deformations that are trivial in the cohomology generated by the twisted super-charge [24, 25, 27]. Since cohomology is dual to homology [116]⁵⁶, we may wonder

⁵⁶The main difference between cohomology and homology is the lack of a ring structure in the latter.

as to whether we can compute functional integrals by deformations that are trivial in homology rather than in cohomology. Were the answer be affirmative, we could get localization without supersymmetry.

While there is no positive answer in local field theory, gauge theories contain many non-local observables, the Wilson loops. Thus the natural arena for homological localization in gauge theories, as opposed to cohomological localization, is the loop equation [46, 47].

In general the loop equation is the sum of a classical equation of motion and of a quantum term, that involves the contour integral along the loop. By homological localization of the loop equation we mean a deformation of the loop that is trivial in homology and for which the quantum term vanishes, in such a way that the loop equation is reduced to a critical equation for an effective action [10]. Hence the needed homological deformation has to satisfy the following properties.

It has to be trivial in homology.

It has to leave the expectation value of the loop invariant.

It has to imply the vanishing of the quantum term in the loop equation, i.e. of the term that contains the contour integral along the loop.

In this homology framework there is a very natural analog of the operation of adding a coboundary in cohomology, that is based on the zig-zag symmetry of Wilson loops. The zig-zag symmetry is the invariance of a Wilson loop by the addition of a backtracking arc ending with a cusp. This deformation is a "vanishing boundary" in singular homology. In a regularized version the arc is the boundary of a tiny strip. While this symmetry holds classically in most of the cases, quantum mechanically the renormalization process may spoil it. The reason is that in general Wilson loops have perimeter and cuspidal divergences. The perimeter divergence is linear in the cutoff scale. The cuspidal divergence is logarithmic, with a coefficient that in turn is divergent for backtracking cusps. In $SUSY$ gauge theories with extended $SUSY$ there are examples of locally- BPS Wilson loops that have no perimeter divergence [105].

We have seen in sect.(4) that twistor Wilson loops share with their supersymmetric cousins these non-renormalization properties and, being trivial in the large- N limit, they have not cuspidal divergences either. Localization by homology leads this kind of non-renormalization properties to their extreme consequences.

One of the virtues of the lattice regularization of the Nicolai map of sect.(7), from the point of view of the homological localization, is to allow identifying the cusps of the aforementioned backtracking arcs with the parabolic singularities of the reduced EK theory. We have noticed in sect.(7) that this point-like parabolic singularities are daughters of codimension-two singularities of surface operators of the parent four-dimensional theory. The lattice version of the holomorphic loop equation of sect.(6) follows ⁵⁷ :

$$\langle Tr(\frac{\delta\Gamma}{\delta\mu(z_p)'}\Psi'(C_{z_p z_p})) \rangle = \frac{1}{\pi} \int_{C_{z_p z_p}} \frac{dw}{z_p - w} \langle Tr\Psi'(C_{z_p w}) \rangle \langle Tr1 \rangle \quad (10.1)$$

where Ψ' is the holonomy of B in the gauge $B'_z = 0$. The lattice points associated to the divisor of surface operators become the cusps that are the end points, p , of the backtracking strings, b_p , that

⁵⁷The holomorphic loop equation is written in linear form since it is assumed that the loop C_{zz} is simple, i.e. it has no self-intersections.

perform the deformation of the loop, C . Adding the backtracking strings implies the homological localization of the holomorphic loop equation:

$$\langle \text{Tr} \left(\frac{\delta \Gamma([b_p])}{\delta \mu'(z_p)} \Psi'(C \cup [b_p]) \right) \rangle = 0 \quad (10.2)$$

The homological localization can be understood in the following geometrical terms. The existence of the regularized residue in Eq.(6.8) is the geometric obstruction to the localization of the loop equation. The regularized residue is the line integral of a current supported on a point and computes the de Rham cohomology of compact support in one dimension, $H_c^1(R)$ ⁵⁸, that coincides with the cohomology of a point, $H^0(pt)$ [116]. It is interesting to display how this cohomology of a point can be obtained by a Mayer-Vietoris argument [116] involving partitions of unity in the right hand side of the loop equation. For a smooth point of the arc (i.e. a point with a continuous tangent), introducing partitions of unity on the arc, $1 = u_1(s) + u_2(s)$, we get:

$$\int dw_+(s) \delta(z_+(s_{sm}) - w_+(s)) = u_1(s_{sm}) \frac{\dot{w}_+(s_{sm}^+)}{|\dot{w}_+(s_{sm}^+)|} + u_2(s_{sm}) \frac{\dot{w}_+(s_{sm}^-)}{|\dot{w}_+(s_{sm}^-)|} = 1 \quad (10.3)$$

Let us suppose now that we try to compute the "cohomology of a backtracking cusp". The same Mayer-Vietoris argument shows that such a cohomology does not exist in a classical sense, since the result depends on the choice of the partition of unity:

$$\begin{aligned} \int dw_+(s) \delta(z_+(s_{cusp}) - w_+(s)) &= u_1(s_{cusp}) \frac{\dot{w}_+(s_{cusp}^+)}{|\dot{w}_+(s_{cusp}^+)|} + u_2(s_{cusp}) \frac{\dot{w}_+(s_{cusp}^-)}{|\dot{w}_+(s_{cusp}^-)|} \\ &= u_1(s_{cusp}) - u_2(s_{cusp}) \end{aligned} \quad (10.4)$$

This is due to the fact that we can integrate distributions on smooth manifolds, but their extension to non-smooth ones depends on arbitrary choices in general. In particular, if the partition of unity is symmetric, $u_1(s_{cusp}) = u_2(s_{cusp})$, the regularized residue vanishes. Therefore a regularization exists, that preserves the zig-zag symmetry of twistor Wilson loops, for which the holomorphic loop equation localizes.

Thus if every marked point of the loop equation can be transformed into a backtracking cusp we can complete our argument about localization. But this is precisely the effect of our lattice, since marked points of the loop contribute to the loop equation in the lattice theory only if they coincide with the lattice points.

We may think that it is a change of the conformal structure around the lattice points that generates the cusps.

Since the Lagrangian submanifold on which twistor Wilson loops are supported on satisfies:

$$|dzd\bar{z}| = |dud\bar{u}| \quad (10.5)$$

this two-dimensional conformal transformation lifts to a conformal rescaling of the four-dimensional metric:

$$ds^2 = dzd\bar{z} + dud\bar{u} \quad (10.6)$$

⁵⁸The integral of the delta function in one dimension can be approximated by the normalized integral of one-forms of small compact support around a point.

Thus it acts by adding a conformal anomaly to the effective action:

$$\Gamma([b_p]) = \Gamma([p]) + \text{ConformalAnomaly}([b_p]) \quad (10.7)$$

that amounts to a local counterterm, i.e. to a change of the subtraction point.

Therefore there is a symmetry of the RG flow that generates the homological deformation of the loop by a vanishing boundary, i.e. by backtracking strings. This is the analog of the action being a closed form in cohomology, since in the last case there is a symmetry of the action (i.e. the twisted supersymmetry) that generates the coboundary.

At this point we would like to clarify why we cannot use the Makeenko-Migdal (MM) loop equation to get localization.

We can write the MM loop equation for unitary Wilson loops in the large- N YM theory as:

$$\begin{aligned} \int_{C_{xx}} dx_\alpha &< \frac{N}{2g^2} \text{Tr} \left(\frac{\delta S_{YM}}{\delta A_\alpha(x)} \Psi(x, x; A) \right) > \\ &= i \int_{C_{xx}} dx_\alpha \int_{C_{xx}} dy_\alpha \delta^{(4)}(x-y) < \text{Tr} \Psi(x, y; A) > < \text{Tr} \Psi(y, x; A) > \end{aligned} \quad (10.8)$$

where

$$\Psi(x, y; A) = P \exp i \int_{C(x,y)} A_\alpha dx_\alpha \quad (10.9)$$

In the case of loops without self-intersections but with cusps the MM loop equation reduces to:

$$\begin{aligned} \int_{C_{xx}} dx_\alpha &< \frac{N}{2g^2} \text{Tr} \left(\frac{\delta S_{YM}}{\delta A_\alpha(x)} \Psi(x, x; A) \right) > \\ &= i \int_{C_{xx}} dx_\alpha \int_{C_{xx}} dy_\alpha \delta^{(4)}(x-y) < \text{Tr} \Psi(x, x; A) > < \text{Tr} 1 > \end{aligned} \quad (10.10)$$

Performing the two contour integrations along the loop in the right hand side, we get [105]:

$$\begin{aligned} \int_{C_{xx}} dx_\alpha &< \frac{1}{2g^2} \text{Tr} \left(\frac{\delta S_{YM}}{\delta A_\alpha(x)} \Psi(x, x; A) \right) > \\ &\sim i(P\Lambda^3 + \sum_{cusp} \frac{\cos \Omega_{cusp}}{\sin \Omega_{cusp}} \Omega_{cusp} \Lambda^2) < \text{Tr} \Psi(x, x; A) > \end{aligned} \quad (10.11)$$

where P is the perimeter of the loop and Ω_{cusp} the cusp angle at a cusp. For our conventions $\Omega_{cusp} = 0$ for no cusp, while $\Omega_{cusp} = \pi$ for a backtracking cusp. The perimeter divergence arises by the double integration of the four-dimensional delta function, i.e. of the contact term, along the loop. Integrating the contact term in a neighborhood of each cusp gives rise to a sub-leading quadratic divergence, since around a cusp there are two independent integrations instead of one, due to the two sides of the cusp. The coefficient of the cusp contribution is proportional to the ratio:

$$\frac{\cos \Omega_{cusp}}{\sin \Omega_{cusp}} \quad (10.12)$$

The numerator arises from the scalar product, the denominator from the two independent integrations of the two-dimensional delta function. In the limit in which the cusp angle Ω_{cusp} reaches

π the cusp backtracks and the cusp contribution to the contact term of the MM loop equation is divergent. It turns out to be proportional to the coefficient of the logarithm in the cusp anomaly [105]. Therefore for backtracking cusps the cusp anomaly and the perimeter divergence mix together [105]. Zig-zag invariance can be implemented in the MM loop equation only by means of a subtle, regularization dependent, cancellation between the extra perimeter due to the backtracking cusp and the cusp anomaly. This is due to the non-cohomological nature of the contact term that arises as the obstruction to localization in the MM loop equation, that is the line integral of a non-integrable distribution.

We can write the MM loop equation also for a planar twistor Wilson loop:

$$\int_{C_{ww}} d\bar{z} \langle \frac{N}{2g^2} \text{Tr} \left(\frac{\delta S_{NC}}{\delta B_z(z, \bar{z})} \Psi(z, z; B) \right) \rangle = i \int_{C_{ww}} d\bar{z} \int_{C_{ww}} dz \delta^{(2)}(z-w) \langle \text{Tr} \Psi(w, z; B) \rangle \langle \text{Tr} \Psi(z, w; B) \rangle \quad (10.13)$$

Also in this case the obstruction to localization of the loop equation is of non-cohomological nature, being the line integral of a $\delta^{(2)}$, as opposed to the $\delta^{(1)}$ that occurs in the regularized holomorphic loop equation. For twistor Wilson loops the contact term in the MM loop equation gives rise to the same divergent contribution⁵⁹ for backtracking cusps as for unitary Wilson loops. Cancellations occur only in the solution of the loop equation. But even if it were possible to get cancellations at the cusps by fine-tuning, a lattice regularization of the MM equation would provide links rather than points, because the functional integral in the MM equation involves the connection instead of the curvature.

Hence the MM loop equation cannot localize in the homological sense that we are discussing, not even for twistor Wilson loops, because it cannot be regularized in a way that does implement the zig-zag symmetry.

11. Canonical beta function of large- N YM by homological localization

11.1 Gluing rules for local systems

In this section we compute the canonical beta function of the large- N YM theory in the scheme defined by the homological localization of the loop equation. We take into account some global constraint that we have disregarded in our computation of sect.(9) of local counterterms in the Wilsonian scheme.

We have seen in the last section that, for getting homological localization of the holomorphic loop equation, we should draw backtracking strings from the loop to the lattice points in order to transform all the marked points into cusps.

We now show that the cusps can be paired by the backtracking strings.

This is a consequence of requiring the consistency of the gluing rules for functional integrals with the localization on local systems. Indeed, to glue together two disks with twistor Wilson loops as boundaries, we need that the twistor Wilson loops on the boundaries, taken with opposite orientation, agree. Moreover, in a theory in which the twistor Wilson loops are localized on local

⁵⁹The missing factor of Λ^2 is in fact hidden in the normalization of S_{NC} (see sect.(4)).

systems, they are determined by the holonomies around the punctures inside the disks, since local systems are the same as representations of the fundamental group.

