Some Aspects of Resummation

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- Collaborations with Leo Almeida, Werner Vogelsang, Ilmo Sung, Alex Mitov and Ozan Erdogan.
- Something of a review, but slanted towards NNLL developments and the special role of the "cusp".

I. Threshold resummation: NLL and beyond

II. Soft anomalous dimension matrices: toward a dipole formula?

III. Graphical exponentiation: from webs to surfaces for the cusp

I. Threshold resummation: NLL and beyond

- A center of attention at the Tevatron & LHC: factorized cross section at fixed final-state 'signal' mass M, rapidity y, and relative rapidity $\hat{\eta}$ in pair c.m.

$$M^4 \frac{d\sigma_{h_1 h_2 \to Q\bar{Q}}}{dM^2 dy d\hat{\eta}} = \sum_f \int_\tau^1 dz \int \frac{dx_a}{x_a} \frac{dx_b}{x_b} \, \phi_{f/h_1}(x_a, \mu^2) \, \phi_{\bar{f}/h_2}(x_b, \mu^2)$$
$$\times \, \delta\left(z - \frac{\tau}{x_a x_b}\right) \delta\left(y - \frac{1}{2} \ln \frac{x_a}{x_b}\right)$$
$$\times \, \omega_{f\bar{f} \to Q\bar{Q}}\left(z, \hat{\eta}, \frac{M^2}{\mu^2}, \frac{m^2}{\mu^2}, \alpha_s(\mu^2)\right) + \ldots$$

- Variable z measures how much partonic energy is used to produce the pair

$$z = \frac{M^2}{x_a x_b S} = \frac{\tau}{x_a x_b} > \tau \equiv \frac{M^2}{S}$$

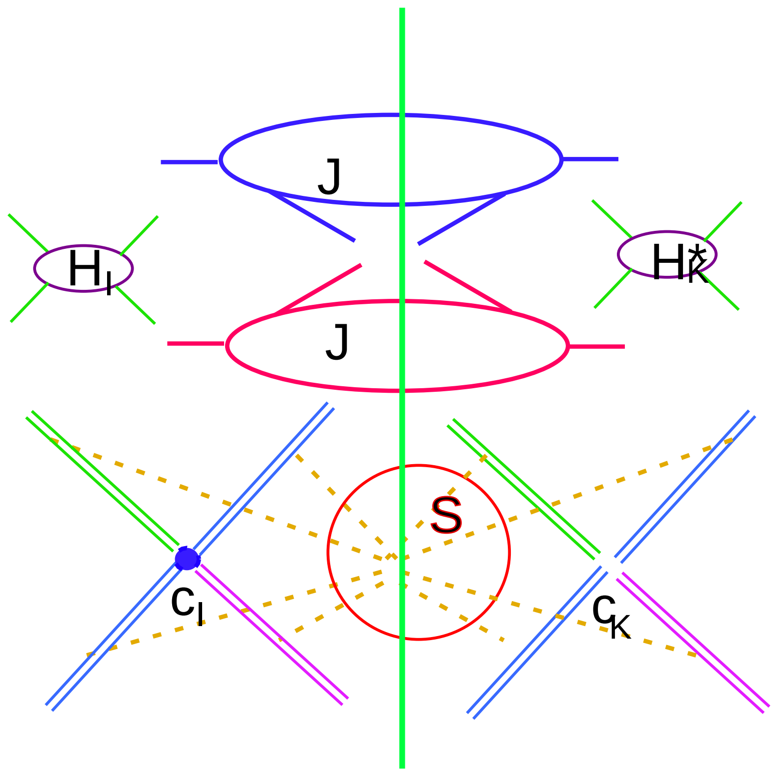
- The limit $z \to 1$ is the subject of threshold resummation.

- **Short-distance ω_{ab} is a generalized function.**
- **Example: at one loop for Drell-Yan ($M^2 = Q^2$):**

$$
\begin{aligned}
&\omega^{(1)}_{q\bar{q}\to\gamma^* g}(z, Q^2, \mu^2) \\
&\quad = \sigma_0(Q^2)\ C_F \left(\frac{\alpha_s(\mu)}{\pi}\right)\ \left\{2(1+z^2)\ \left[\frac{\ln(1-z)}{1-z}\right]_+ \right. \\
&\qquad \left. - \frac{(1+z^2)\ln z}{(1-z)} + \left(\frac{\pi^2}{3} - 4\right)\ \delta(1-z)\right\} \\
&\qquad + \sigma_0(Q^2)\ C_F\ \frac{\alpha_s}{\pi}\ \left[\frac{1+z^2}{1-z}\right]_+\ \ln\left(\frac{Q^2}{\mu^2}\right)
\end{aligned}
$$

