Work inspired by studies of Rajamani Narayanan and Herbert Neuberger

Spectral shock waves in QCD

Maciej A. Nowak
(In collaboration with Jean-Paul Blaizot and Piotr Warchoł)

Mark Kac Complex Systems Research Center,
Marian Smoluchowski Institute of Physics,
Jagiellonian University, Kraków, Poland

Large N Gauge Theories, Galileo Galilei Institute, Florence
May 9th, 2011

Outline

- Diffusion of large (huge) matrices
- Non-linear Smoluchowski-Fokker-Planck equations and shock waves
- Finite N as viscosity in the spectral flow – Burgers equations
- Order-disorder phase transition in large N YM theory, colored catastrophes and universality
- Shock waves in large N SYM?
- Chiral shock waves
- Summary

Motivation

- Matricial ($N \times N$, $N \sim \infty$) analogue of classical probability calculus (physics, telecommunication, life science itd)
- Large N QFT in 0+0 dimensions (on one space-time point)
- Building in the dynamics: systems evolve as a function of some exterior parameters (time, length of the wire, area of the surface, temperature ...)
- Finding out universality windows where this simplified dynamics is shared by non-trivial theories

Two probability calculi

CLASSICAL

- probability density distribution $< \ldots >= \int \ldots p(x) dx$
- Fourier transform $F(k)$ of pdf generates moments
- ln $F(k)$ of Fourier tr. generates additive cumulants
- Gaussian – Non-vanishing second cumulant only, $\ln F(k) = c_2 k^2$

MATRICIAL (FRV for $N = \infty$)

- spectral measure of matrix-valued ensemble $< \ldots >= \int \ldots P(H) dH$
- Resolvent $G(z) = \left\langle \mathrm{Tr} \frac{1}{z-H} \right\rangle$
- R-transform generates additive cumulants $G[R(z) + 1/z] = z$
- Wigner semicircle – Non-vanishing second cumulant only, $R(z) = C_2 z$

Spectral observables in RMT (FRV)

- $P(H)dH = e^{-N\mathrm{Tr}V(H)}dH = \prod_{i=1}^{N} dx_i e^{-N\sum_i V(x_i)} \prod_{i<j}(x_i - x_j)^2$
- Jacobian (Vandermonde determinant) triggers interactions between eigenvalues
- All nontrivial correlations in the spectral functions reflect this interaction
- One-point function $G(z) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z-H} \right\rangle = \sum_k \frac{1}{z^{k+1}} m_k$, where $m_k = \frac{1}{N}\left\langle \mathrm{Tr} H^k \right\rangle = \int dx x^k \rho(x)$
- Note that $-\frac{1}{\pi}\Im G(z)|_{z=x+i\epsilon} = \rho(x)$

Inviscid Burgers equation

After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler...... from F.J. Dyson, J. Math. Phys. 3 (1962) 1192

- $H_{ij} \rightarrow H_{ij} + \delta H_{ij}$ with $< \delta H_{ij} = 0 >$ and $< (\delta H_{ij})^2 >= (1 + \delta_{ij})\delta t$
- For eigenvalues x_i, random walk undergoes in the "electric field" (Dyson) $< \delta x_i > \equiv E(x_i)\delta t = \sum_{i \neq j} \left(\frac{1}{x_j - x_i} \right) \delta t$ and $< (\delta x_i)^2 >= \delta t$
- Resulting SFP equation for the resolvent in the limit $N = \infty$ and $\tau = Nt$ reads $\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0$
- Non-linear, inviscid complex Burgers equation, very different comparing to Fick equation for the "classical" diffusion $\partial_\tau p(x,\tau) = \frac{1}{2}\partial_{xx} p(x,\tau)$

Inviscid Burgers equation - details

- SFP eq:
  $\partial_t P(\{x_j\},t) = \frac{1}{2}\sum_i \partial^2_{ii} P(\{x_j\},t) - \sum_i \partial_i (E(x_i) P(\{x_j\},t))$
- Integrating, normalizing densities to 1 and rescaling the time $\tau = Nt$ we get
  $\partial_\tau \rho(x) + \partial_x \rho(x) P.V. \int dy \frac{\rho(y)}{x-y} =$
  $\frac{1}{2N}\partial^2_{xx}\rho(x) + P.V. \int dy \frac{\rho_c(x,y)}{x-y}$
- r.h.s. tends to zero in the large N limit
- $\frac{1}{x \pm i\epsilon} = P.V. \frac{1}{x} \mp i\pi\delta(x)$
- Note that contrary to Dyson we consider free diffusion and not Ornstein-Uhlenbeck process, since we focus on non-equilibrium phenomena.