Therefore the precise condition for gluing is that the product of the holonomies around the points inside the two disks are the inverse of each other, in such a way that the total product is 1. This constraint is satisfied naturally gluing a lattice of vortices in one disk with a lattice of the same number of antivortices⁶⁰ in the other disk. Hence in this case the number of punctures in the two disks to be glued must be equal in general. Another possibility is to pair vortices in one disk with vortices in the other disk and to impose the condition that the product of all the holonomies is 1 after gluing, for example requiring that the total number of points is an integer multiple of N . This can be certainly done for disks that have the same number of punctures and the same cutoff. Afterward one of the disks is rescaled together with the cutoff to a large size in order to perform the thermodynamic limit (see below). The procedure does not spoil localization since all the punctures are paired by backtracking arcs, that can be glued on the common boundary of the disks since the punctures are equal in number.

This looks like a holographic correspondence [3] that is the consequence that all the punctures can be paired by arcs connecting punctures in different disks⁶¹.

The resulting configuration has not a translational invariant cutoff, as in the computation of the Wilsonian beta function, but what really matters for that computation to hold is that the lattice on which the background field lives has a very large number of punctures and it is locally uniform.

If the punctures of the two disks are paired, the density of the punctures in the two disks cannot be the same in the thermodynamic limit, since the area of one disk must be much larger than the area of the other one. Thus we get two cutoff scales, a and \tilde{a} , on the two disks, that we refer to as the ultraviolet and infrared scale respectively and that we identify with the lattice spacing. Thus each puncture carries a weight that is the lattice scale. Since the weights are all equal inside the two disks, because we require at least locally uniform lattice spacing, and the punctures are equal in number, the two collections of weights are in fact projectively equal, i.e. equal up to a common rescaling factor.

Therefore the localization of the loop equation for twistor Wilson loops on local systems leads to a kind gluing that coincides with the one implied by a weighted arc families with projectively equal weights [54]. Surprisingly these are precisely the string gluing axioms [54, 55, 56] for weighted arc families at topological level (see fig.4 I of [56]). Hence we may say that open strings solve the YM loop equation for the twistor Wilson loops, in the sense that they localize the loop equation on a saddle point for an effective action.

We have just come to the conclusion that if we combine the gluing properties of the functional integral with the vanishing requirement for the contact term in the holomorphic loop equation, i.e. the requirement of localization, we get the string gluing axioms at topological level. Thus the proper graph to get localization is a weighted graph [54] similar to a Mandelstam graph [57, 58] (see fig.4 I of [56]). The weights in this language are the sizes of the strips.

⁶⁰By antivortices we mean surface operators that have precisely the inverse holonomy of vortices. They have the same classical action, if evaluated at the same cutoff scale, and a neighborhood with moduli space of the same dimension as for vortices.

⁶¹These are the "hairs" associated to the holographic correspondence between the ultraviolet and the infrared (see below).

A subtle point arises about the cutoff of the localized effective action. We recall that the introduction of a lattice is essential for localization, since it allows us to transform every non-trivial marked lattice point into a backtracking cusp. We have seen that the gluing rules imply a different cutoff at the cusps of the two disks of the sphere in the quantum effective action.

Hence the local part of the effective action, as a consequence of the stringy nature of localization in the loop equation, is in fact bilocal with two local fields living at different scales, one at the ultraviolet cutoff and one at the infrared cutoff, paired by the backtracking strings. We will see momentarily that the field at the ultraviolet, but not the one at the infrared, affects the renormalization of the Wilsonian coupling constant, as it is expected from its very definition. Instead the field at the infrared together with the one at the ultraviolet affects the renormalization of the canonical coupling constant.

To summarize, we draw our weighted graph [54, 56, 57, 58], that is made by two charts with the boundary loop in common. The charts are a conformal transformation of two topological disks with punctures (see fig.(1) and fig.(2) of [55]). This introduces a conformal transformation in the EK two-dimensional reduced theory. However, this transformation lifts to a conformal rescaling in the four-dimensional parent theory because on the Lagrangian submanifold that is the support of twistor Wilson loops Eq.(10.6) holds. Thus the four-dimensional metric changes conformally and the effective action changes by the appropriate conformal anomaly (see Eq.(10.7)). This implies that, in addition to the explicit cutoff dependence, the effective action on the weighted graph is related to the one on the sphere with marked points by the addition of a divergent conformal anomaly, because of the singularity of the conformal transformation. This divergent conformal anomaly plays a key role to reconcile our computation of the anomalous dimension in the canonical beta function with the general properties of the RG group and, more generally, in the interpretation a posteriori of localization as a RG flow to the ultraviolet.

11.2 Z factor and canonical normalization

It is useful to write the effective action of the commutative Morita equivalent theory before the EK reduction, that amounts to dividing by the factor of N_2 (see sect.(9)). This is convenient to define the canonical normalization of the effective action in analogy to the $SUSY$ case of sect.(3.2).

The contribution of one surface operator of Z_N holonomy, that we call a vortex, to the local part of the Morita equivalent Wilsonian effective action reads:

$$\begin{aligned}
 & \exp(-\Gamma_q(\text{one} - \text{vortex})) \\
 &= \exp\left(-\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k(N-k)\right) \exp\left(\frac{2\pi}{Ha_T^2} \frac{k(N-k)}{2} \log\left(\frac{1}{Ha^2}\right)\right) \\
 & \exp\left(-\frac{2\pi}{H\tilde{a}^2} \frac{8\pi^2}{2g_W^2} k(N-k)\right) \exp\left(\frac{2\pi}{Ha_T^2} \frac{k(N-k)}{2} \log\left(\frac{1}{H\tilde{a}^2}\right)\right) \tag{11.1}
 \end{aligned}$$

where the factor of $\frac{2\pi}{Ha_T^2}$ is the transverse measure over the zero modes two-dimensional vortex sheet that is the singular divisor of a surface operator in four dimensions, and $H = \frac{1}{\theta}$ by definition. The factor in the second line of the right hand side is actually the contribution of the antivortex in the infrared ⁶², that is paired to the vortex by a backtracking string. Equating the ultraviolet

⁶²Or of the vortex in the infrared.

cutoff on the transverse and longitudinal planes, because of rotational invariance, we get $a = a_T$. Another way of formulating this condition is that the longitudinal measure on the size of vortices that we read from the classical action and the transverse measure on the vortices two-dimensional sheet should coincide. The Wilsonian beta function of sect.(9) follows, since the contribution of the antivortex at the infrared is finite.

For a twistor Wilson loop in the adjoint representation it is possible to pair the holomorphic integral to the antiholomorphic one. In this case the counting of zero modes corresponds to the real dimension of the orbits. Thus we get for the bilocal part of the Wilsonian effective action:

$$\begin{aligned} \exp(-\Gamma_q) &= \prod_p \exp\left(-\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k_p(N-k_p) + c.c.\right) \exp\left(\frac{2\pi}{Ha_T^2} \frac{k_p(N-k_p)}{2} \log\left(\frac{1}{Ha^2}\right) + c.c.\right) \\ &\quad \exp\left(-\frac{2\pi}{H\tilde{a}^2} \frac{8\pi^2}{2g_W^2} k_p(N-k_p) + c.c.\right) \exp\left(\frac{2\pi}{Ha_T^2} \frac{k_p(N-k_p)}{2} \log\left(\frac{1}{H\tilde{a}^2}\right) + c.c.\right) \end{aligned} \quad (11.2)$$

We now come to the canonical coupling. We observe that the fields at the ultraviolet and at the infrared are not canonically normalized in the Wilsonian effective action. In order to obtain the canonical β function, as in the $\mathcal{N} = 1$ SUSY YM case of sect.(3.2), we have to rescale the fields on the ultraviolet divisor by a factor of $gZ^{\frac{1}{2}}$ and on the infrared divisor by a factor of g , since Z is finite in the infrared and can be normalized to 1. This can be achieved by rescaling a and H as $a = a_c Z^{-\frac{1}{2}}$ and $H = g^{-2} H_c$ and setting $a_c = a_T$ to preserve an equal longitudinal and transverse cutoff. Rescaling H is equivalent to rescale inversely L^2 , the area of a surface operator. The canonically normalized effective action reads:

$$\begin{aligned} \exp(-\Gamma_q) &= \prod_p \exp\left(-\frac{2\pi}{H_c a_c^2} \frac{8\pi^2}{2g_W^2} g^2 k_p(N-k_p) + c.c.\right) \exp\left(\frac{2\pi}{H_c a_c^2} g^2 \frac{k_p(N-k_p)}{2} \log\left(\frac{1}{H_c g^{-2} Z^{-1} a_c^2}\right) + c.c.\right) \\ &\quad \exp\left(-\frac{2\pi}{H_c \tilde{a}^2} \frac{8\pi^2}{2g_W^2} g^2 k_p(N-k_p) + c.c.\right) \exp\left(\frac{2\pi}{H_c a_c^2} g^2 \frac{k_p(N-k_p)}{2} \log\left(\frac{1}{H_c g^{-2} \tilde{a}^2}\right) + c.c.\right) \end{aligned} \quad (11.3)$$

This produces the following rescaling factors at each point from the contribution of the zero modes:

$$[(gZ^{\frac{1}{2}})^{k(N-k)} g^{k(N-k)}]_{H_c a_c^2}^{\frac{2\pi}{g}} g^2 \quad (11.4)$$

Thus defining as in sect.(3.2)

$$-\frac{8\pi^2 k(N-k) g^2}{2g^2(\Lambda)} = -\frac{8\pi^2 k(N-k) g^2}{2g_W^2(\Lambda)} + 2k(N-k) g^2 \log g + g^2 \frac{1}{2} k(N-k) \log Z \quad (11.5)$$

and factorizing the term $8\pi^2 k(N-k) g^2$ we get:

$$\frac{1}{2g_W^2(\Lambda)} = \frac{1}{2g^2(\Lambda)} + \frac{4}{(4\pi)^2} \log g + \frac{1}{(4\pi)^2} \log Z \quad (11.6)$$

Taking the derivative with respect to $\log \Lambda$ and using the fact that g_W is 1-loop exact we obtain:

$$\beta_0 = -\frac{1}{g^3} \frac{\partial g}{\partial \log \Lambda} + \frac{4}{(4\pi)^2} \frac{1}{g} \frac{\partial g}{\partial \log \Lambda} + \frac{1}{(4\pi)^2} \frac{\partial \log Z}{\partial \log \Lambda} \quad (11.7)$$

from which it follows:

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{g^3}{(4\pi)^2} \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2} \quad (11.8)$$

Since

$$\frac{\partial \log Z}{\partial \log \Lambda} \sim \frac{10}{3} \frac{g_W^2}{(4\pi)^2} \frac{1}{1 + \frac{10}{3} \frac{g_W^2}{(4\pi)^2} \log \frac{\Lambda}{\mu}} \sim \frac{10}{3} \frac{g_W^2}{(4\pi)^2} \sim \frac{10}{3} \frac{g^2}{(4\pi)^2} \quad (11.9)$$

we get:

$$\frac{\partial g}{\partial \log \Lambda} = \left(-\beta_0 g^3 + \frac{10}{3} \frac{g^5}{(4\pi)^4} \right) \left(1 + \frac{4}{(4\pi)^2} g^2 + \dots \right) = -\beta_0 g^3 - \beta_1 g^5 + \dots \quad (11.10)$$

where

$$\beta_1 = \frac{4\beta_0}{(4\pi)^2} - \frac{10}{3} \frac{1}{(4\pi)^4} = \frac{34}{3} \frac{1}{(4\pi)^4} \quad (11.11)$$

that agrees with the perturbative result up to two-loops [12, 13, 14, 15].

Now we come to the computation of the anomalous dimension. The one-loop exactness of Z implies:

$$\frac{\partial \log Z}{\partial \log a} = - \frac{\frac{1}{(4\pi)^2} \frac{10}{3} g_W^2}{1 - g_W^2 \frac{1}{(4\pi)^2} \frac{10}{3} \log \left(\frac{\tilde{a}}{\sigma a} \right)} \quad (11.12)$$

where now we have included the contribution of the conformal anomaly, that rescales the subtraction point by the factor of σ to give a finite but arbitrary result for the higher order contributions to the anomalous dimension. Assuming that the beta function is independent on the subtraction point, as required by general principles of the RG , the RG trajectory must be followed along the line $c = -\frac{1}{(4\pi)^2} \frac{10}{3} \log \left(\frac{\tilde{a}}{\sigma a} \right) = \text{const}$. We observe that it is precisely the contribution of the conformal anomaly that allow the anomalous dimension to be a function of the coupling only, according to the RG .