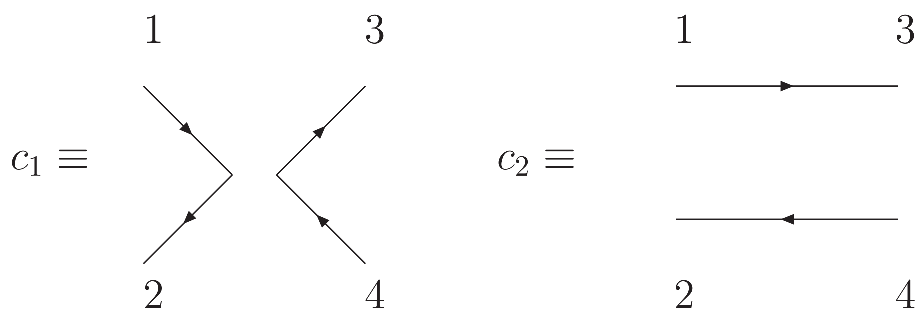
- **The '$+$' distributions tend to increase the cross section and at nth order we get $\frac{(C_F\alpha_s/\pi)^n}{n!}\left[\frac{\ln^{2n-1}(1-z)}{1-z}\right]_+$.**
- **Why $\frac{1}{n!}$? Because the cross section factorizes near $z = 1\ \ldots$**

- We compute $\omega_{ab}(z)$ from a regularized, partonic cross section. When $\tau \to 1$, it forces $z \to 1$.
- In this limit, we find factorization into $x \to 1$ (partonic) distributions, soft radiation and nearly lightlike outgoing jets and/or non-recoiling heavy quarks (squarks, gluinos).



- Color tensor c_I describes color exchange in H_I.

- An example of color tensor c_I: $q\bar{q}$ tensors $(c_L)_{\{r_i\}}$:



- To leading power in $1 - z$, the coupling of soft radiation to scattered partons is through ordered exponentials,

$$P \ \exp\left[-ig \int_0^\infty d\lambda \, \beta \cdot A(\lambda\beta)\right]$$

In SCET, summarized by appropriate field redefinitions. For final-state heavy quarks, this may be in HQET, or even NRQCD at low relative velocity.

- As factorization scales change, color exchange at the hard scattering evolves.

- Independence of factorization scales $\Rightarrow$ evolution equations.

- **Hard/jet/soft radiation $\Rightarrow$ double-logarithmic (Sudakov) exponentiation. A number of approaches can be used, including SCET, with other applications to less inclusive quantities** (Collins, Soper (1981) ... Stewart, Tackmann, Waalewijn (2009))

- **Threshold resummation kinematics: in c.m., z is $\propto$ energy of soft radiation:**

$$1 - z = \sum_{i\ \mathrm{soft}} \frac{2E_i^*}{\sqrt{\hat{s}}} \sim \frac{2k_{\mathrm{soft}} \cdot Q_{\mathrm{obs}}}{\hat{s}}$$

- **The partonic cross section then becomes a convolution in soft gluon energy, factorizes under moments of z: $1 - z \sim -\ln z$ and $z^N \sim \exp[-N(1-z)]$.**

$$\begin{aligned} \sigma(N) &= \int_0^1 dz z^{N-1} \sigma(z) \\ &= \int_0^1 dz e^{-(N-1)(1-z)} \sigma(z) + \mathcal{O}(1/N) \end{aligned}$$

- **This allows us to compute ...**

- a rather general form for threshold resummation, with leading, "universal" N-dependence in jets $\mathcal{J}$ (Catani, Mangano, Nason, Grazzini, de Florian, Kidonakis, Oderda, GS, Vogelsang; circa 1997 – pres.)

$$\begin{aligned}
\omega_{ab\to cd}&\left(N,\hat{\eta},\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right)\\
&= \prod_{i=a,b;c,d}\Delta_i(N,M/\mu,m/\mu,\alpha_s(\mu^2))\\
&\times \sum_{IK}[\mathrm{H}^{ab\to cd}_{IK}\left(\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\hat{\eta},\alpha_s(\mu^2)\right)\\
&\times \mathrm{S}^{ab\to cd}_{KI}\left(\frac{N^2\mu^2}{M^2},\frac{M^2}{m^2},\hat{\eta},\alpha_s(\mu^2)\right)] + \mathcal{O}(1/N)
\end{aligned}$$

- Different processes differ in list of (leading-logarithmic) jets and in the (single-logarithmic) "soft matrices" S. Matrix indices label color exchange c_I at the hard scattering: singlet, octet ... in amplitude and complex conjugate.

