Gluon condensate corrections to Wilson loops from holography

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Abstract

In this talk we study the relation between gluon condensate and Wilson loop expectation value from the point of view of holography. We find the related coefficient in the operator product exapnsion of the small supersymmetric Wilson loop and speculate on its possible form in nonsupersymmetric QCD. [1]

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I. INTRODUCTION

The object of our study is the gluon condensate, the nonperturbative vacuum expectaion value of the scalar gluon operator

$$\langle \alpha_s Tr(G^2) \rangle, \tag{1}$$

introduced in the framework of QCD sum rules in [1]. This value plays a major role in the QCD sum rules method and has a variety of phenomenological applications. For instance, one can show, that the ρ and other meson masses are proportional to $\langle \alpha_s Tr(G^2) \rangle^{1/4}$ and the energy of QCD vacuum equals $\epsilon = -\frac{b}{32} \langle \frac{\alpha_s}{\pi} Tr(G^2) \rangle$. So there are strong reasons to believe, that in the real QCD the vacuum expectation value under consideration do not vanish. The usually used phenomenological value of the gluon condensate is of order

$$\langle \alpha_s Tr(G^2) \rangle \sim (200 MeV)^4. \tag{2}$$

In the framework of the QCD sum rules the gluon condensate is a free parameter, which can be adjusted, to get better phenomenological results. It was then interesting to try to measure the condensate in the fundamental theory on the lattice, to find if it really exist. For this purpose one can study the expectation value of the small Wilson loop operator.

$$W(\mathcal{C}) = \frac{1}{N_c} \langle Tr \ Pexp[\oint_{\mathcal{C}} ig A_\mu dx^\mu] \rangle \tag{3}$$

As was shown in [2], the sufficiently small Wilson loop can be presented in the form of the operator product expansion, which looks like (omitting the input of the perturbation theory)

$$W(\mathcal{C}) = 1 - \frac{1}{48} \frac{\langle \alpha_s tr(G^2) \rangle}{N_c} S^2 + \dots, \tag{4}$$

where S is the area of the Wilson contour $\mathcal{C}$. This relation was a basis for the lattice study of the gluon condensate, which has got a reasonable value, while the procedure of subtracting of the perturbatie input to the Wilson loop is not well defined [3].

II. THE GLUON CONDENSATE IN THE HOLOGRAPHY

In the gauge/gravity duality the gluon condensate can be calculated in two, for the first sight unrelated, ways. First of all, as the gluon condensate is a vacuum expectation value of certain operator, it can be extracted as a coefficient in front of the normalizeble mode of its dual field in the supergravity [4]. In fact there is a general way to show, that the field, dual to the $Tr(G^2)$ operator is dilaton [15]. That means, that to describe a field theory with nonzero gluon condensate we need to consider the supergravity background with nontrivial normalizeble mode of the dilaton field

$$\phi = \phi_0 + \phi_4 z^4, \tag{5}$$

where z is a distance from the boundary of holographic background.

From the other hand, one can study the area of the minimal sufrace, spanned on the contour with a small radius on the boundary, which relates to the expectation value of the field theoretic Wilson loop on this contour, and get the value of the gluon condensate as a coefficient in its small size expansion in the spirit of lattice calculation.

To find out, if these ways are equivalent, we will take the supergravity background with nontrivial dilaton, compute the gluon condensate there and then calculate the small Wilson loop to check the relation (4).

We note here, that the applicability of our approach to QCD is rather subtle, because the OPE (4) is valid at small enough size of the loop (namely $r \ll \langle \alpha_s Tr(G^2)\rangle^{(-1/4)}$), what implies large enough energy scale. From the other hand, the supergravity description is valid only when the coupling constant is large, what in the QCD implies a small energy scale. Recall thought, that in the QCD the coupling becomes large already at the scale of $1GeV$, that means, that there is a window in the energy scale, where our approach is applicable.

III. THE SUPERSYMMETRIC CASE

The simple supergravity background, possessing the feature 5 and thus involving nonzero gluon condensate, is that of D-instanton smeared on the D3-branes [5]. This background in the string frame has the metric

$$ds^2_{D3}|_{str} = \frac{L^2}{z^2}\sqrt{h_{-1}}(dx^\mu dx_\mu + dz^2 + z^2 d\Omega_5^2) \tag{6}$$

and dilaton

$$e^\phi = g_s h_{-1}, \qquad h_{-1} = 1 + \frac{q}{\lambda} z^4 \tag{7}$$

The curvature radius of the AdS is related to the string coupling g_s (the dilaton asymptotic value) as $L^4 = \lambda l_s^4$, where $l_s = \sqrt{\alpha'}$ is a string length scale and $\lambda = g_s = N_c\alpha_s$ is a t'Hooft coupling in the field theory.