12. Glueballs spectrum from the localized effective action

12.1 Glueballs propagators and anomalous dimensions in perturbation theory at large- N

We start this section recalling some features and conjectures about the large- N limit of pure YM .

To the leading large- N order and to every order in the 't Hooft coupling constant g , the expectation value of a product of normalized local gauge invariant operators factorizes [117]. For example:

$$\left\langle \frac{1}{N} \sum_{\alpha\beta} \text{tr} F_{\alpha\beta}^2(x_1) \dots \frac{1}{N} \sum_{\alpha\beta} \text{tr} F_{\alpha\beta}^2(x_k) \right\rangle = \left\langle \frac{1}{N} \sum_{\alpha\beta} \text{tr} F_{\alpha\beta}^2(x_1) \right\rangle \dots \left\langle \frac{1}{N} \sum_{\alpha\beta} \text{tr} F_{\alpha\beta}^2(x_k) \right\rangle \quad (12.1)$$

Thus to this order the only information that survives is the value of the condensate. In the case of $\langle g^{-2} \sum_{\alpha\beta} \frac{1}{N} \text{Tr} F_{\alpha\beta}^2(x) \rangle$, it must be proportional for a suitable regularization to the appropriate power of the renormalization group invariant scale, Λ_{YM} , since it coincides up to numerical factors with the action density.

In turn Λ_{YM} encodes the information on the beta function of the large- N theory. In addition it is believed that to the next to leading $\frac{1}{N}$ order the connected two-point functions of local gauge invariant operators are saturated by a sum of pure poles.

For example, for the scalar glueballs propagator it is conjectured that the equation holds [71]:

$$\int \langle \frac{1}{N} \sum_{\alpha\beta} \text{tr} F_{\alpha\beta}^2(x) \sum_{\alpha\beta} \frac{1}{N} \text{tr} F_{\alpha\beta}^2(0) \rangle_{conn} e^{ipx} d^4x = \sum_r \frac{Z_r}{p^2 + M_r^2} \quad (12.2)$$

The sum of pure poles is constrained by the perturbative operator product expansion [72]. It must agree asymptotically for large momentum with the "anomalous dimension" of the glueballs propagator as computed by perturbation theory plus the sum over condensates that occur in the operator product expansion [72]. Indeed the scalar glueballs propagator behaves in perturbation theory at large momentum, within two-loop accuracy, up to contact terms, i.e. polynomials in the momentum squared, p^2 , as [72]:

$$g^4(p) p^4 \log\left(\frac{p^2}{\mu^2}\right) \quad (12.3)$$

The logarithm explicitly displayed in the two-loop computation is necessary to reproduce the conformal behavior:

$$\int d^4p e^{ipx} p^4 \log\left(\frac{p^2}{\mu^2}\right) \sim \frac{1}{x^8} \quad (12.4)$$

The factors of g , the renormalized 't Hooft coupling at momentum p , occur because of the canonical normalization of the glueballs propagator, that involves $\langle \sum_{\alpha\beta} \frac{1}{N} \text{Tr} F_{\alpha\beta}^2(x) \rangle$ rather than the action density $\langle g^{-2} \sum_{\alpha\beta} \text{Tr} F_{\alpha\beta}^2(x) \rangle$, and they account for the one-loop anomalous dimension that in this case is determined by the one-loop coefficient of the beta function. They imply logarithmic corrections to the conformal behavior. Glueballs propagators for other normalized gauge invariant operators of naive dimension L involve in general one-loop anomalous dimensions that are independent on the one-loop coefficient of the beta function [16, 17]:

$$\begin{aligned} & \int \langle \frac{1}{N} \text{tr} O(x) \frac{1}{N} \text{tr} O(0) \rangle_{conn} e^{ipx} d^4x \\ &= \sum_r \frac{Z_r}{p^2 + M_r^2} \\ &= G_O(p^2) \sim Z_O^2(p^2) p^{2L-4} \log\left(\frac{p^2}{\mu^2}\right) \end{aligned} \quad (12.5)$$

and satisfy the following RG equation:

$$\left(\frac{\partial}{\partial \log p} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_O(g) \right) G_O(p^2) = 0 \quad (12.6)$$

where

$$\gamma_O(g) = \frac{\partial \log Z_O}{\partial \log p} \quad (12.7)$$

Sometimes it is convenient to factorize out the contribution of the anomalous dimension:

$$G'_O(p^2) = Z_O^{-2}(p^2) G_O(p^2) \quad (12.8)$$

in such a way that the suitably normalized glueballs propagator, $G'_O(p^2)$, is RG invariant⁶³:

$$\left(\frac{\partial}{\partial \log p} + \beta(g) \frac{\partial}{\partial g} \right) G'_O(p^2) = 0 \quad (12.9)$$

In the large- N limit there is a sector of the theory that is integrable at one loop [16, 17], that is made by operators of ASD or SD type and their covariant derivatives. The corresponding anomalous dimensions can be computed as the eigenvalues of a Hamiltonian spin chain.

In the ASD one-loop integrable sector the anomalous dimension of $\frac{1}{N} \sum_{\alpha\beta} Tr F_{\alpha\beta}^{-2}(x)$ is the same as the one of $\frac{1}{N} \sum_{\alpha\beta} Tr F_{\alpha\beta}^2(x)$ since $\langle g^{-2} \sum_{\alpha\beta} \frac{1}{N} Tr(F_{\alpha\beta} \tilde{F}_{\alpha\beta}) \rangle$ and $\langle g^{-2} \sum_{\alpha\beta} \frac{1}{N} Tr F_{\alpha\beta}^2(x) \rangle$ are both RG invariant [16]. Therefore the renormalization factor of $\frac{1}{N} \sum_{\alpha\beta} Tr F_{\alpha\beta}^{-2}(x)$ is g^2 and the one-loop anomalous dimension coincides with $2\beta_0$.

The anomalous dimensions of a number of operators can be computed explicitly solving by the Bethe ansatz the Hamiltonian spin chain in the thermodynamic limit, that corresponds to operators of large naive dimension $2L$ and length L ⁶⁴. In particular the anomalous dimensions of the antiferromagnetic ground states turn out to be of the form (see Eq.(27) of [16] and Eq.(5.23) of [17]):

$$Z_L = 1 - Lg^2 \frac{5}{3} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right) + O(L^0) \quad (12.10)$$

The ground state of the spin chain corresponds to the operators with the most negative anomalous dimension, that turn out to be all scalars constructed by certain contractions involving only the ASD part of the curvature [17].

12.2 Localized effective action and holomorphic/antiholomorphic fusion

We are now ready to start our computation of the glueballs spectrum. The large- N one-loop integrable sector contains the correlators that can be computed by localization. This is perhaps not completely surprising since localization involves the change of variables from the gauge connection to the ASD part of the curvature, that lives in the one-loop integrable sector.

In the previous sections we have obtained the localization of trivial twistor Wilson loops, in the sense that we have reduced the loop equation for them to a critical equation for an effective action and we have renormalized the effective action in Euclidean space, in such a way that the effective action is finite.

⁶³We learned this suggestion by an unpublished talk of Gabriele Ferretti, "Applying the BES trick to QCD", Newton Institute, Cambridge (2007).

⁶⁴We would like to thank Konstantin Zarembo for explaining to us this result at the GGI. We would like to thank also Gabrielle Ferretti for explanations about the same subject at Chalmers University.

Now the question arises as to which infrared information the renormalized effective action actually contains. We have seen that to make sense of the holomorphic loop equation of sect.(6) after the renormalization of the Euclidean effective action it is necessary to continue analytically the Euclidean loop equation to Minkowski space-time. Then at operator level the twistor Wilson loops become:

$$\text{Tr}_{\mathcal{N}} \Psi(\hat{B}_\rho; L_{ww}) \rightarrow \text{Tr}_{\mathcal{N}} P \exp i \int_{L_{ww}} (\hat{A}_{z_+} + i\rho \hat{D}_u) dz_+ + (\hat{A}_{z_-} + i\rho^{-1} \hat{D}_{\bar{u}}) dz_- \quad (12.11)$$

as shown in sect.(4). The support of the twistor Wilson loops analytically continued in this way becomes:

$$(z, \bar{z}, u, \bar{u}) = (z_+, z_-, -\rho z_+, -\rho^{-1} z_-) \quad (12.12)$$

For computational simplicity we set $\rho = -1$ in the following. This is just an irrelevant change of sign of the antihermitian part of μ_1 , since $\mu_\rho = \mu^0 + \rho n - \rho^{-1} \bar{n}$. Thus in this section $\mu = \mu_{-1} = \frac{1}{2}(\mu_{01} - i\mu_{03})$.

It is in the spirit and in the letter of the localization idea that the renormalized effective action analytically continued to Minkowski space-time contains information on local observables supported on the Lagrangian submanifold displayed in Eq.(12.12) since this is the analytic continuation of the support of Euclidean twistor Wilson loops from which the effective action has been obtained by localization. However, getting the spectrum of fluctuations exceeds by far the limited framework of localization of the homology of 1 in the same sense in which the statement that the prepotential of $\mathcal{N} = 2$ SUSY YM determines the low energy effective action in the Coulomb branch exceeds by far the framework of the localization of the cohomology of 1.

We have seen in sect.(10) that localization by homology requires implicitly extending diagonally the action of the conformal group from two to four dimensions, an extension that can occur meaningfully only on Lagrangian submanifolds of the kind displayed in Eq.(12.12). Thus the aforementioned Lagrangian submanifold is in a sense the only object for which we can hope that the localized effective action has a physical meaning beyond the leading large- N approximation.

More physically homological localization requires that the ASD field be singular and therefore by necessity be of magnetic type, in a theory in which smooth fields are of electric type. Thus homological localization realizes in a technical sense long-standing ideas on the YM vacuum as a dual superconductor [42, 43, 44], whose analog are the duality ideas [26] that lead to the physical justification of the prepotential as the low energy effective action in the Coulomb branch.

In particular 't Hooft duality leads us to hope that homological localization may capture the mass gap via the implied condensation of the magnetic charge.

We write now the localized effective action in the Wilsonian scheme. From the effective action we extract the glueballs spectrum restricted to the Lagrangian Minkowski section of space-time. To compute fluctuations we need to extend the effective action from the fixed points, characterized by $\partial_{\hat{A}} \hat{D} = \bar{\partial}_{\hat{A}} \hat{D} = 0$, to fluctuations in neighborhood of them. The fluctuations correspond generically to surface operators of semisimple type, i.e. with different eigenvalues of the (diagonalizable) holonomy and of the ASD curvature, μ .

For a twistor Wilson loop in the fundamental representation this extension can be performed only in a holomorphic way via the holomorphic loop equation. The corresponding effective action

is not Hermitian. Therefore we couple the twistor Wilson loop in the fundamental representation with the one in the complex conjugate representation⁶⁵. As a result the effective action is Hermitian⁶⁶. Thus we are realizing a sort of holomorphic/antiholomorphic fusion that is reminiscent and in fact technically very similar to holomorphic/antiholomorphic fusion [52, 53] in conformal field theory.

Initially, thanks to large- N factorization, we treat μ and $\bar{\mu}$ as independent variables that define two different chiral sectors. But then we choose the section in function space where $\bar{\mu}$ is actually the Hermitian conjugate of μ for the effective action to be Hermitian. This enables us to compute $\frac{1}{N}$ fluctuations.

Yet, this $\mu/\bar{\mu}$ sector is only a special sector of the theory, that involves certain correlators constructed by $\mu = \frac{1}{2}(\mu_{01}^- - i\mu_{03}^-)$ and its Hermitian conjugate. These are the special correlators accessible to homological localization of the loop equation. We cannot say anything about correlators of μ_{02}^- .