- **Incoming jet functions in moment space (parton a), with full leading-power N dependence (N^nLL):**

(Almeida, Sung, GS, Vogelsang, forthcoming)

$$
\begin{aligned}
&\ln \Delta^a_{\rm in}(N,Q) \\
&\quad = \int_0^{Q^2} \frac{du^2}{u^2}\, 2A_a\left(\alpha_s(u)\right) \left[K_0\left(\frac{2Nu}{Q}\right) - \ln\left(\frac{Q}{\bar{N}u}\right)\right] \\
&\qquad + \int_0^{Q^2} du^2\, D_a^{(\psi)}\left(\alpha_s(u)\right) \frac{\partial}{\partial u^2}\left[K_0\left(\frac{2Nu}{Q}\right) - \ln\left(\frac{Q}{\bar{N}u}\right)\right] \\
&\quad = \frac{1}{2}\int_0^1 dx \frac{x^{N-1}-1}{1-x} \left\{ \int_{(1-x)^2}^{Q^2} \frac{dq^2}{q^2}\, 2\, A_a\left(\alpha_s\left[q^2\right]\right) \right. \\
&\qquad\qquad \left. + D_a\left(\alpha_s\left[(1-x)^2 Q^2\right]\right) \right\}
\end{aligned}
$$

- **A_a the cusp anomalous dimension: $A_a \ln N$ in DGLAP evolution kernel. $D_a$and $D_a^{(\psi)}$ begin at NNLL, and include information on threshold phase space.**

- **The soft function mixes color exchange. It is generated from a matrix of anomalous dimensions.**
- **Effective theory approach** (Becher, Neubert, Schwartz, 2007–)**: moments are used on jet functions separately; inverted to derive an evolution equation in x. Soft anomalous dimension matrix is the same as here . . .**

- **In terms of the anomalous dimension matrix:**

$$\mathrm{S}\left(\frac{N^2\mu^2}{M^2}, \beta_i\cdot\beta_j, \alpha_s(\mu^2)\right)\Big|_{\mu=M}$$
$$= \overline{\mathcal{P}}\exp\left\{\int_0^1 dx \frac{x^{N-1}-1}{1-x}\,\Gamma_S^\dagger\left(\beta_i\cdot\beta_j, \alpha_s\left((1-x)^2M^2\right)\right)\right\}$$
$$\times \mathrm{S}\left(1, \beta_i\cdot\beta_j, \alpha_s\left(M^2/N^2\right)\right)$$
$$\times \mathcal{P}\exp\left\{\int_0^1 dx \frac{x^{N-1}-1}{1-x}\,\Gamma_S\left(\beta_i\cdot\beta_j, \alpha_s\left((1-x)^2M^2\right)\right)\right\}$$

- $\Gamma_S^{(1)} \rightarrow$ **NLL resummation** ($\alpha_s{}^n \ln^n N$)
- **Boundary condition,** $\mathrm{S}\left(1, \beta_i\cdot\beta_j, \alpha_s\left(M^2/N^2\right)\right)$ **starts at** $\alpha_s{}^0$**, and is a constant up to NNLL. Computed for inclusive heavy quark production (Czakon, Mitov, GS, 2009), and also for light parton** $2\rightarrow 2$ **processes (Almeida, Sung, GS, Vogelsang, forthcoming).**

II. Soft anomalous dimension matrices. IR structure of amplitudes.

- **Multiloop scattering amplitudes in dimensional regularization** (Catani (1998) Tejeda-Yeomans & GS (2002) Kosower (2003) Aybat, Dixon & GS (2006); Becher, Neubert (2008), Gardi, Magnea (2008), Dixon, Gardi, Magnea (2009), Del Duca *et al* (2011))