To proceed with our study of the gluon condensate $\langle \alpha_s tr(G^2)\rangle$ we need to fix the relation between the dilaton field and $\alpha_s tr(G^2)$ as it was done in [6]. By the symmetry arguments we can state, that the supergravitational dilaton field is dual to the gauge theory operator O_ϕ, which is proportional to scalar gluon operator: $O_\phi = c_\phi \alpha_s tr(G^2)$. Hence, by the standard AdS/CFT recipe [4, 7] the boundary value of the dilaton field is treated as a source of O_ϕ and its normalizable mode as a vacuum expectation value $\langle O_\phi\rangle$. We can compute the two-point function $\langle O_\phi O_\phi\rangle$ to fix the coefficient c_ϕ. In order to do this we need to calculate the classical action of the dilaton fluctuation and take the second variation with respect to its boundary value. The kinetic term of the dilaton field has a canonical form in the Einstein frame metric, which is defined as $G_E = \sqrt{g_s}e^{-\frac{1}{2}\phi}G_S$. In the Einstein frame the part of the supergravity action, that we are interested in, is

$$S_E = \frac{1}{(2\pi)^7 l_s^8}\int d^{10}x\sqrt{G_E}\left(-\frac{1}{2}\partial_A\phi\partial^A\phi\right) \tag{8}$$

$$= -\frac{N_c^2}{4(2\pi)^2}\int d^4x dz\ \frac{1}{2z^3}\left((\partial_z\phi)^2 + (\partial_\mu\phi)^2\right) \tag{9}$$

In the last line we assumed, that there is no dynamics along S_5-sphere, and used the definition of L given above. The solution to the equation of motion near the boundary is

$$\varphi(z,Q) = \frac{Q^2z^2}{2}K_2(Qz), \qquad \varphi(0,Q) = 1, \tag{10}$$

where K_i is McDonald function of the second kind. Substituting this solution back into the action and taking the second variation we get the expected two-point function. In the leading order of large Q expansion it is just the conformal result

$$\langle O_\phi O_\phi\rangle = \frac{N_c^2}{4(2\pi)^2}\frac{1}{2}\varphi(z,Q)\ \frac{\partial_z\varphi(z,Q)}{z^3}\Bigg|_{z=\epsilon} = \frac{N_c^2}{4(2\pi)^2}\frac{1}{8}Q^4 ln(Q^2\epsilon), \tag{11}$$

where ϵ is a AdS space cutoff, which is related to the UV cutoff in the quantum field theory. This expression can be compared with the leading order in the large Q expansion of the correlator of scalar gluon operators, found in the QCD sum rule approach [8]

$$\langle tr(G^2)tr(G^2)\rangle = \frac{N_c^2-1}{4\pi^2}Q^4 ln(Q^2\epsilon^2), \tag{12}$$

and this comparison in the large N_c limit allows us to fix the operator, dual to the dilaton

$$O_\phi = \frac{1}{4\sqrt{2}} tr G^2 \tag{13}$$

Similarly, we can compute the vacuum expectation value of O_ϕ. As in the considered background the classical profile of the dilaton has a normalizable branch, $\phi = \frac{q}{\lambda} z^4$, we find nonzero result

$$\langle O_\phi \rangle = \frac{N_c}{4(2\pi^2)} \varphi(z,Q) \left. \frac{\phi(z,Q)}{z^3} \right|_{z=\epsilon} = \frac{N_c}{\alpha_s (2\pi^2)} q \tag{14}$$

Thus, we find the desired expression for the gluon condensate in this model

$$\langle \alpha_s tr(G^2) \rangle \equiv 4\sqrt{2} \alpha_s O_\phi = N_c \frac{4\sqrt{2}}{(2\pi)^2} q \tag{15}$$

Note that the final relation doesn't include α_s, and as $\langle \alpha_s tr(G^2) \rangle \sim N_c$, q do not depend on N_c. Therefore all the dependancies on N_c and α_s are reproduced by hologrphy.