Realizing holomorphic/antiholomorphic fusion in the non-commutative Eguchi-Kawai reduced theory (see sect.(4.1)) involves therefore the product of the holomorphic and antiholomorphic partition functions, extended to the holomorphic and antiholomorphic neighborhoods of the fixed points, that are thought to be each the Hermitian conjugate of the other one:

$$\begin{aligned}
 Z &= \int e^{-\Gamma} \left| \prod_p \delta \hat{\mu}'_p \right|^2 \\
 &= \left| \int \hat{\delta}_A \delta \hat{A} \delta \hat{D} \delta \hat{D} \delta(-iF_{\hat{B}} - \sum_p \hat{\mu}_p \delta^{(2)}(z - z_p) - \theta^{-1} \mathbf{1}) \right. \\
 &\quad \left. \delta(\partial_{\hat{A}} \hat{D} + \bar{\partial}_{\hat{A}} \hat{D}) \delta(\partial_{\hat{A}} \hat{D} - \bar{\partial}_{\hat{A}} \hat{D} + i \sum_p (\hat{\mu}_p - \hat{\bar{\mu}}_p) \delta^{(2)}(z - z_p)) \right. \\
 &\quad \left. \exp\left(-\frac{4N\hat{N}}{g_W^2} \sum_p \text{tr}_N \hat{T} r_{\hat{N}}(\hat{\mu}_p \hat{\bar{\mu}}_p)\right) \text{Det}\left(\frac{\delta \hat{\mu}}{\delta \hat{\mu}'}\right) \prod_p \delta \hat{\mu}'_p \right|^2
 \end{aligned} \tag{12.13}$$

We then proceed as in sect.(7). In the reduced theory the classical action is finite on surface operators because of the rescaling explained in sect.(4.1). The $U(N)$ non-commutative theory can be compactified on a torus of large area, L^2 . For rational values of the dimensionless non-commutative parameter:

$$2\pi\theta L^{-2} = \frac{\hat{M}}{\hat{N}} \tag{12.14}$$

the non-commutative $U(N)$ theory is Morita equivalent to an $U(N \times \hat{N})$ commutative gauge theory on the smaller torus, $L^2 \hat{N}^{-2}$, with non-trivial 't Hooft flux through the commutative torus and with the same 't Hooft coupling constant, g . Thus the large- N limit of the partition function reduces to

⁶⁵This coupling is automatic via large- N factorization if we start with twistor Wilson loops in the adjoint representation. In this representation the condensation of the magnetic charge via surface operators of Z_N holonomy is compatible with triviality of twistor Wilson loops, because of the pairwise cancellation in the holonomy of every Z_N factor by its complex conjugate for any shape of the loop.

⁶⁶The complex conjugate field and the Hermitian conjugate contain the same information since they differ by transposition of the indices.

a finite dimensional inductive sequence:

$$\begin{aligned}
 Z &= \lim_{N, \hat{N} \rightarrow \infty} \left| \int \delta A \delta \bar{A} \delta D \delta \bar{D} \delta (-iF_B - \sum_p U(u, \bar{u}) \mu_p U(u, \bar{u})^{-1} \delta^{(2)}(z - z_p)) \right. \\
 &\quad \delta(\partial_A D - \bar{\partial}_A D + i \sum_p (U(u, \bar{u}) \mu_p U(u, \bar{u})^{-1} - U(u, \bar{u})^{-1} \bar{\mu}_p U(u, \bar{u})) \delta^{(2)}(z - z_p)) \\
 &\quad \left. \delta(\partial_A D + \bar{\partial}_A D) \exp\left(-\frac{4N\hat{N}}{g_W^2} \sum_p \text{tr}_N \text{Tr}_{\hat{N}}(\mu_p \bar{\mu}_p)\right) \frac{\delta \mu}{\delta \mu'} \prod_p |\delta \mu'_p|^2 \right| \quad (12.15)
 \end{aligned}$$

The unitary matrices, $U(u, \bar{u})$, account for the twisted boundary conditions on the commutative torus. In the thermodynamic limit, $L^2 \rightarrow \infty$, the effect of the twist of the torus disappears, at least for fixed large \hat{N} , since the Morita equivalent theory must coincide in the large- N limit with the commutative theory on R^2 . The equivalence with the untwisted theory can be seen also choosing the gauge where there is no twist in front of the delta functions. Of course this gauge is singular on the torus, in the sense that it is defined by a gauge transformation that is not single-valued on the torus, precisely because it is not continuous on the boundary of the fundamental domain that is employed to define the torus. But the boundary becomes irrelevant in the thermodynamic limit. However, some care is necessary to match in the untwisted theory the exact scaling with \hat{N} of the trace of Wilson loops in the twisted theory (see Eq.(7.6)). This is taken into account in the untwisted commutative theory by the ansatz $U(N) \times 1_{\hat{N}}$ for the local holonomy at the localized locus.

We require that the fields on the commutative R^2 so obtained in the thermodynamic limit have well defined limits at infinity, in such a way that the theory can be compactified on S^2 . Thus the fields on $R^2 \times R^2$ can be extended to $S^2 \times S^2$ and the resolution of the identity reduces to the one for ordinary surface operators. Thus we get:

$$\begin{aligned}
 Z &= \lim_{N, \hat{N} \rightarrow \infty} \left| \int \delta A \delta \bar{A} \delta D \delta \bar{D} \delta (-iF_B - \sum_p \mu_p \delta^{(2)}(z - z_p)) \right. \\
 &\quad \delta(\partial_A \bar{D} + \bar{\partial}_A D) \delta(\partial_A \bar{D} - \bar{\partial}_A D + i \sum_p (\mu_p - \bar{\mu}_p) \delta^{(2)}(z - z_p)) \\
 &\quad \left. \exp\left(-\frac{4N\hat{N}}{g_W^2} \sum_p \text{tr}_N \text{Tr}_{\hat{N}}(\mu_p \bar{\mu}_p)\right) \frac{\delta \mu}{\delta \mu'} \prod_p |\delta \mu'_p|^2 \right| \quad (12.16)
 \end{aligned}$$

that in the Feynman gauge, as shown in sect.(3.3) and sect.(8), reduces to:

$$\begin{aligned}
 Z &= \left| \int [Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - i\mu_{\alpha\beta}^-) Det(-\Delta_A)]_{\mu=\frac{1}{2}(\mu_{01}^- - i\mu_{03}^-)} \right. \\
 &\quad \left. \Lambda^{n_b[\mu']} \exp\left(-\frac{4N\hat{N}}{g_W^2} \sum_p \text{tr}_N \text{Tr}_{\hat{N}}(\mu_p \bar{\mu}_p)\right) \frac{\delta \mu}{\delta \mu'} \wedge \omega'^{\frac{n_b[\mu']}{2}} \prod_p |\delta \mu'_p|^2 \right| \quad (12.17)
 \end{aligned}$$

In this formula ω' should be chosen in a way compatible with the holomorphic/antiholomorphic fusion. In particular ω' should depend holomorphically on the complex eigenvalues, λ , of μ . This requires that ω' be the pullback on the moduli space of the symplectic form associated to the twistor connection for $\rho = -1$, $\hat{B} = \hat{B}_{-1}$ (see sect.(7)):

$$\omega' = \frac{1}{2\pi} \int d^2 z \text{tr}_f \hat{T}r(\delta \hat{B}_z \wedge \delta \hat{B}_{\bar{z}}) \quad (12.18)$$

The superscript in ω' refers to the version of ω defined on the punctured sphere as discussed in sect(7). This is not restrictive, since excluding the singular divisor is equivalent to omitting in ω the sum of the Kirillov forms of the adjoint orbits on the singular divisor [118], because the volume form on these orbits is already taken into account by the product measure on the adjoint orbits, $|\prod_p \delta\mu'_p|$, (see sect.(12.3)) that occurs by the resolution of identity. As a consequence:

$$\omega^{\frac{n_b[\mu']}{2}} \wedge \bar{\omega}^{\frac{n_b[\mu']}{2}} \wedge \prod_p \delta\mu'_p \delta\bar{\mu}'_p = \omega'^{\frac{n_b[\mu']}{2}} \wedge \bar{\omega}'^{\frac{n_b[\mu']}{2}} \wedge \prod_p \delta\mu'_p \delta\bar{\mu}'_p \quad (12.19)$$

In any case the version that involves ω' is the one that leads to the glueballs potential.

12.3 The glueballs potential

The glueballs potential arises as a term in the localized effective action that is the logarithm of the modulus of the Jacobian for the change of variables of the complex field of ASD type from a unitary, μ , to a holomorphic gauge, μ' .

This term cannot arise in perturbation theory and the only way to derive it in our approach is via the holomorphic loop equation, since the choice of the holomorphic gauge is necessary in order to produce the Cauchy kernel in the right hand side of the holomorphic loop equation.

In turn the Cauchy kernel is essential, because after analytic continuation to Minkowski space-time leads to localization by homology, since the corresponding regularized distribution is zig-zag invariant in a neighborhood of a cusp and therefore its contour integral vanishes for arcs that backtrack at the cusps.

The physical meaning of this Jacobian is the following.

For surface operators, because of the specific features of the Hitchin equations, there is a mismatch between the degrees of freedom carried by the ASD curvature at a point in a unitary and in a holomorphic gauge.

While at moduli level the unitary and holomorphic gauges are completely equivalent as shown in sect.(7), it does not hold true for the local degrees of freedom that are manifest looking at the ASD curvature in the unitary or holomorphic gauge.

In the holomorphic gauge there is an essentially ⁶⁷ one-to-one correspondence between the moduli that occur in the holonomy, M_p , of the connection around a point, p , and the moduli of the local curvature, μ'_p , at p , given by the equation:

$$M_p = e^{2i\mu'_p} \quad (12.20)$$

Thus in the holomorphic gauge μ'_p is parameterized by orbits in the complexification of the gauge group, $\mu'_p = G_p \lambda_p G_p^{-1}$, with λ_p the (complex) eigenvalues of μ'_p , that we assume all different in the semisimple case relevant to compute fluctuations.

Therefore the integral over $\mu'_p = G_p \lambda_p G_p^{-1}$ is on an orbit of the complexification of the gauge group with measure:

$$\delta\mu'_p = \text{Det}(ad\lambda_p) \delta\lambda_p \delta G_p \quad (12.21)$$

⁶⁷The moduli depend actually only on the conjugacy class of the eigenvalues of the holonomy, that determines the eigenvalues of the curvature only up to shifts of 2π .

where the integration on δG_p is actually on the moduli of the orbit, that can be parametrized as $G_p = g_p P_p$, by factorizing the complexification of the gauge group into its compact and parabolic factor⁶⁸. In fact it is not restrictive to project the parabolic factor, P_p , to its unipotent subgroup, P'_p , since the diagonal factor of P_p acts trivially by conjugation on the eigenvalues.

Thus the preceding equation can be rewritten as:

$$\delta \mu'_p = \Delta(\lambda_p)^2 \delta \lambda_p \delta g_p \delta P'_p \quad (12.22)$$

where the square of the Vandermonde determinant of the eigenvalues, $\Delta(\lambda)$:

$$\Delta(\lambda) = \prod_{i>j} (\lambda_i - \lambda_j) \quad (12.23)$$

arises by the usual Faddeev-Popov procedure, as in the holomorphic matrix models [48, 49].

In the unitary gauge this does not hold true. It is a fundamental result of the theory of surface operators [80, 81] that in a unitary gauge $[\mu_p, \bar{\mu}_p] = 0$, so that generically μ_p and $\bar{\mu}_p$ can be diagonalized simultaneously by a unitary transformation (see sect.(7)). Thus only the moduli associated to the adjoint action of the compact unitary group, g_p , are manifest in the curvature in the unitary gauge, while the remaining degrees of freedom are hidden in the moduli of the Hermitian metric whose choice is implicit in the unitary gauge [81].

Now the non-Hermitian matrix, $\mu_p = g_p \lambda_p U_p g_p^{-1}$, is generically conjugated by a unitary transformation to a triangular matrix, $\lambda_p U_p$, with U_p triangular and unipotent, in such a way that the induced measure is:

$$\begin{aligned} \delta \mu_p &= Pf(ad \mu_p) \delta \lambda_p \delta U_p \delta g_p \\ &= \Delta(\lambda_p) \delta \lambda_p \delta U_p \delta g_p \end{aligned} \quad (12.24)$$

The different power of the Vandermonde determinant in Eq.(12.24) with respect to Eq.(12.22) arises by the Faddeev-Popov procedure for triangularizing rather than diagonalizing μ_p ⁶⁹.

But the unitary gauge transformation, that sets the complex matrix μ_p in triangular form in a unitary gauge, does actually diagonalize those μ_p that arise as the ASD curvature on the dense locus of the surface operators in the resolution of identity of the non-SUSY Nicolai map. Therefore in the induced measure the integration over the unipotent factor, U_p , actually factorizes, since the integrand does not depend in fact on U_p for those μ_p that arise by surface operators in a unitary gauge. On the other hand the integration over the moduli P'_p contributes to the zero modes and produces the measure:

$$\omega'^{\frac{n_b[\mu']}{2}} = Pf(\omega') \prod_p \delta P'_p \quad (12.25)$$

in such a way that:

$$\begin{aligned} \frac{\delta \mu}{\delta \mu'} \wedge \omega'^{\frac{n_b[\mu']}{2}} &= Pf(\omega') \prod_p \frac{Pf(ad \mu_p)}{Det(ad \mu_p)} \frac{\delta \lambda_p \delta P'_p \delta g_p}{\delta \lambda_p \delta G_p} \\ &= \frac{Pf(\omega')}{\prod_p Pf(ad \mu_p)} \end{aligned} \quad (12.26)$$

⁶⁸This is the Iwasawa decomposition.