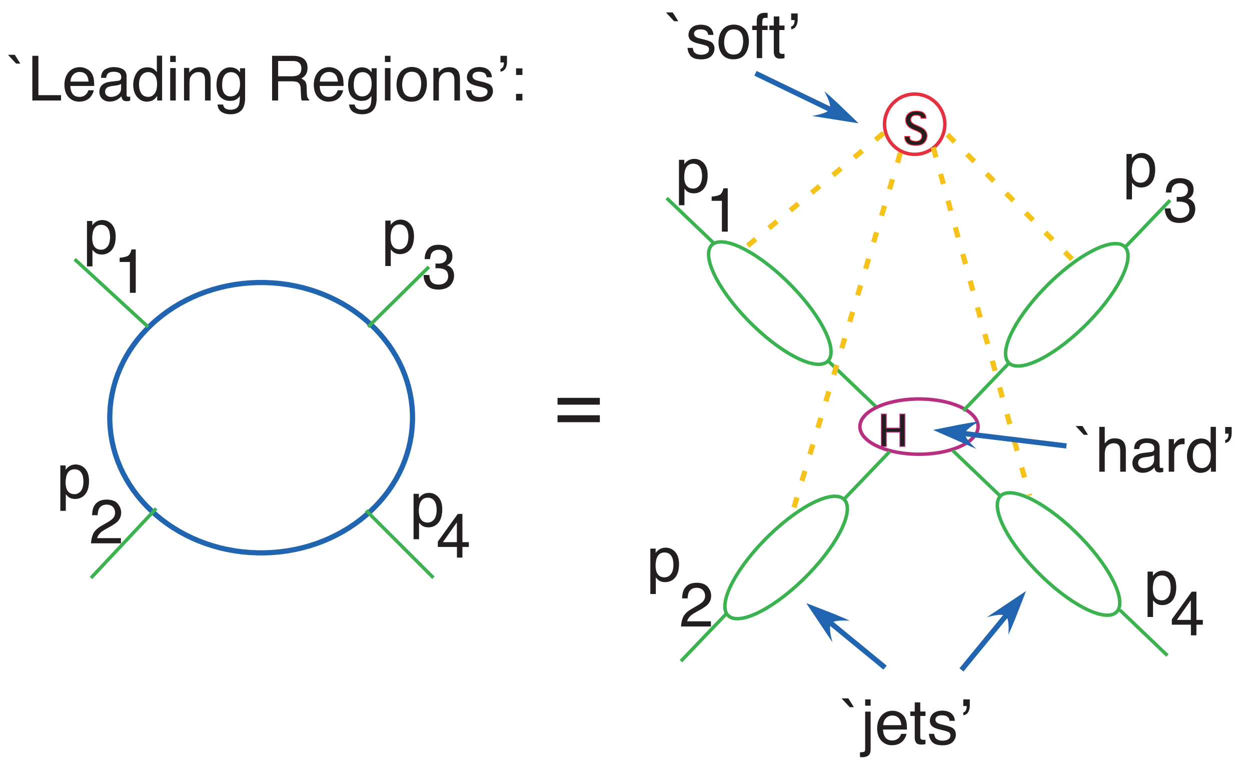
- **Amplitude for partonic process**

$$\mathrm{f}: \quad f_A(p_A, r_A) + f_B(p_B, r_B) \rightarrow f_1(p_1, r_1) + f_2(p_2, r_2)$$

$$\mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(p_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{M}^{[\mathrm{f}]}_{L}\left(p_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) (c_L)_{\{r_i\}}$$

- **Need to control poles in ϵ for factorized calculations at fixed order and for resummation.**

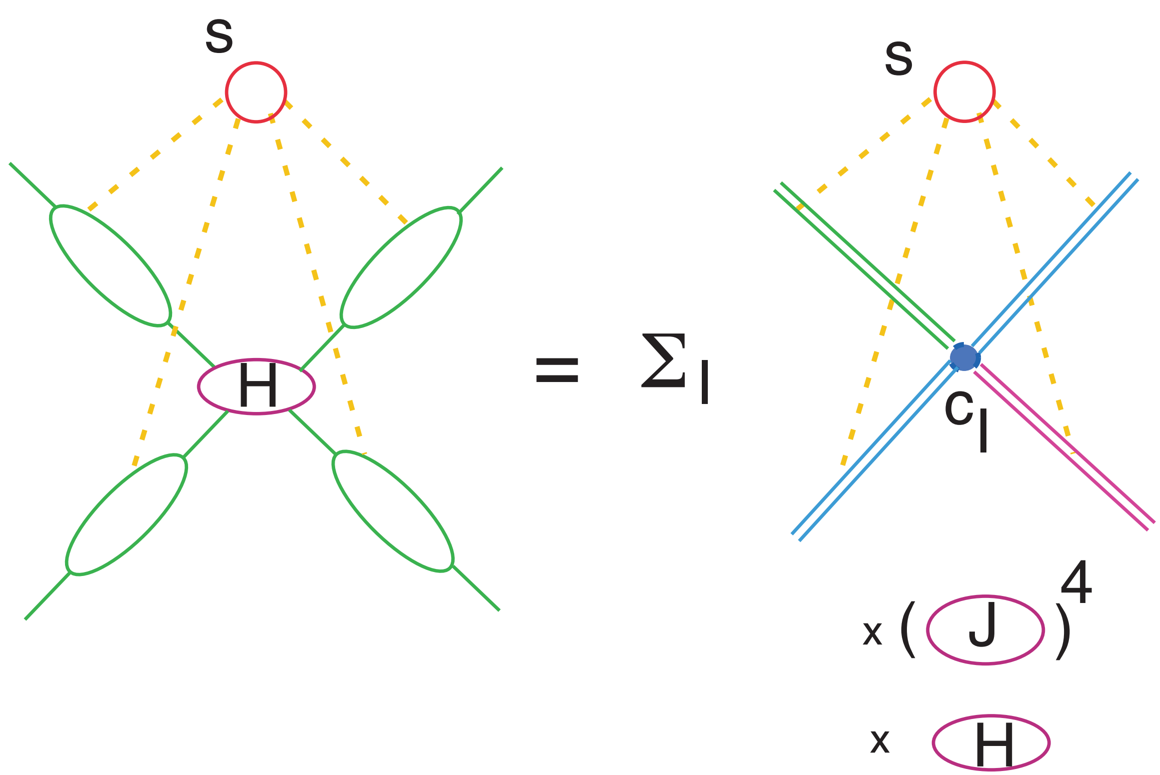
• **Source of double logs and poles in dimensional reg.:**