Dolphins wisdom - surfing the shock wave

Tracing the singularities of the flow allows to understand the pattern of the evolution of the complex system without explicit solutions of the complicated hydrodynamic equations...

UK Daily Mail, July 11th 2007

Complex Burgers Equation

- Burgers equation $\partial_\tau G + G\partial_z G = 0$
- Complex characteristics
  $G(z,\tau) = G_0(\xi[z,\tau]])$ $\qquad G_0(z) = G(\tau = 0, z) = \frac{1}{z}$
  $\xi = z - G_0(\xi)\tau \quad (\xi = x - vt)$, so solution reads
  $G(z,\tau) = G_0(z - \tau G(z,\tau))$
- Shock wave when $\frac{d\xi}{dz} = \infty$
- Since explicit solution reads $G(z,\tau) = \frac{1}{2\pi\tau}(z - \sqrt{z^2 - 4\tau})$, i.e. $\rho(x,\tau) = \frac{1}{2\pi\tau}\sqrt{4\tau - x^2}$, shock waves appear at the edges of the spectrum $(x = \pm 2\sqrt{\tau})$.
- But we can infer the same information from the condition $dz/d\xi = 0$, since $\xi_c = \pm\sqrt{\tau}$, so $z_c = \xi_c + G_0(\xi_c)\tau = \pm 2\sqrt{\tau}$

Universal preshock – relaxing $N = \infty$ condition

- $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z-H(\tau)}\right\rangle = \partial_z \left\langle \frac{1}{N}\mathrm{Tr}\ln(z-H(\tau))\right\rangle = \partial_z \left\langle \frac{1}{N}\ln\det(z-H(\tau))\right\rangle$
- We define $f(z,\tau) = \frac{1}{N}\partial_z \ln < \det(z-H(\tau)) >$
- Note that f and G coincide only when $N=\infty$ (cumulant expansion)
- Remarkably f fulfills for any N an exact equation $\partial_\tau f + f\partial_z f = -\nu\partial_{zz} f \qquad \nu = \frac{1}{2N}$
- Exact viscid Burgers equation with negative (!) viscosity
- Positive viscosity smoothens the shocks, negative is "roughening" them
- $\pm x = 2\sqrt{\tau} + \nu^{2/3}s$ and $f_N(x,\tau) \sim \pm\frac{1}{\sqrt{\tau}} + \nu^{1/3}\xi_N(s,\tau)$, where $\xi_N \sim \partial_s \ln Ai(\frac{s}{2\sqrt{\tau}})$
- Preshock: "soft edge" (Airy) universality

Multiplicative matricial random walk

- classically: $y_{i+1} - y_i = y_i \eta$ (η -noise)
- matricially: product of $< \prod_k (1 + H_k) >$ in general has complex spectra. But we can impose the constraint of unitarity $< \prod_k \exp iH_k >$, then eigenvalues are complex, but always confined to the unit circle ($x = e^{i\theta}$)
- Resolvent $G(z, \tau) = \int_{-\pi}^{\pi} d\theta \frac{\rho(\tau,\theta)}{z - e^{i\theta}}$.
- Related function $F(z = e^{i\theta}, \tau) = i(zG(z, \tau) - \frac{1}{2}) = i(\frac{1}{2} + \sum_{n=1}^{\infty} w_n(\tau) e^{-in\theta})$

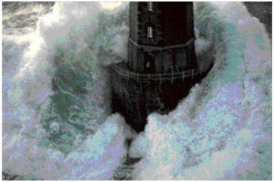
Diffusion of unitary matrices:

- Burgers equation for $F(z = e^{i\theta}, \tau)$ Durhuus, Olesen, Migdal, Makeenko, Kostov, Matytsin, Gross, Gopakumar, Douglas, Rossi, Kazakov, Voiculescu, Pandey, Shukla, Janik, Wieczorek, Neuberger, Biane...
- Collision of two shock waves, since they propagate on the circle
- Universal preshock - expansion at the singularity for finite N
- Universal, wild oscillations anticipating the shock – Pearcey universality

Three phases

If we encounter branch singularity $(\theta - \theta_c)^\mu$ on the complex plane, then for large n, $w_n = |n|^{-\mu-1} e^{-n\Delta} \Re e^{in\theta^*}$, where $\theta_c = \theta^* + i\Delta$

Gapped phase
$\tau < 4$
real singularities
$\mu = 1/2$
moments oscillate in time
modulo power law



Closure of the gap
$\tau = 4$
inflection point, so
$\mu = 1/3$
Durhuus-Olesen phase transition
different power law

Gappless phase
$\tau > 4$
complex singularities,
$\mu = 1/2$
moments decay exponentially
modulo power law

Photos by Jean Guichard (La Jument lighthouse, Brittany)

Central limit theorem

Nontrivial evolution from order ($\rho(\theta,0)=\delta(\theta)$) to disorder ($\rho(\theta,\infty)=\frac{1}{2\pi}$) (Haar measure), unravelled due to $\tau = Nt$

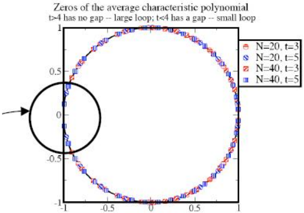
- Gapped phase: laminar "flow"
- Critical point: inflection point
- Gapless phase: Inverse spectral cascade

L. Da Vinci, Florence (?), ca 1506

Wilson loops in large N Yang-Mills theories (time $\equiv$ area)

Studies by Narayanan, Neuberger, 2006-2011

- $W(c) = \left\langle P \exp(i \oint A_\mu dx^\mu) \right\rangle_{YM}$
- $Q_N(z, \mathcal{A}) \equiv \langle det(z - W(\mathcal{A})) \rangle$
- Double scaling limit...
- $z = -e^y$
  $y = \frac{2}{12^{1/4} N^{3/4}} \xi$
  $\mathcal{A}^{-1} = \mathcal{A}^{*-1} + \frac{\alpha}{4\sqrt{3}} \frac{1}{N^{1/2}}$
- $Q_N(z, \mathcal{A}) \rightarrow$
  $\lim_{N \to \infty} \left(\frac{4N}{3}\right)^{1/4} Z_N(\Theta, \mathcal{A}) =$
  $= \int_{-\infty}^{+\infty} due^{-u^4 - \alpha u^2 + \xi u}$



universality!

Closing of the gap is universal in $d = 2, 3, 4$

Viscid Burgers equation

- $\phi_N \equiv -\frac{1}{N}\partial_y \ln(e^{N(\tau/8-y/2)} < \det(e^y + W(y,\tau) >)$ fulfills viscid Burgers equation $\partial_\tau + \phi\partial_y\phi = \frac{1}{2N}\partial_{yy}\phi$ [Neuberger]
- In our conventions, $z = e^{i\theta} = -e^y$, $\phi_N = if_N$, where $\partial_\tau f_N + f_N\partial_\theta f_N = -\frac{1}{2N}\partial_{\theta\theta} f_N$
- Collision of two universal oscillating preshocks (Airy) at critical time (area) produces novel universal oscillatory pattern (Pearcey).
- Airy: $Ai(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp i(t^3/3 + \xi t)$
- Pearcey: $P(\xi,\eta) = \int_{-\infty}^{+\infty} dt \exp i(t^4/4 + \xi t^2/2 + \eta t)$

Universal scaling visualization - "classical" analogy

Caustics, illustration from Henrik Wann Jensen

Fold and cusp fringes, illustrations by Sir Michael Berry

Morphology of singularity (Thom, Berry, Howls)