⁶⁹We may get the same result noticing that the integration measure $d\mu d\bar{\mu}$ on normal non-Hermitian matrices is $|\Delta(\lambda)|^2 d\lambda d\bar{\lambda} dg$ by the standard Faddeev-Popov procedure and then taking the "square root".

where we used $G_p = g_p P'_p$. The same result is obtained in the singular gauge where μ_p is actually diagonal, $\mu_p = \lambda_p$, for which therefore $\delta U_p \delta g_p$ factorizes from the measure $\delta \mu_p$, but the measure on the moduli is:

$$\omega'^{\frac{n_b[\mu']}{2}} = Pf(\omega') \prod_p \delta P'_p \delta g_p \quad (12.27)$$

Therefore the effective action reads:

$$\begin{aligned} \Gamma &= \frac{8N\hat{N}}{g_W^2} \sum_p \text{tr}_N \text{Tr}_{\hat{N}}(\mu_p \bar{\mu}_p) + \sum_p \log |\Delta(\mu_p)|^2 \\ &\quad - \log |Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - i\mu_{\alpha\beta}^-) Det(-\Delta_A)|_{\mu=\frac{1}{2}(\mu_{01}^- - i\mu_{03}^-)}|^2 \\ &\quad - 2n_b[\mu'] \log \Lambda - \log |Pf(\omega')|^2 \end{aligned} \quad (12.28)$$

We anticipated that the second term turns out to be the glueballs potential, that generates the glueballs masses. The term $\log |Pf(\omega')|^2$ is irrelevant for the glueball potential, since ω' depends on the eigenvalues only through the holonomy of the connection (see sect.(7)). Indeed, since the holonomy at the critical points lives in Z_N , the second derivative of $\log |Pf(\omega')|^2$ at the critical points couples only to the trace of the fluctuations, $\text{Tr}(\delta \lambda_p)$, and thus decouples in the large- N limit. The Pauli-Villars factors do not contribute to fluctuations. The remaining terms are considered in the following subsections.

It is very instructive to reinsert the factor of N_2 to recover the unreduced theory and to introduce a regularization by the density of the lattice of surface operators that is more suitable for the continuum limit. Reinserting the factor of N_2 we get:

$$\begin{aligned} \Gamma &= \frac{8N\hat{N}}{g_W^2} N_2 \sum_p \text{tr}_N \text{Tr}_{\hat{N}}(\mu_p \bar{\mu}_p) + \sum_p N_2 \log |\Delta(\mu_p)|^2 \\ &\quad - \log |Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - i\mu_{\alpha\beta}^-) Det(-\Delta_A)|_{\mu=\frac{1}{2}(\mu_{01}^- - i\mu_{03}^-)}|^2 \\ &\quad - 2N_2 n_b[\mu'] \log \Lambda - N_2 \log |Pf(\omega')|^2 \\ &= \frac{8N\hat{N}}{g_W^2} \delta^{(2)}(0) \int d^2 u \sum_p \text{tr}_N \text{Tr}_{\hat{N}}(\mu_p \bar{\mu}_p) + \sum_p \delta^{(2)}(0) \int d^2 u \log |\Delta(\mu_p)|^2 \\ &\quad - \log |Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - i\mu_{\alpha\beta}^-) Det(-\Delta_A)|_{\mu=\frac{1}{2}(\mu_{01}^- - i\mu_{03}^-)}|^2 \\ &\quad - 2\delta^{(2)}(0) \int d^2 u \sum_p n_b[\mu'_p] \log \Lambda - \delta^{(2)}(0) \int d^2 u \log |Pf(\omega')|^2 \end{aligned} \quad (12.29)$$

with the traces that define the functional determinants properly rescaled. We can now introduce the density of lattice points:

$$\rho = \sum_{p'} \delta^{(2)}(z - z_{p'}) \quad (12.30)$$

normalized in such a way that

$$\int d^2 z \sum_{p'} \delta^{(2)}(z - z_{p'}) = N'_2 \quad (12.31)$$

is the number of lattice points at the scale at which the density is ρ . Notice that the density is not normalized necessarily to N_2 , the number of lattice points at the cutoff scale, because the primed sum is only on lattice points where the holonomy of the surface operator is non-trivial. This allows ρ to scale non-trivially with the RG . Assuming translational invariance at $N = \infty$ the effective action reads:

$$\begin{aligned} \Gamma &= \frac{8N\hat{N}}{g_W^2} \int d^2ud^2z\rho^2 \text{tr}_N \text{Tr}_{\hat{N}}(\mu\bar{\mu}) + \int d^2ud^2z\rho^2 \log |\Delta(\mu)|^2 \\ &\quad - \log |Det^{-1/2}(-\Delta_A \delta_{\alpha\beta} - i\mu_{\alpha\beta}^-) Det(-\Delta_A)|_{\mu=\frac{1}{2}(\mu_{01}^- - i\mu_{03}^-)}|^2 \\ &\quad - 2 \int d^2ud^2z\rho^2 n_b[\mu'] \log \Lambda - \int d^2u\rho \log |Pf(\omega')|^2 \end{aligned} \quad (12.32)$$

This form of the effective action is of the utmost importance, because it shows that the coefficient of the glueballs potential is in fact a RG invariant scale as it should be.

12.4 The effective action is degenerate at the fixed points

We can compute ρ in terms of Λ_W , the RG invariant scale in the Wilsonian scheme, by minimizing the renormalized effective action as a function of ρ for a given Λ and g_W . In fact we know already from sect.(8) that the effect of the third and fourth term in the effective action is to renormalize the coupling constant. Hence the local divergent part of the effective action for surface operators with Z_N holonomy of magnetic charge k and density ρ reads:

$$\begin{aligned} \Gamma_k &= \frac{k(N-k)\hat{N}^2(4\pi)^2}{2g_W^2} \left(1 - g_W^2 \frac{10}{3} \frac{1}{(4\pi)^2} \log \frac{\Lambda}{M}\right) \int d^2ud^2z\rho^2 \\ &\quad - 2k(N-k)\hat{N}^2 \int d^2ud^2z\rho^2 \log \frac{\Lambda}{M} + \dots \\ &= k(N-k)\hat{N}^2(4\pi)^2 \left(\frac{1}{2g_W^2} - \frac{11}{3} \frac{1}{(4\pi)^2} \log \frac{\Lambda}{M}\right) \int d^2ud^2z\rho^2 + \dots \\ &= k(N-k)\hat{N}^2(4\pi)^2 \left(-\beta_0 \log \frac{\Lambda e^{-\frac{1}{2\beta_0 g_W^2}}}{M}\right) \int d^2ud^2z\rho^2 + \dots \\ &= -\beta_0 \hat{N} \log \frac{\Lambda e^{-\frac{1}{2\beta_0 g_W^2}}}{M} \int d^2ud^2z M^4 + \dots \\ &= -\beta_0 \hat{N} \log \frac{\Lambda_W}{M} \int d^2ud^2z M^4 + \dots \end{aligned} \quad (12.33)$$

where we have chosen the subtraction point at the scale of the action density:

$$M^4 = \hat{N}k(N-k)(4\pi)^2\rho^2 \quad (12.34)$$

This condition ensures that all the sectors labelled by k are degenerate in the large- N, \hat{N} limit, in such a way that the renormalized effective action at the subtraction scale is large and negative and equal for all of them. In this case the trivial solution with magnetic charge $k = 0$ is excluded since it has greater action and therefore there is condensation of surface operators with all the magnetic charges. Indeed the critical equation:

$$\frac{\delta\Gamma_k}{\delta M} = 4M^3 \log \frac{\Lambda_W}{M} - M^3 = M^3(4 \log \frac{\Lambda_W}{M} - 1) = 0 \quad (12.35)$$

has solution:

$$\log \frac{\Lambda_W}{M} = \frac{1}{4} \quad (12.36)$$

in such a way that the effective action reaches its negative minimum:

$$-\beta_0 \frac{\hat{N}}{4} \int d^2 u d^2 z M^4 \quad (12.37)$$

with M^4 given by:

$$M^4 = e^{-1} \Lambda_W^4 \quad (12.38)$$

This also means that for large N the square density of surface operators scales as $\frac{1}{k}$ and as $\frac{1}{N}$. It implies also that the glueballs propagators are a sum of pure poles as we will see momentarily.

We may wonder as to whether finite terms, i.e. not Λ divergent, may affect this picture. It is easy to see that terms proportional to $k(N-k)$ simply redefine Λ_W . However it is not obvious that the contributions from all finite parts have this form. Thus finite terms may affect the dependence of ρ on k .

In fact we have chosen the action density of the condensate, $\rho^2 \text{Tr}_{\hat{N}} \text{tr}_N (\lambda^2 1_N 1_{\hat{N}})$, as the infrared subtraction scale, M^4 ⁷⁰. This is the same prescription as for the Veneziano-Yankielowicz effective action [119] of $\mathcal{N} = 1$ $SUSY$ YM . This subtraction point implies that the renormalized action is the same in every sector of magnetic charge k , in such a way that all the Z_N magnetic charges "condense at once" with a renormalized square density that scales as $\frac{1}{k}$.

While this is the same prescription that occurs in the effective action of $SUSY$ theories in fact its justification at fundamental level may imply a certain fine-tuning of the finite parts in the renormalized effective action. This fine-tuning is always possible if the surface operators of Z_N holonomy that occur at the fixed points are viewed as limit points in the closure of orbits with unipotent holonomy in the Lagrangian cone mentioned in sect.(7) and sect.(9). Indeed in this case the relative scale of $|Pf(\omega')|^2$ can be suitably adjusted for different k in approaching the limit points where the nilpotent residue of the Higgs field vanishes.

This is essentially due to the fact that the moduli space of such orbits is non-compact and that we are suitably choosing the size of the neighborhood of the fixed points in order to satisfy certain conditions. Indeed every "compactification" is to some extent arbitrary⁷¹. On the contrary, the unitary orbits in the Lagrangian cone, that are the other orbits that saturate the beta function (see sect.(7) and sect.(9)), do not allow such fine-tuning since they are compact.

12.5 The kinetic term

We are now ready to compute the glueballs propagator. It is quite clear that the classical action cannot furnish the kinetic term for the glueballs since it is ultralocal. Therefore the kinetic

⁷⁰Finite changes of the subtraction point affect the normalization of the glueballs propagator but do not affect the glueballs spectrum.

⁷¹In Nekrasov theory of cohomological localization a certain (to some extent arbitrary) compactification of the moduli spaces of instantons has to occur too. Such compactification turns out to be compatible with the Seiberg-Witten ideas on the electric/magnetic duality in $\mathcal{N} = 2$ $SUSY$ YM . In our case we can argue similarly that we can choose the size of the neighborhoods of the fixed points to avoid that only a proper subgroup of Z_N condenses, spoiling 't Hooft ideas on electric/magnetic duality.

term must be generated by radiative corrections around surface operators. This turns out to be the case for the fluctuations of Lagrangian-embedded surface operators analytically continued to Minkowski space-time. They can be obtained by diagonally embedded Euclidean surface operators continued to Minkowski space-time.

The diagonal embedding can be described as follows. We choose the surface ($z = u, \bar{z} = \bar{u}$) diagonally embedded in R^4 . Since we have defined a lattice in the (z, \bar{z}) plane, this defines a lattice also in the (u, \bar{u}) plane by the diagonal map ($z_p = u_p, \bar{z}_p = \bar{u}_p$). This lattice in the (u, \bar{u}) plane has a set of dual plaquettes in such a way that the (u, \bar{u}) plane is the union of the plaquettes. Now we define the function $z_p(u, \bar{u}) = u$ with domain the interior of the plaquette dual to p and analogously for the complex conjugate. We also define a lattice fluctuating field supported on the plaquette dual to p and locally constant as (u, \bar{u}) vary in the support, $\delta\mu_p(u, \bar{u})$. $\delta\mu_p(u, \bar{u})$ is zero outside its support. We are now ready to do computations. We suppose that in addition to the translational invariant background of surface operators there are locally-defined fluctuating surface operators⁷² diagonally embedded in space-time:

$$-iF_B = \sum_p \mu \delta^{(2)}(z - z_p) + \sum_p \delta\mu_p(u, \bar{u}) \delta^{(2)}(z - z_p(u, \bar{u})) \quad (12.39)$$

Since the kinetic term for the glueballs propagator must arise by the radiative corrections we examine the loop expansion of the functional determinants restricted to the fluctuating lattice of surface operators. We have seen in sect.(9) and sect.(11) that the divergent parts determine the beta function. We are now interested in the finite parts of the loop expansion to second order.