• **The same cast of characters as for cross section.**

• **Same separation, from Ward identities** **(Sen (1983))**:

Factorization of soft gluons:



- Jet/soft factorization for amplitude. :

$$\mathcal{M}_L^{[\mathrm{f}]}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \prod_{i=A,B,1,2} J_i^{[\mathrm{virt}]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$
$$\times\ \mathrm{s}_{LI}^{[\mathrm{f}]}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) h_I^{[\mathrm{f}]}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

- Jet function: $J = \sqrt{\Gamma_{\mathrm{singlet}}(Q^2)}$ (Tejeda-Yeomans & GS (1982))

- Soft function labelled by color exchange (singlet, octet . . .)

- Factors require dimensional regularization

- Same factorization $\rightarrow$ resummation

- Relation to supersymmetric Yang-Mills theories

  (Bern, Czakon, Dixon, Kosower & Smirnov (2006) N=4 crosscheck to 4 loops.)

- **Dimensionally-regularized soft matrix**

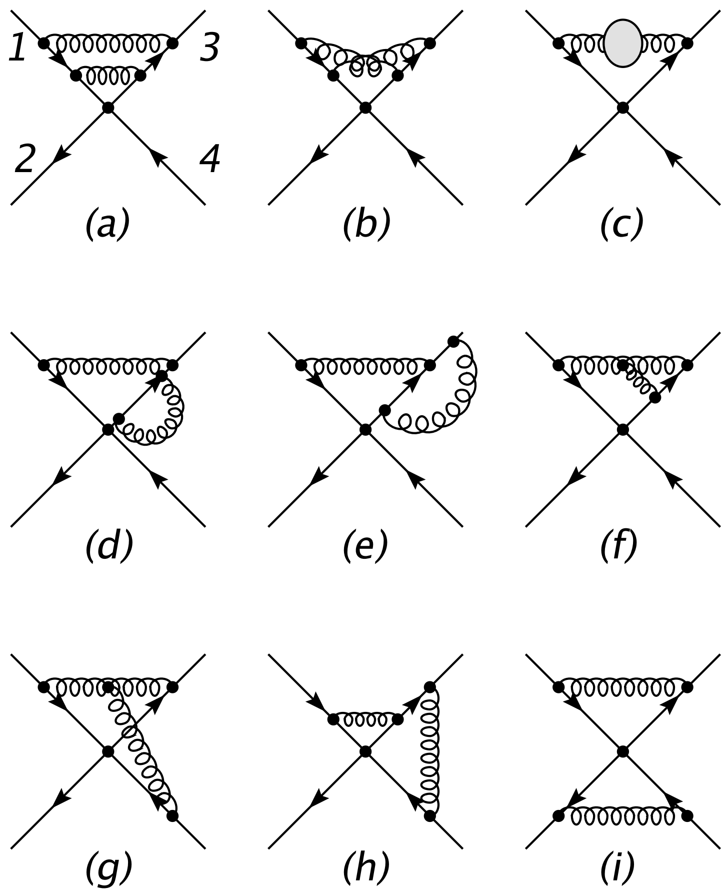
(Tejeda-Yeomans & GS (2002))

$$
\begin{aligned}
&\mathrm{s}^{[\mathrm{f}]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \\
&\qquad = \mathrm{P}\,\exp\left[-\frac{1}{2}\int_0^{-Q^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\Gamma^{[\mathrm{f}]}\left(\bar{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2}, \alpha_s(\mu^2), \epsilon\right)\right)\right]
\end{aligned}
$$

- $\Gamma^{[\mathrm{f}]}$: **anomalous dimension; color mixing**
- **Same anomalous dimension as in the cross section!**
- **For all massless $2 \to n$ processes** (Aybat, Dixon, GS (2006))

$$
\Gamma_S = \frac{\alpha_s}{\pi}\left(1 + \frac{\alpha_s}{\pi}K\right)\,\Gamma_{S'}^{(1)} + \cdots
$$

$\Gamma^{(2)} = (K/2)\Gamma^{(1)}$ with same K as in the DGLAP splitting.