GEOMETRIC OPTICS
(wavelength $\lambda = 0$)

- trajectories: rays of light
- intensity surface: caustic

WAVE OPTICS ($\lambda \to 0$)

$N \to \infty$ Yang-Mills
($\nu = \frac{1}{2N} = 0$)

- trajectories: characteristics
- singularities of spectral flow

FINITE N YM (viscosity $\nu \to 0$)

Universal scaling, Arnold (μ) and Berry (σ) indices

"Wave packet" scaling
(interference regime)

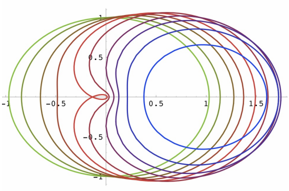
- $\Psi = \frac{C}{\lambda^\mu}\Psi(\frac{x}{\lambda^{\sigma_x}}, \frac{y}{\lambda^{\sigma_y}})$
- fold $\mu = \frac{1}{6}$ $\sigma = \frac{2}{3}$ Airy
- cusp $\mu = \frac{1}{4}$ $\sigma_x = \frac{1}{2}$ $\sigma_y = \frac{3}{4}$ Pearcey

Yang-Lee zeroes scaling with N
(for $N \to \infty$)

- YL zeroes of Wilson loop
- $N^{2/3}$ scaling at the edge
- $N^{1/2}$ and $N^{3/4}$ scaling at the closure of the gap

Shocks in SYM

- Complex dissusion $< \prod_k (1 + H_k(\tau)) >$ leads to "topological phase transition at $\tau = 4$ [Gudowska-Nowak, Janik, Jurkiewicz, MAN]



- Confirmed for equivalent complex diffusion $< \prod_k e^{H_k(\tau)} >$[Lohmayer,Neuberger,Wettig]
- Bijection between unitary and complex realizations of random walk [Biane]
- Phase transition for the complexification of the gauge potential – e.g. in SYM beyond conformal window

Hard edge universality

- Random walk of chiral Gaussian matrices: mirror eigenvalues due to "chiral symmetry", zero modes (fermion determinant) from "rectangularity"
- $H = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}$, where H is $M \times N$ complex Gaussian random matrix.
- Note that $[H, \gamma_5]_+ = 0$, where $\gamma_5 = \mathrm{diag}(1_N, -1_M)$ (chiral symmetry)
- Change of variables converts the evolution onto complex Bru (Wishart) evolution for $K^\dagger K$

Hard edge universality -cont.

- Burgers alike equation for the resolvent: e.g. $r = 1$ $\partial_\tau G(z,\tau) + 2zG(z,\tau)\partial_z G(z,\tau) = -G^2(z,\tau)$
- Riccati eq. for Airy transmutes into Riccati-Bessel eq.
- Crucial role of $G^2(0) = -\pi^2\rho^2(0)$
- Banks-Casher relation $< q\bar{q} > \sim \pi\rho(0)/V$,
- Spectral shocks and spontaneous symmetry breaking in QCD, universal preshock for finite volume ($N \leftrightarrow V$), in the guise of analysis of Stony Brook group [Shuryak, Verbaarschot, Zahed]

More details: J.-P. Blaizot, MAN, P. Warchoł, to be published;

[International Ph.D. project "Physics of Complex Systems" of the Foundation of Polish Science and cofinanced by the European Regional Development Fund in the framework of the Innovative Economy Programme]

Conclusions

- New insight for order-disorder transitions in strong interactions (e.g. Durhuus-Olesen transition, chiral symmetry breakdown)
- Multiple realizations of the universality, presumably also in several real complex systems
- Turbulence (in Kraichnan sense) as a dynamical mechanism for Haar measure in CUE interpreted as a Gibbs state
- Hint for new mathematical structures? (similar shocks for averaged inverse determinants)

More details: J.-P. Blaizot, MAN: Phys. Rev. Lett. 101, (2008)102001; Acta Phys. Pol. B40(2009) 3321; Phys. Rev. E82 (2010) 051115 and references therein.