The justification is as follows. Every term of the loop expansion contains a trace in the adjoint representation and thus it is proportional to N^2 , that of course diverges for large N . However, the loop expansion is in fact an expansion in powers of the density ρ of surface operators. But since the density scales as $N^{-\frac{1}{2}}$ only the leading quadratic term survives the double large- N, \hat{N} limit.

We will see later that \hat{N} should change with the RG to maintain the equality of scales along the RG flow according to the diagonal embedding of the cutoff on the lattice in the (u, \bar{u}) and (z, \bar{z}) planes.

The spin contribution to the effective action is:

$$-\frac{2N\hat{N}'4}{(4\pi^2)^2} \sum_{p \neq p'} \int d^2u d^2v \frac{tr_N Tr_{\hat{N}'}(\delta\mu_p(u, \bar{u}) \delta\bar{\mu}_{p'}(v, \bar{v}))}{(|z_p(u, \bar{u}) - z_{p'}(v, \bar{v})|^2 + |u - v|^2)^2} \quad (12.40)$$

plus the complex conjugate term that we add only at the end of the computation. The orbital contribution has the same structure and a different coefficient and sign in order to reproduce the divergent Z^{-1} factor when evaluated on constant surface operators⁷³:

$$\frac{\frac{1}{3}N\hat{N}'4}{(4\pi^2)^2} \sum_{p \neq p'} \int d^2u d^2v \frac{tr_N Tr_{\hat{N}'}(\delta\mu_p(u, \bar{u}) \delta\bar{\mu}_{p'}(v, \bar{v}))}{(|z_p(u, \bar{u}) - z_{p'}(v, \bar{v})|^2 + |u - v|^2)^2} \quad (12.41)$$

⁷²Both the background and the fluctuations are diagonal matrices in color space.

⁷³The Z^{-1} factor, contrary to the beta function, does not depend on sitting on surface operators of Z_N holonomy.

Indeed the orbital contribution to second order, up to constant overall factors, is ⁷⁴:

$$\begin{aligned}
 & \int d^4x d^4y \text{Tr}(A_z(x) \partial_{\bar{z}} \frac{1}{(x-y)^2} A_{\bar{z}}(y) \partial_z \frac{1}{(x-y)^2} + A_u(x) \partial_{\bar{u}} \frac{1}{(x-y)^2} A_{\bar{u}}(y) \partial_u \frac{1}{(x-y)^2}) \\
 &= \int d^4x d^4y \text{Tr}(A_z(x) \frac{2(z-w)}{(x-y)^4} A_{\bar{z}}(y) \frac{2(\bar{z}-\bar{w})}{(x-y)^4} + A_u(x) \frac{2(u-v)}{(x-y)^4} A_{\bar{u}}(y) \frac{2(\bar{u}-\bar{v})}{(x-y)^4}) \\
 &= \int d^2z d^2w d^2u d^2v \text{Tr}(A_z(x) A_{\bar{z}}(y) \frac{4|z-w|^2}{(|z-w|^2 + |u-v|^2)^4} + A_u(x) A_{\bar{u}}(y) \frac{4|u-v|^2}{(|z-w|^2 + |u-v|^2)^4}) \\
 &\sim - \int d^2z d^2w d^2u d^2v \text{Tr}(A_z(x) A_{\bar{z}}(y) \partial_{\bar{z}} \partial_w \frac{4}{6(|z-w|^2 + |u-v|^2)^2} + A_u(x) A_{\bar{u}}(y) \partial_{\bar{u}} \partial_v \frac{4}{6(|z-w|^2 + |u-v|^2)^2}) \\
 &= - \int d^2z d^2w d^2u d^2v \text{Tr}(\partial_{\bar{z}} \partial_w (A_z(x) A_{\bar{z}}(y)) \frac{4}{6(|z-w|^2 + |u-v|^2)^2} + \partial_{\bar{u}} \partial_v (A_u(x) A_{\bar{u}}(y)) \frac{4}{6(|z-w|^2 + |u-v|^2)^2}) \\
 &= - \int d^2z d^2w d^2u d^2v \text{Tr}(\partial_{\bar{z}} A_z(x) \partial_w A_{\bar{z}}(y) \frac{4}{6(|z-w|^2 + |u-v|^2)^2} + \partial_{\bar{u}} A_u(x) \partial_v A_{\bar{u}}(y) \frac{4}{6(|z-w|^2 + |u-v|^2)^2}) \\
 &\sim \sum_{p,p'} \int d^2u d^2v \frac{N \text{Tr}(\delta \mu_p \delta \bar{\mu}_{p'})}{(|z_p(u, \bar{u}) - z_{p'}(v, \bar{v})|^2 + |u-v|^2)^2} \tag{12.42}
 \end{aligned}$$

where in the last line we used $\partial_{\bar{z}} A_z(x) \sim \sum_p \delta \mu_p^0 \delta^{(2)}(z - z_p(u, \bar{u}))$, $\partial_w A_{\bar{z}}(y) \sim -\sum_p \delta \mu_p^0 \delta^{(2)}(w - z_p(v, \bar{v}))$, $\partial_{\bar{u}} A_u(x) \sim \sum_p \delta n_p \delta^{(2)}(z - z_p(u, \bar{u}))$ and $\partial_v A_{\bar{u}}(y) \sim -\sum_p \delta \bar{n}_p \delta^{(2)}(w - z_p(v, \bar{v}))$ for the fluctuations of surface operators on the diagonal Lagrangian submanifold, with $x = (z, \bar{z}, u, \bar{u})$ and $y = (w, \bar{w}, v, \bar{v})$. The total result is:

$$- \frac{5N\hat{N}'4}{3(4\pi^2)^2} \sum_{p \neq p'} \int d^2u d^2v \frac{\text{tr}_N \text{Tr}_{\hat{N}'}(\delta \mu_p(u, \bar{u}) \delta \bar{\mu}_{p'}(v, \bar{v}))}{(|z_p(u, \bar{u}) - z_{p'}(v, \bar{v})|^2 + |u-v|^2)^2} \tag{12.43}$$

The coefficient can be found without direct computation since, for the surface operators that are constant on the (u, \bar{u}) plane and translational invariant, Eq.(12.43) must give rise to the logarithmic divergence that produces Z^{-1} . Expressing Eq.(12.3) in terms of the density of the surface operators, ρ , we get:

$$\begin{aligned}
 & - \frac{20N\hat{N}'}{3(4\pi^2)^2} \left(\int d^2v d^2u \rho^2 \right) \int d^2u d^2v \frac{\text{tr}_N \text{Tr}_{\hat{N}'}(\delta \mu(u, \bar{u}) \delta \bar{\mu}(v, \bar{v}))}{(|u-v|^2 + |u-v|^2)^2} \\
 &= - \frac{20N}{3(4\pi^2)^2 k(N-k)(4\pi)^2} \left(\int d^2v d^2u e^{-1} \Lambda_W^4 \right) \int d^2u d^2v \frac{\text{tr}_N \text{Tr}_{\hat{N}'}(\delta \mu(u, \bar{u}) \delta \bar{\mu}(v, \bar{v}))}{(|u-v|^2 + |u-v|^2)^2} \tag{12.44}
 \end{aligned}$$

that for large N reduces to ⁷⁵ :

$$- \frac{20e^{-1}}{3(4\pi^2)^2 2k(4\pi)^2} N_2'^2 \int d^2u d^2v \frac{\text{tr}_N \text{Tr}_{\hat{N}'}(\delta \mu(u, \bar{u}) \delta \bar{\mu}(v, \bar{v}))}{(u-v)^2 (\bar{u}-\bar{v})^2} \tag{12.45}$$

⁷⁴The term involving $\partial_{\alpha} A_{\alpha}$ vanishes identically around the local singularity, while the term involving A_{α}^2 is quadratically divergent and does not contribute because of cancellations due to gauge invariance.

⁷⁵It is not restrictive to require $k = 1, \dots, \frac{N}{2}$ for N even, because of the symmetry of all our formulae for the exchange $k \rightarrow N-k$.

This term generates the kinetic term of the glueballs propagator after analytic continuation to Minkowski space-time ⁷⁶. Inserting the complex conjugate term we get:

$$\begin{aligned}
 & -\frac{20e^{-1}}{3(4\pi^2)^2 2k(4\pi)^2} N_2'^2 \int du_+ du_- dv_+ dv_- \frac{\text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu(u_+, u_-) \delta\bar{\mu}(v_+, v_-))}{(u_+ - v_+ + i\varepsilon)^2 (u_- - v_- + i\varepsilon)^2} + c.c. \\
 & = (2\pi)^2 \frac{20e^{-1}}{3(4\pi^2)^2 2k(4\pi)^2} N_2'^2 \int du_+ du_- \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu(u_+, u_-) \partial_+ \partial_- \delta\bar{\mu}(u_+, u_-)) + c.c.
 \end{aligned} \tag{12.46}$$

We notice that for obtaining this result it is crucial that fluctuations occur as surface operators, that the support of fluctuations is embedded as a Lagrangian submanifold in Euclidean space-time and that analytic continuation to the Minkowskian Lagrangian submanifold is performed.

It is interesting to compare the kinetic term just obtained with the contribution of the classical action:

$$\begin{aligned}
 & \frac{8N\hat{N}'}{g_W^2} \int d^2u d^2z \rho^2 \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu(u, \bar{u}) \delta\bar{\mu}(u, \bar{u})) \\
 & = \frac{8Ne^{-1}}{g_W^2 k(N-k)(4\pi)^2} \left(\int d^2z \Lambda_W^2 \right) \int d^2u \Lambda_W^2 \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu(u, \bar{u}) \delta\bar{\mu}(u, \bar{u}))
 \end{aligned} \tag{12.47}$$

that at large N reduces to:

$$\frac{8e^{-1}}{g_W^2 k(4\pi)^2} N_2' \int d^2u \Lambda_W^2 \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu(u, \bar{u}) \delta\bar{\mu}(u, \bar{u})) \tag{12.48}$$

Hence the classical action is irrelevant with respect to the term generated by radiative corrections for the fluctuations in the large- N limit and in the thermodynamic limit in which N_2' diverges. In fact the real source of the mass term is the glueballs potential whose contribution to the effective action is:

$$\begin{aligned}
 & \int d^2u d^2z \rho^2 \log |\Delta(\mu)|^2 \\
 & = \frac{e^{-1}}{\hat{N}'k(N-k)(4\pi)^2} \left(\int d^2z \Lambda_W^2 \right) \int d^2u \Lambda_W^2 \log |\Delta(\mu)|^2 \\
 & = \frac{e^{-1}}{\hat{N}'k(N-k)(4\pi)^2} N_2' \int d^2u \Lambda_W^2 \log |\Delta(\mu)|^2
 \end{aligned} \tag{12.49}$$

Apparently the glueballs potential in the thermodynamic limit, $N_2' \rightarrow \infty$, is as suppressed with respect to the kinetic term as the classical action is. Therefore the only possibility for the theory to have a mass gap is that the glueballs potential is singular. In fact these singularities of the glueballs potential arise precisely when some eigenvalues coincide. This looks encouraging because it implies

⁷⁶It is necessary to assume that $\delta\mu_{01}^-(u_+, u_-)$ and $\delta\mu_{03}^-(u_+, u_-)$ are the boundary values of holomorphic functions on the upper-half plane for each of the independent variables, (u_+, u_-) , with suitable properties at infinity.

that the mass gap may arise only by configurations for which the gauge group is unbroken. In addition these configurations for which the eigenvalues coincide must have infinite degeneracies in the thermodynamic limit, in order to compensate the $\frac{1}{N^2}$ suppression with respect to the kinetic term. The needed degeneracies follow by the ansatz that the center Z_N of $SU(N)$ occurs with multiplicity \hat{N} in the commutative theory Morita equivalent to the non-commutative one.