- **Essential result: three-line $f_{abc} T_a^I T_b^J T_c^K$ terms vanish.**

(diagrams (g), (h))

- To NNLO, "single-web" exchange generalizes single gluon. (C.F. Berger, 2002). A 'radiation scheme' for the running coupling (Catani, Dokshitzer, Marchesini)

- Hints of unexpected simplicity for IR gluons, at least with massless Wilson lines. Web exchange $\leftrightarrow$ "sum over dipoles". (Becher, Neubert (2008), Gardi, Magnea (2008), Dixon (2008) and Dixon, Gardi, Magnea (2009)).

- Much can be understood in terms of scale-independence of Wilson lines: $\beta \to \zeta\beta^\mu$. This is quite restrictive for massless lines. Scale invariance eliminates almost all non-jet velocity dependence up to three loops, where "conformal cross ratios"

$$\frac{\beta_i \cdot \beta_j \, \beta_k \cdot \beta_l}{\beta_i \cdot \beta_l \, \beta_k \cdot \beta_j}$$

can appear. What really happens beyond three loops isn't yet known.

- **As a practical matter, the IR structure of two-loop single-pole are now understood as**

$$\frac{1}{\varepsilon}\left[\sum_{i\in \mathrm{f}} E_1^{[i]\,(2)} + \frac{1}{4}\Gamma_S^{[\mathrm{f}]\,(2)}\right] \times \mathrm{LO}$$

where $E_1^{[i]\ (2)}$ is residue of the 2 loop single pole in the Sudakov form factor. **(Ravindran, Smith, van Neerven (2005); Jantzen, Kuhn, Penin, Smirnov (2005, 2006) EW logs.)**

- For the massless case, the 'radical' simplicity of a dipole form to all orders explored in 1108.5947, 1109.3581 (Del Duca, Duhr, Gardi, Magnea, White):

$$\mathcal{M}_L^{[\mathrm{f}]}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = J_i^{[\mathrm{virt}]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$
$$\times \exp\Bigg[-\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\frac{A(\alpha_s(\lambda^2))}{4} \sum_{\{i,j\}} \ln\left(\frac{s_{ij}}{\lambda^2}\right) \mathbf{T}_i\cdot\mathbf{T}_j$$
$$+\sum_i \gamma_{J,i}(\alpha_s(\lambda^2))\Bigg]\, \mathcal{H}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

- A sum of two-eikonal 'cusps' in the exponent.
- Whether or not this holds beyond NNLpole, the simple two-eikonal, singlet 'cusp' becomes even more interesting.

III. The graphical interpretation of exponentiation: the 'cusp'

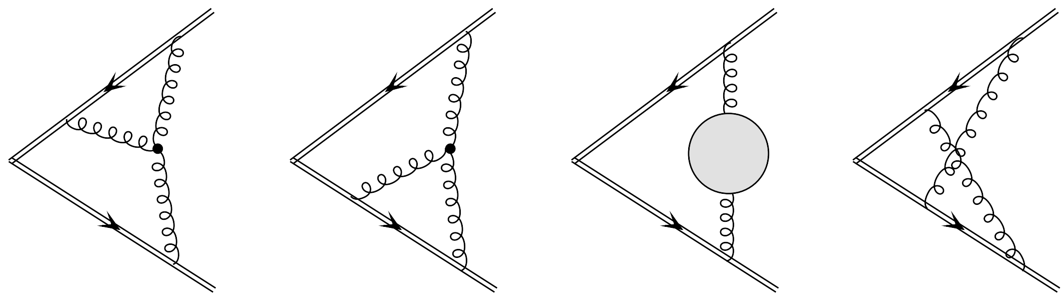
(Erdogan, Mitov, GS, Sung)

- An graphical interpretation of exponentiation. An alternative/interpretation of anomalous dimensions. (It's more general, but we'll consider just the cusp.)
- Will find an interesting 'geometrical' interpretation.
- The 2-line eikonal form factor is the exponential of a sum of two-eikonal irreducible diagrams, the "webs" with modified color factors:

  (Gatheral, Frenkel Taylor, GS)

$$A = \exp\left[\sum_{i=1}^{\infty} w^{(i)}\right]$$

- “‘Webs” in the exponent, $w^{(i)}$. are 2-eikonal irreducible diagrams. At 2 loops:



- **All have color factor (fundamental representation) $C_F C_A$ (only – no C_F^2).**