Now we can check the validity of the relation (4) in the D3-model. In order to compute the Wilson loop via holography one need to study the minimal area of the string worldsheet spanned on the Wilson contour on the boundary of the bulk space [9]

$$\langle W(\mathcal{C}) \rangle = e^{-Area(\mathcal{C})}. \tag{16}$$

The area of the string worldsheet, that we are interested in, is described by the Nambu-Goto action in the string frame

$$S_{NG} = \frac{1}{2\pi l_s^2} \int d^2\sigma \sqrt{g}, \tag{17}$$

where g is the induced two-dimensional metric on the worldsheet. In the string frame metric of the D3-model (6) the action (17) for the circular Wilson contour of the radius R takes the form (we expand the metric in q, because $q \sim \Lambda_{QCD}^4$ and we are considering the worldsheet in the region of the bulk space $z \sim R \ll \Lambda_{QCD}^{-1}$)

$$S_{NG} = \frac{2\pi}{2\pi l_s^2} \int\limits_0^R dr \; \frac{L^2}{z^2} \; r\sqrt{\dot{z}^2 + 1} \; \left(1 + \frac{qz^4}{2} + O(q^2 z^8)\right). \tag{18}$$

We assumed here the parametrization of the worldsheet by the radius and angle variables and performed the integration over the latter due to the symmetry of the problem. At this stage we end up with one-dimensional problem of finding the solution to the equation of motion, following from (18). This solution at $q = 0$ has a simple form [10]

$$z^0(r) = \sqrt{R^2 - r^2}, \tag{19}$$

And after the substraction of linear divergence gives a constant contribution to the area of the minimal surface. We can calculate the correction due to the gluon condensate to the Wilson loop expectation value using the perturbation theory on qz^4. Plugging this correction into the action (18) gives us the final result:

$$\delta\langle W\rangle \sim \frac{L^2}{l_s^2}\,\frac{1}{18}\frac{q}{\lambda}R^4 \sim \frac{1}{\sqrt{\lambda}}\frac{\langle\alpha_s tr(G^2)\rangle}{N_c}, \tag{20}$$

where we used the result (15) and expanded the exponent in (16). On the first sight the result is rather discouraging, because we get the wrong power of λ. But one should not be surprised, because in fact from the very beginning the D3-model does not describe real QCD. The holographic dual of D3-model under consideration is instead the N=4 supersymmetric Yang-Mills theory in the background self-dual field, which breaks half supersymmetry [5]. We note however, that the obtained relation is sensitive only to the near boundary behavior of the dual geometry, so it will hold in other D3-based models too (for instance [11, 12]). Hence, we state that what we have obtained (20), is the value of the coefficient in the operator product exapnsion of the small Wilson loop in the strongly coupled, supersymmetric quantum field theory with gluon condensate.

IV. QCD CASE

To describe QCD holographicaly one need to introduce the gravitational background, where supersymmetry and conformal symmetry are broken. As we have already mentioned, in the models, based on D3 brane background [11, 12], the OPE of the Wilson loop will look like (20). But these are not the only case. For instance, the Sakai-Sugimoto AdS/QCD model [13] is based on the D4-brane background, developed in [14]. For generality, we will consider the case of N_c p-branes here. For arbitrary p this supergravity solution in the string frame looks as (see [15] for a review)

$$ds_p^2|_{str} = h^{-\frac{1}{2}}\eta_{ab}dx^a dx^b + h^{\frac{1}{2}}(du^2 + u^2 d\Omega_{8-p}^2), \tag{21}$$

with dilaton

$$e^{\Phi} = g_s h(u)^{-\frac{p-3}{4}}, \tag{22}$$

where in the throat limit

$$h(u) = \left(\frac{L}{u}\right)^{7-p}, \tag{23}$$

$$L^{7-p} = d_p g_s N_c l_s^{7-p}, \tag{24}$$

$$d_p = (4\pi)^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right), \tag{25}$$

and indices a, b denote the (p+1)-dimensional space parallel to p-brane. We can make the coordinate transformation

$$u = \left(\frac{2}{5-p}\right)^{-\frac{2}{5-p}} L^{\frac{7-p}{5-p}} z^{-\frac{2}{5-p}} \tag{26}$$

which brings the metric to the conformaly flat form (Poincare form), similar to (6)

$$ds_p^2|_{str} = \left(\frac{5-p}{2}\frac{L}{z}\right)^{\frac{7-p}{5-p}} \left(\eta_{ab} dx^a dx^b + dz^2 + \left(\frac{2}{p-5}\right)^2 z^2 d\Omega_{8-p}^2\right). \tag{27}$$

Given this expression we can easily compute the induced metric on the string worldsheet with circular boundary and hence the Nambu-Goto action (17).

$$S_{NG}|_{Dp} = \frac{1}{2\pi l_s^2} \int\limits_0^R dr \ \left(\frac{5-p}{2}\frac{L}{z}\right)^{\frac{7-p}{5-p}} \ r\sqrt{\dot{z}^2 + 1} \tag{28}$$