12.6 The mass gap

We are now ready to compute the mass matrix. With our normalizations $e^{2i\mu} \in Z_N$ with $\mu \in su(N)$ for μ the translational invariant condensate. Thus:

$$2\mu = \text{diag}(\underbrace{2\pi(k-N)/N}_k, \underbrace{2\pi k/N}_{N-k}) \quad (12.50)$$

The mass matrix in the $\mu/\bar{\mu}$ sector is the second derivative of the logarithm of the modulus of the Vandermonde determinant:

$$\begin{aligned} M_{ij}^2 &= \frac{\partial^2}{\partial \mu_i \partial \bar{\mu}_j} \log |\Delta(\mu)|^2 = \frac{\partial^2}{\partial \mu_i \partial \bar{\mu}_j} \sum_{\alpha > \beta} \log |\mu_\alpha - \mu_\beta|^2 \\ &= \frac{\partial}{\partial \bar{\mu}_j} \sum_{\alpha > \beta} \left(\frac{1}{\mu_\alpha - \mu_\beta} \delta_{\alpha i} - \frac{1}{\mu_\alpha - \mu_\beta} \delta_{\beta i} \right) + c.c. \\ &= \frac{\partial}{\partial \bar{\mu}_j} \left(\sum_{\beta < i} \frac{1}{\mu_i - \mu_\beta} - \sum_{\alpha > i} \frac{1}{\mu_\alpha - \mu_i} \right) + c.c. \\ &= \frac{\partial}{\partial \bar{\mu}_j} \left(\sum_{\beta < i} \frac{1}{\mu_i - \mu_\beta} + \sum_{\beta > i} \frac{1}{\mu_i - \mu_\beta} \right) + c.c. \\ &= \frac{\partial}{\partial \bar{\mu}_j} \sum_{\beta \neq i} \frac{1}{\mu_i - \mu_\beta} + c.c. \\ &= \pi \sum_{\beta \neq i} (\delta^{(2)}(\mu_i - \mu_\beta) \delta_{ij} - \delta^{(2)}(\mu_i - \mu_\beta) \delta_{j\beta}) + c.c. \\ &= \pi \sum_{\beta \neq i} \delta^{(2)}(\mu_i - \mu_\beta) \delta_{ij} - \delta^{(2)}(\mu_i - \mu_j) (1_{ij} - \delta_{ij}) + c.c. \end{aligned} \quad (12.51)$$

where we used:

$$\frac{\partial}{\partial \bar{z}_i} \frac{1}{z_j} = \pi \delta_{ij} \delta^{(2)}(z_j) \quad (12.52)$$

and 1_{ij} stands for a matrix with all entries equal to 1, $1_{ij} = 1, \forall i, j$.

$\delta^{(2)}(\mu_i - \mu_j)$ is a distribution in color space. It is nonvanishing when the two eigenvalues are degenerate.

Let us now specialize to the Z_N vortices in Eq.(12.50) with multiplicity \hat{N}' .

For the the first block, $i, j = 1, \dots, k$, Eq.(12.50) gives:

$$M_{ij}^2 = \pi \delta^{(2)}(0) ((\hat{N}'k - 1) \delta_{ij} - (1_{ij} - \delta_{ij})) = \pi \delta^{(2)}(0) (\hat{N}'k \delta_{ij} - 1_{ij}) \quad (12.53)$$

with $\delta^{(2)}(0) = \frac{N\hat{N}'}{(2\pi)^2}$ the delta function at zero in color space.

Since the diagonal terms scale as \hat{N}' , the non-diagonal corrections to the mass term are negligible in the large- \hat{N}' limit, in such a way that the theory has a mass gap ⁷⁷.

Similarly, for the second block, $i, j = k + 1, \dots, \frac{N}{2}$, one gets

$$M_{ij}^2 = \pi \delta^{(2)}(0) (\hat{N}'(N-k) \delta_{ij} - 1_{ij}) \quad (12.54)$$

that implies that the glueballs masses are at the cutoff scale ⁷⁸ for large N in this block. Thus the glueballs mass term is:

$$\frac{N \hat{N}' e^{-1}}{(2\pi)^2 \hat{N}' k (N-k) (4\pi)^2} \hat{N}' N_2' \int d^2 u k \pi \Lambda_W^2 \text{tr}_N \text{Tr}_{\hat{N}'} (\delta \mu \delta \bar{\mu}) + c.c. \quad (12.55)$$

to be added to the kinetic term to get:

$$\begin{aligned} & \frac{e^{-1}}{(4\pi)^2} \left(\frac{(2\pi)^2 10}{3(4\pi^2)^2} N_2'^2 \frac{1}{k} \int du_+ du_- \text{tr}_N \text{Tr}_{\hat{N}'} (\delta \mu \partial_+ \partial_- \delta \bar{\mu}) + \right. \\ & \left. + \frac{1}{(2\pi)^2} \hat{N}' N_2' \int du_+ du_- \pi \Lambda_W^2 \text{tr}_N \text{Tr}_{\hat{N}'} (\delta \mu \delta \bar{\mu}) \right) + c.c. \\ & = \frac{e^{-1}}{(4\pi)^2 4\pi} \left(\frac{10}{3\pi} N_2'^2 \frac{1}{k} \int du_+ du_- \text{tr}_N \text{Tr}_{\hat{N}'} (\delta \mu \partial_+ \partial_- \delta \bar{\mu}) \right. \\ & \left. + \hat{N}' N_2' \int du_+ du_- \Lambda_W^2 \text{tr}_N \text{Tr}_{\hat{N}'} (\delta \mu \delta \bar{\mu}) \right) + c.c. \end{aligned} \quad (12.56)$$

Now we have the identification (Eq.(4.14)):

$$\hat{N} \left(\frac{2\pi}{\Lambda} \right)^2 = 2\pi\theta \quad (12.57)$$

with

$$2\pi\theta = L^2 \frac{\hat{M}}{\hat{N}} \quad (12.58)$$

and

$$N_2 = \left(\frac{\Lambda}{2\pi} \right)^2 L^2 \quad (12.59)$$

Therefore:

$$\hat{N} = \left(\frac{\Lambda}{2\pi} \right)^2 L^2 \frac{\hat{M}}{\hat{N}} \quad (12.60)$$

at the cutoff scale and thus for $\hat{M} = \hat{N} - 1$ (see sect.(7)):

$$N_2 = \hat{N} \quad (12.61)$$

⁷⁷For $\hat{N}' = 1$ and $k = 1$ the theory has a massless eigenvalue in addition to the trivial diagonal $U(1)$ that decouples. We would like to thank Daniele Dorigoni for working out this case during our course at SNS.

⁷⁸It is not restrictive to require $k = 1, \dots, \frac{N}{2}$ for N even since our formulae are symmetric for the exchange $k \rightarrow N - k$.

and by the diagonal embedding:

$$N'_2 = \hat{N}' \quad (12.62)$$

at the renormalized scale. Finally in the reduced theory in the sector labelled by k the glueballs effective action is:

$$\begin{aligned} & \frac{e^{-1}}{(4\pi)^2 2\pi} \left(\frac{10}{3\pi} N'_2 \frac{1}{k} \int du_+ du_- \text{tr}_N \text{Tr}_{\hat{N}'} (\delta\mu \partial_+ \partial_- \delta\bar{\mu}) \right. \\ & \left. + \hat{N}' \int du_+ du_- \Lambda_W^2 \text{tr}_N \text{Tr}_{\hat{N}'} (\delta\mu \delta\bar{\mu}) \right) + c.c. \end{aligned} \quad (12.63)$$

12.7 Glueballs propagators in the Wilsonian scheme

Thus we find the following propagator in the twistor sector of the large- N theory for the Wilsonian normalization of the EK reduced commutative Morita equivalent effective action:

$$\begin{aligned} & \Lambda_W^6 \int \langle \frac{1}{N\hat{N}'} \text{tr}_N \text{Tr}_{\hat{N}'} (\mu\bar{\mu})(x_+, x_-, x_+, x_-) \frac{1}{N\hat{N}'} \text{tr}_N \text{Tr}_{\hat{N}'} (\mu\bar{\mu})(0, 0, 0, 0) \rangle_{\text{conn}} e^{i(p_+ x_- + p_- x_+)} dx_+ dx_- \\ & \sim \frac{1}{N^2 \hat{N}'^2} \sum_{k=1}^{\infty} \frac{k^2 \Lambda_W^6}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & \sim \frac{1}{N^2 \hat{N}'^2} \alpha'^2 (-p_+ p_-)^2 \log \frac{-p_+ p_-}{\Lambda_W^2} \end{aligned} \quad (12.64)$$

with:

$$\alpha' = \frac{10}{3\pi} \quad (12.65)$$

Indeed it is not hard to see that, setting $k^2 \Lambda_W^4 = [(k\Lambda_W^2 + \alpha' p_+ p_-)(k\Lambda_W^2 - \alpha' p_+ p_-) + (-\alpha' p_+ p_-)^2]$, the second line in Eq.(12.64) can be written as a logarithmic divergent sum that reproduces the correct logarithmic behavior of perturbation theory:

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{k^2 \Lambda_W^6}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & = \sum_{k=1}^{\infty} \frac{((k\Lambda_W^2 + \alpha' p_+ p_-)(k\Lambda_W^2 - \alpha' p_+ p_-) + (-\alpha' p_+ p_-)^2) \Lambda_W^2}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & = \alpha'^2 \sum_{k=1}^{\infty} \frac{(-p_+ p_-)^2}{-\alpha' p_+ p_- - \Lambda_W^{-2} + k} + \dots, \end{aligned} \quad (12.66)$$

up to a divergent sum of condensates, proportional to a power of Λ_W , and up to a divergent sum of contact terms. We can now define glueball composite operators in the following way. Operators of the form:

$$\text{tr}_N \text{Tr}_{\hat{N}'} (\mu_{\alpha\beta}^- \mu_{\alpha\beta}^-)^L \quad (12.67)$$

restricted to surface operators become:

$$\delta^{(2)}(0)^{2L-2} \sum_p \delta^{(2)}(0) \delta^{(2)}(z - z_p) \text{tr}_N \text{Tr}_{\hat{N}'} (\mu_p \bar{\mu}_p)^L \quad (12.68)$$

They can be evaluated, in the same fashion as the action density, as :

$$\delta^{(2)}(0)^{2L-2} \rho^2 \text{tr}_N \text{Tr}_{\hat{N}'}(\mu \bar{\mu})^L \quad (12.69)$$

We set:

$$\Lambda^2 = N \Lambda_W^2 \quad (12.70)$$

in such a way that the factors of N^{2L-2} from $\Lambda^{2(2L-2)}$ are cancelled by the powers of N^{-1} from the denominator in $\text{tr}_N \text{Tr}_{\hat{N}'} \delta(\mu \bar{\mu})^L$ evaluated on the condensate of surface operators. We rescale also ρ^2 by a factor of $N \hat{N}'$ in order to get a quantity on the order of 1. Thus our operators are:

$$\delta^{(2)}(0)^{2L-2} \hat{N}' N \rho^2 \text{tr}_N \text{Tr}_{\hat{N}'}(\mu \bar{\mu})^L \quad (12.71)$$

The corresponding glueballs propagators in the Wilsonian scheme are:

$$\begin{aligned} & \left(\frac{\Lambda}{2\pi}\right)^{8L-8} \hat{N}'^2 N^2 \rho^4 \int \langle \text{tr}_N \text{Tr}_{\hat{N}'}(\mu \bar{\mu})^L(x_+, x_-) \text{tr}_N \text{Tr}_{\hat{N}'}(\mu \bar{\mu})^L(0, 0) \rangle_{\text{conn}} e^{i(p_+ x_- + p_- x_+)} d^4 x \\ & \sim \left(\frac{\Lambda_W}{2\pi}\right)^{8L-8} N^{4L-4} \hat{N}'^2 N^2 \rho^4 \int \langle \text{tr}_N \text{Tr}_{\hat{N}'}(\delta \mu \mu^{L-1} \bar{\mu}^L)(x_+, x_-) \text{tr}_N \text{Tr}_{\hat{N}'}(\mu^L \bar{\mu}^{L-1} \delta \bar{\mu})(0, 0) \rangle_{\text{conn}} e^{i(p_+ x_- + p_- x_+)} d^4 x \\ & \sim \frac{\Lambda_W^2 L^2}{N} \sum_{k=1}^{\infty} \frac{\Lambda_W^2 k^{4L-2} \Lambda_W^{8L-4}}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & \sim \sum_{k=1}^{\infty} \frac{\Lambda_W^2 k^{2(2L-1)} \Lambda_W^{4(2L-1)}}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & \sim \sum_{k=1}^{\infty} \frac{\Lambda_W^2 ((k \Lambda_W^2 + \alpha' p_+ p_-)(k \Lambda_W^2 - \alpha' p_+ p_-) + (-\alpha' p_+ p_-)^2)^{2L-1}}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & \sim (-p_+ p_-)^{4L-2} \sum_{k=1}^{\infty} \frac{\Lambda_W^2}{-\alpha' p_+ p_- + k \Lambda_W^2} \\ & \sim (-p_+ p_-)^{4L-2} \log \frac{-P_+ P_-}{\Lambda_W^2} \end{aligned} \quad (12.72)$$

up to a sum of RG invariant condensates and contact terms. Only the leading singularity for large momentum has been displayed. In fact the subleading singularities in powers of the momentum have divergent coefficients. We can "renormalize" these singularities, for which we have not an interpretation, as follows. Localization of twistor Wilson loops admits shifting the eigenvalues of surface operators by adding $2\pi 1$ at every point⁷⁹. Indeed this does not modify anything in the $SU(N)$ sector, but shifts the diagonal $U(1)$ part of the action by a central term, that can be cancelled by a counterterm, in such a way that the effective action is flat for this $U(1)$.