- **How it works. Say we know the exponent $w^{(i)}$ to order N. Expand to $N+1$st order, as a sum of diagrams and in terms of the exponential**

$$\boldsymbol{A}^{(N+1)} = \left(\exp\left[\sum_{i=1}^{N+1} \boldsymbol{w}^{(i)} \right] \right)^{(N+1)}$$

$$\boldsymbol{A}^{(N+1)} = \sum_{D^{(N+1)}} \boldsymbol{D}^{(N+1)} \,.$$

- **This gives a formula for the highest order in the exponent:**

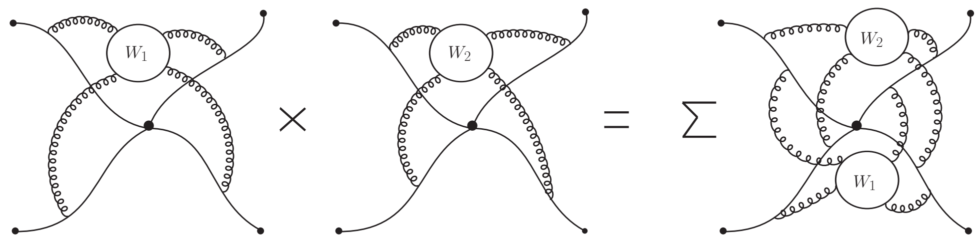
$$\boldsymbol{w}^{(N+1)} = \sum_{D^{(N+1)}} \boldsymbol{D}^{(N+1)} - \Big[\sum_{m=2}^{N+1} \frac{1}{m!} \times \sum_{i_m=1}^{N} \dots \sum_{i_1=1}^{N} \boldsymbol{w}^{(i_m)} \boldsymbol{w}^{(i_{m-1})} \dots \boldsymbol{w}^{(i_1)} \Big]^{(N+1)}$$

- To the relation

$$w^{(N+1)} = \sum_{D^{(N+1)}} D^{(N+1)} - \Big[\sum_{m=2}^{N+1} \frac{1}{m!}$$

$$\times \sum_{i_m=1}^{N} \ldots \sum_{i_1=1}^{N} w^{(i_m)} w^{(i_{m-1})} \ldots w^{(i_1)} \Big]^{(N+1)}$$

- Now apply the 'abelian' graphical identity, which holds for any number, length or shape of Wilson lines, simply an expression of the path ordering of $\exp[\int d\lambda \beta \cdot A(\beta\lambda)]$:



- The identity allows us to interpret the products of lower-order w's in terms of $N+1$st order diagrams, so that the effect of all the lower orders is to modify color factors, since these are unaffected by the identity.

- **Important: Consistency with the exponential in terms of anomalous dimensions requires that the webs automatically subtract all divergences where there is more than one 'subjet' in the web. The entire web is either hard, soft, or collinear to one line or the other.**
- **The web acts like a single gluon (consistent with the dipole exponentiation).**

- **As ordered exponentials, the webs can be constructed in coordinate space. Care must be taken to preserve gauge invariance, or double-logarithmic exponentiation fails in general. With this in mind, the result, with an IR cutoff L, is:**

(Erdogan, GS, forthcoming)

$$E(L,\varepsilon) = \int_0^L \frac{d\lambda}{\lambda}\frac{d\sigma}{\sigma}\, w\left(\alpha_s\left(1/\lambda\sigma\right)\right)$$

- **Here σ and λ are distances along the eikonal lines. The invariant size of the web fixes the running coupling.**

- **n.b. Systematic subtractions are necessary to construct this "web function" w. When this is done,**

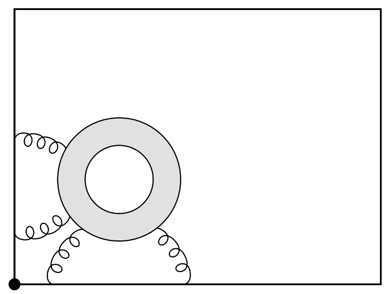
$$w = -\frac{1}{4}A(\alpha_s) + \mathcal{O}(\varepsilon)$$

- **Again,**

$$E(L,\varepsilon) = \int_0^L \frac{d\lambda}{\lambda}\frac{d\sigma}{\sigma}\, w\left(\alpha_s\left(1/\lambda\sigma\right)\right)$$

- **A "surface" interpretation is tempting. Indeed this is the same formula found from gauge/gravity duality for strongly coupled conformal gauge theory by minimizing a 5-dimensional surface (Alday, Maldacena (2007)). But here it's QCD with a coupling the runs with the size of the web. In a sense, it is just the invariances of the theory driving the result.**

- **Another interesting correspondence: polygonal Wilson lines** (Korchemskaya, Korchemsky (1993), Drummond et al (2008). . .). **Webs appear in "corners, and when defined in a gauge-invariant fashion give the leading singularities:**