We note, that the string coupling constant g_s as well as N_c enter this expression only via the parameter L. Making this dependence explicit, we find

$$S_{NG}|_{Dp} \sim (g_s N_c)^{\frac{1}{5-p}} \sim \lambda^{\frac{1}{5-p}} \tag{29}$$

We see now, that the dependance of the Nambu-Goto action on $\sqrt{\lambda}$ is rigidly related to the dimension of the underlying D-brane worldvolume. Let's consider the case of p=4, the Sakai-Sugimoto model. The string frame metric of this setting in the conformal coordinates (26) is

$$ds_{str}^2 = \left(\frac{L}{2z}\right)^3 \left(\eta_{\mu\nu} dx^\mu dx^\nu + f(z) d\tau^2 + \frac{1}{f(z)} dz^2 + 4z^2 d\Omega_4^2\right). \tag{30}$$

Here the τ dimension is compactified with the period $\delta\tau = \frac{16\pi}{3} z_0$, the black hole warp factor is $f(z) = 1 - \frac{z^6}{z_0^6}$ and μ, ν stand for usual 4D space indices. The dilaton is

$$e^{\Phi} = g_s \left(\frac{L}{2z}\right)^{\frac{3}{2}} \tag{31}$$

The parameter L is related to string coupling constant as

$$L^3 = \pi g_s N_c l_s^3 \tag{32}$$

The relation between g_s and field theory coupling constant can be established similarly to the D3 case

$$g_s = \frac{2}{3\pi} \frac{z_0}{l_s} g_{YM}^2. \tag{33}$$

Let us note here, that the metric (30) is not conformal from the very beginning, as the rescaling of 4D coordinates can't be absorbed now in the rescaling of z. It is not asymptotically AdS and its curvature behaves as $\frac{1}{z}$, making the classical gravity approach inapplicable near the boundary. Therefore the quantum gauge theory, dual to this geometry is not asymptotically free. This is a significant obstacle for us, because now our method of the dilaton operator normalization (11) is useless. Indeed, in the asymptotically non-free theory the consideration of perturbation theory leading divergence (12) is not legitimate even in the limit of asymptotically large momenta. Thus, if we consider the perturbation of dilaton profile by the dimensionfull quantity q (as in the preceding section), we can not unambiguously fix its relation to the gluon condensate.

Nevertheless, let us assume, that in QCD the similar to (15) simple relation between the dilaton profile and gluon condensate holds at least parametricaly. In this case, we can proceed in our calculations.

The Nambu-Goto action in the D4-brane background (30) is (from now we omit all numerical factors, as we are interested on parametrical dependence only), see (29),

$$S_{NG}|_{D4} \sim (g_s N_c) l_s \int_0^R \frac{r}{z^3} \sqrt{1 + f(z)\dot{z}^2} \tag{34}$$

Note, that the scale of compactification z_0, enters now the equation of motion for $z(r)$ via the warp factor $f(z)$. We do not know the solution to the classical equation of motion, but we can analyze it from the dimensional considerations. The integral in (34) has a dimension (-1) in the length units. There are two scales: R and z_0, that can compose this dimension, but in the limit $R \to 0$ the integral must remain finite, to agree with the leading term of (4). Therefore the value of the integral, computed on the classical solution must have the form $\frac{1}{z_0}g(R)$, where $g(R)$ has no negative powers of R. If we introduce now the parameter q/λ of dimension (-4) in the dilaton profile, similarly to (20) it will enter this function as $g(R) = const + \frac{q}{\lambda}R^4 + O(q^2R^8)$. At the end of the day we can find the parametric structure of the gluon condensate correction to the small Wilson loop

$$\delta\langle W\rangle \sim (g_s N_c)\,\frac{l_s}{z_0}(\frac{q}{\lambda}R^4 + \dots) \sim \frac{\langle \alpha_s tr(G^2)\rangle}{N_c}S^2, \tag{35}$$

where we used the expressions for g_s (33) and q (15). Interestingly, it is just the behavior, presented in (4).

V. CONCLUSION

In this talk we showed, that the OPE coefficients of the small Wilson loop, computed in holography, depend strongly on the basic features of the backround of the model under consideration, namely the dimension of the underlying D-brane and are insensitive to the deformations of the background, vanishing at the boundary. We calculated the strong coupling behaviour of the coefficient in the case of supersymmetric field theory, dual to D3 background, and studied its parametrical structure for the nonsupersymmetric, nonconformal theory, dual to D4 background. We found the interesting coincidence of the latter result with the QCD relation (4), athough this result should be considered only as a hint for future model building, as the D4-based holographic model does not have an asymptotic freedom behaviour and can not pretend on describing the QCD coupling constant properly.

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