We can now construct composite surface operators as in Eq.(12.71), but for the shifted curvature:

$$2\mu = \text{diag} \left(\underbrace{2\pi k/N}_k, \underbrace{2\pi(k+N)/N}_{N-k} \right) \quad (12.73)$$

⁷⁹This shift may be related to the split central extension that we disregarded in Eq.(7.15) as opposed to Eq.(7.11).

that has the same Z_N holonomy. Let us call the two dimensional Fourier transform of these operators $O_L(p_+, p_-)$. In the Wilsonian scheme we get:

$$\begin{aligned}
 & \langle Tr_{\mathcal{N}} O^L(p_+, p_-) Tr_{\mathcal{N}} O^L(-p_+, -p_-) \rangle_{\text{conn}}^{(W)} \\
 & \sim \sum_{k=1}^{\frac{N}{2}} \frac{\Lambda_W^2 k^{2(2L-1)} \Lambda_W^{4(2L-1)}}{-\alpha' p_+ p_- + k \Lambda_W^2} \\
 & + \sum_{k=1}^{\frac{N}{2}} \frac{\frac{N^2}{k^2} \Lambda_W^2 (N+k)^{2(2L-1)} \Lambda_W^{4(2L-1)}}{-\alpha' p_+ p_- + (N-k) \Lambda_W^2} \\
 & \sim (-p_+ p_-)^{4L-2} \sum_{k=1}^{\infty} \frac{\Lambda_W^2}{-\alpha' p_+ p_- + k \Lambda_W^2} + \dots \\
 & \sim (-p_+ p_-)^{4L-2} \log \frac{-p_+ p_-}{\Lambda_W^2} + \dots
 \end{aligned} \tag{12.74}$$

where the dots stand for contact terms, i.e. distributions whose inverse Fourier transform is supported at coinciding points. But now there are not anymore subleading singularities in momentum with divergent coefficients.

The natural interpretation is that this computation in the Wilsonian scheme furnishes the RG invariant version of some glueballs propagators for which the anomalous dimensions or the powers of the gauge coupling have been factored out.

12.8 Glueballs propagators in the canonical scheme

If we wish to recover the anomalous dimensions we should choose a canonical scheme, in which the fields are normalized in such a way to include the renormalization factors.

Localization of twistor Wilson loops is just a statement about the homology of 1 and in principle does not provide a dictionary to identify fluctuations of surface operators with specific glueballs propagators in perturbation theory. However, heuristically we can construct a dictionary on the following basis. We notice that the one-loop anomalous dimensions in the ground state of the Hamiltonian spin chain coincide within one-loop accuracy with the anomalous dimensions computed by means of the localization on surface operators by the change to the ASD variables:

$$Z_L = 1 - L g^2 \frac{5}{3} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right) + O(L^0) \sim Z^{-\frac{1}{2}} = \left(1 - g^2 \frac{10}{3} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda}{\mu}\right)\right)^{\frac{1}{2}} \tag{12.75}$$

Indeed, choosing a canonical scheme and following the definitions of sect.(11), we rescale the cutoff by a factor of $Z^{\frac{1}{2}}$ and the area L^2 by a factor of g^2 . Thus $\Lambda = Z^{\frac{1}{2}} \Lambda_c$ and $L^2 = g^2 L_c^2$. This changes the normalization of the operators insertions by factors of Z and the normalization of the effective action of the fluctuations by a power of g^4 , since the effective action is quadratic in the area of surface operators (see Eq.(12.44)). Hence we get:

$$\begin{aligned}
 & \left(\frac{\Lambda}{2\pi}\right)^{8L-8} \hat{N}^2 N^2 \rho^4 \int \langle tr_N Tr_{\hat{N}'} (\mu \bar{\mu})^L(x_+, x_-) tr_N Tr_{\hat{N}'} (\mu \bar{\mu})^L(0, 0) \rangle_{\text{conn}}^{(W)} e^{i(p_+ x_- + p_- x_+)} d^4 x \\
 & = \left(\frac{Z^{\frac{1}{2}} \Lambda_c}{2\pi}\right)^{8L-8} \hat{N}^2 N^2 \rho^4 \int \langle tr_N Tr_{\hat{N}'} (\mu \bar{\mu})^L(x_+, x_-) tr_N Tr_{\hat{N}'} (\mu \bar{\mu})^L(0, 0) \rangle_{\text{conn}}^{(W)} e^{i(p_+ x_- + p_- x_+)} d^4 x \\
 & = g^{-4} \left(\frac{Z^{\frac{1}{2}} \Lambda_c}{2\pi}\right)^{8L-8} \hat{N}^2 N^2 \rho^4
 \end{aligned}$$

$$\int \langle \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu\mu^{L-1}\bar{\mu}^L)(x_+,x_-)\text{tr}_N \text{Tr}_{\hat{N}'}(\mu^L\bar{\mu}^{L-1}\delta\bar{\mu})(0,0) \rangle_{\text{conn}}^{(C)} e^{i(p_+x_-+p_-x_+)} d^4x \quad (12.76)$$

where the canonical expectation value for the fluctuations is computed with respect to the effective action in Eq.(12.44) with L replaced by L_c . It follows that:

$$\begin{aligned} & \left(\frac{\Lambda_c}{2\pi}\right)^{8L-8} \hat{N}'^2 N^2 \rho^4 \int \langle \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu\mu^{L-1}\bar{\mu}^L)(x_+,x_-)\text{tr}_N \text{Tr}_{\hat{N}'}(\mu^L\bar{\mu}^{L-1}\delta\bar{\mu})(0,0) \rangle_{\text{conn}}^{(C)} e^{i(p_+x_-+p_-x_+)} d^4x \\ &= g^4 Z^{-\frac{8L-8}{2}} \left(\frac{\Lambda_W}{2\pi}\right)^{8L-8} N^{4L-4} \hat{N}'^2 N^2 \rho^4 \\ & \int \langle \text{tr}_N \text{Tr}_{\hat{N}'}(\delta\mu\mu^{L-1}\bar{\mu}^L)(x_+,x_-)\text{tr}_N \text{Tr}_{\hat{N}'}(\mu^L\bar{\mu}^{L-1}\delta\bar{\mu})(0,0) \rangle_{\text{conn}}^{(W)} e^{i(p_+x_-+p_-x_+)} d^4x \end{aligned} \quad (12.77)$$

Thus the perturbative anomalous dimensions for long operators in the ground state of the Hamiltonian spin chain in the thermodynamic limit [16, 17] are correctly reproduced by correlations of long surface operators in the canonical scheme. Actually they agree also for $L = 1$, since in this case the anomalous dimension is determined by the beta function via the factor of g^4 . This suggests also that the states which surface operators factorize on by homological localization are all scalars, although in principle they may couple also to tensors since $\mu\bar{\mu}$ is not a scalar. Analogously in the canonical scheme the same anomalous dimensions arise for the operators O^L :

$$\begin{aligned} & \langle \text{Tr}_{\mathcal{N}} O^L(p_+,p_-)\text{Tr}_{\mathcal{N}} O^L(-p_+,-p_-) \rangle_{\text{conn}}^{(C)} \\ &= g^4 (-p_+p_-) Z^{-\frac{8L-8}{2}} (-p_+p_-) \langle \text{Tr}_{\mathcal{N}} O^L(p_+,p_-)\text{Tr}_{\mathcal{N}} O^L(-p_+,-p_-) \rangle_{\text{conn}}^{(W)} \\ & \sim g^4 (-p_+p_-) Z^{-\frac{8L-8}{2}} (-p_+p_-) (-p_+p_-)^{4L-2} \log \frac{-p_+p_-}{\Lambda_W^2} \end{aligned} \quad (12.78)$$

13. Wild local systems and Regge trajectories

The hyper-Kahler reduction induced by the restriction to local systems can be extended to representations of the wild fundamental group [80, 91]. Hence in principle we can extend the computation of the fluctuations of surface operators in Eq.(12.39) to curvatures that involve derivatives of the delta function:

$$\begin{aligned} -iF_B &= \sum_p \mu \delta^{(2)}(z-z_p) + \sum_p \sum_n \delta\mu_p^{(n)}(u,\bar{u}) \partial^n \delta^{(2)}(z-z_p(u,\bar{u})) \\ & \quad + \sum_p \sum_{\bar{n}} \delta\mu_p^{(\bar{n})}(u,\bar{u}) \bar{\partial}^{\bar{n}} \delta^{(2)}(z-z_p(u,\bar{u})) \end{aligned} \quad (13.1)$$

This corresponds naturally to Regge trajectories of higher spins. We leave the computation for the future.

14. Conclusions and outlook: QCD-like theories and the twistorial string theory

The main conclusion of this paper is that there exist twistor Wilson loops that can be localized in large- N pure YM on local systems, i.e. on representations of the fundamental group of a punctured Riemann sphere immersed in space-time, i.e. on surface operators. These surface operators turn out to be connections with Z_N holonomy around the punctures. The localization on surface

operators leads to the one-loop exactness of the large- N Wilsonian beta function and to a canonical beta function of $NSVZ$ type.

Some understanding of the mass gap and of the glueballs spectrum occurs in a certain sector of the theory associated to twistor Wilson loops.

By certain changes of variables, that imply integrating on the moduli of surface operators, the loop equation for twistor Wilson loops is written in a holomorphic gauge in which a non-trivial glueballs potential is generated by the change of variables. The second derivative of the glueballs potential implies mass terms for the glueballs that are non-vanishing precisely for the surface operators with degenerate eigenvalues that occur for Z_N holonomy.

In this language glueballs arise as massive fluctuations of magnetic surface operators supported on the Lagrangian submanifold of space-time that is the support of the twistor Wilson loops.

This is the picture that follows from the localization of the loop equation for twistor Wilson loops and that realizes a new version of some long-standing ideas about dual superconductivity in pure YM [42, 43, 44].

On the field theory side we may wonder as to whether the methods of this paper extend to $\mathcal{N} = 1$ $SUSY$ YM , once we observe that twistor Wilson loops are not in the Parisi-Sourlas cohomology of the Nicolai map of $\mathcal{N} = 1$ $SUSY$ YM and therefore there is no reason for which they should be localized on instantons. In fact from the point of view of the standard folklore the glueballs spectrum of large- N $\mathcal{N} = 1$ $SUSY$ YM should not differ in a qualitative way from the one of YM .

Another extension would be to QCD -like theories, such as YM minimally coupled to N_f massless Dirac fermions in the fundamental representation in the large- N limit, keeping the ratio $\frac{N_f}{N}$ fixed. The computation of the associated glueballs spectrum would imply the determination of the lower side of the conformal window, as the point at which the mass gap disappears.

The basic issue involved in such extensions is the realization of the correct Wilsonian beta function around the critical points provided by the localization, and involves a crucial understanding of the fermion zero modes in a neighborhood of the critical points.

On the string theory side the results of this paper suggest a new string program for the YM theory, if we look for exact solvability.

It has been known for some time that $\mathcal{N} = 4$ $SUSY$ YM admits a partially equivalent twistorial string [94, 120, 121, 122]. The triviality of twistor Wilson loops and the fact that they are supported on Lagrangian submanifolds in twistor space suggests the existence of a stringy interpretation of our results in terms of open topological strings ending on Lagrangian submanifolds in twistor space [123, 124] in presence of surface operators. The occurrence of topological strings, as opposed to the usual strings, is due to the trivial nature of twistor Wilson loops at large N .

Since such twistorial topological string would be solvable by cohomological localization [125, 126], morally this conjectured topological string/gauge theory duality would provide the string cohomology dual to the field theory homology. Some hints about this conjectured twistorial string of YM can be found in [127].

We may wonder what such a twistorial string theory would be suited for, since it is supposed to be equivalent to the field theoretical results of this paper. The answer is found in the old fashioned unitarization program of string theory. The field theory computation provides the free glueballs spectrum in the twistor sector but does not furnish easily information about the glueballs interactions. However, in the string approach interactions are fixed by the geometry of the string

world-sheet, once the free theory is known, and thus the conjectured gauge theory/ topological string duality would open the way to computing the glueballs S -matrix.

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