- **Again, an interesting correspondence to gauge/gravity duality, and neglecting the running of the coupling, derive formulas based on minimal surfaces in 5 dimensions, like:**

$$\sum_{a=1}^{4} W_a(\beta_a, \beta_a') = \int_{-1}^{1} dy_1 \int_{-1}^{1} dy_2 \frac{4w_{\text{conformal}}}{(1-y_1^2)(1-y_2^2)}$$

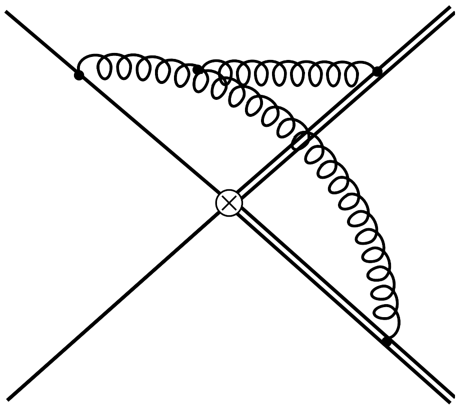
Conclusions

- **Many applications have been left out, especially transverse momentum and related resummations, low-x and BFKL, and power corrections.**
- **Where is this going? A most optimistic conjecture: the graphical interpretation of anomalous dimensions may offer another window to non-partonic degrees of freedom.**

Extra:

- **An equivalent form for the soft function:**

$$
\begin{aligned}
&\mathrm{S}\left(\frac{N^2\mu^2}{M^2}, \beta_i \cdot \beta_j, \alpha_s(\mu^2)\right) \Big|_{\mu=M} \\
&= \overline{\mathcal{P}} \exp\left\{ - \int_{M/N}^{M} \frac{d\mu}{\mu} \, \Gamma_S^{\dagger}\left(\beta_i \cdot \beta_j, \alpha_s\left(\mu^2\right)\right)\right\} \\
&\qquad \times \mathrm{S}\left(1, \beta_i \cdot \beta_j, \alpha_s\left(M^2/N^2\right)\right) \\
&\qquad \times \mathcal{P} \exp\left\{ - \int_{M/N}^{M} \frac{d\mu}{\mu} \, \Gamma_S\left(\beta_i \cdot \beta_j, \alpha_s\left(\mu^2\right)\right)\right\}
\end{aligned}
$$

- **Massive lines, for heavy quark and new physics production . . . ?**
- **Could**  **vanish with massive outgoing lines?**

- **Antisymmetric color combinations with massive Wilson lines** (Mitov, GS, Sung (2009), Becher, Neubert (2009))

- **Explicit forms of form-factor like diagrams** (Kidonakis (2009))

- **Coordinate-space representation can be useful, and give the same results as momentum space. Integrals over Wilson line vertices are trivial.**

- **Nonzero result in Euclidean space (sufficient).**

$$F_{3g}^{(2)}(\beta_I,\varepsilon) = - \int d^D x \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \zeta_k \zeta_i \frac{\beta_i \cdot \beta_j}{\left(\sqrt{x^2}\right)^{4-6\varepsilon}} \times g(\zeta_j,\varepsilon) g(\zeta_k,\varepsilon) \frac{\partial g(\zeta_i,\varepsilon)}{\partial \zeta_i}.$$

with $\zeta_i = \beta_i / \sqrt{\beta^2 x^2}$ **and**

$$g(\zeta_i,\varepsilon) = \int_0^\infty d\lambda' \frac{1}{\left(1 - 2\lambda'\zeta_i + \lambda'^2\right)^{1-\varepsilon}}$$

- Subsequently,elegant explicit results for $\Gamma_S^{(2)}(m_i)$. (Ferrogli, Neubert, Yang (2009)), which check numerically against position space calculations but can be continued to Miknowski space.

- The "3g" diagram, with $\cosh\beta_{IJ} = \frac{(p_I+p_J)^2}{m_I m_J}$:

$$\sum_{IJK} \epsilon_{IJK}\, \beta_{IJ}^2\, \beta_{JK}\, \coth\beta_{JK}$$

  Amazingly simple! The full result not much more complicated.

- Has a puzzling feature: nonuniform limit from $\beta = 1 - 4m^2/M^2 \to 0$:

$$\Gamma_S^{(2)}(m,\beta) \sim \frac{1}{\beta} \ln\left(\frac{1-\beta\cos\theta^*}{1+\beta\cos\theta^*}\right) \sim \cos\theta^*$$

  But not to worry: threshold factorization with recoil-less Wilson lines for outgoing quarks holds only for radiation whose energy is much smaller than quark relative four-momenta. Thus as $\beta \to 0$, the range of applicability of this result shrinks toward zero.