Exact corner free energies for two-dimensional integrable lattice models

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Summary

1. Corner free energy
   - Introduction
   - Finite-lattice method
2. Results
   - Potts model
   - Ising model
   - Other models
3. Asymptotic analysis
4. Conclusion

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Boundary effects in two-dimensional lattice models

- Consider free energy $f_{M,N} = \log Z_{M,N}$ on large $M \times N$ rectangle with specified (usually free) boundary conditions
- Decomposition in bulk, surface and corner parts:

$$f_{M,N} = MNf_{\rm b} + (M+N)f_{\rm s} + f_{\rm c} \,,$$

valid for $M, N \to \infty$ with fixed M/N.

Lessons from integrability

- f_b and f_s known exactly for many integrable models
  - Bethe Ansatz diagonalizes transfer matrix on cylinder (resp. strip) to give f_b (resp. f_s)
  - Series (resp. integral) formula in non-critical (resp. critical) regimes
- Very little known about f_c
  - How to implement boundary condition in terms of Bethe states?
  - Would need *all* eigenstates, not only dominant one!

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Lessons from conformal field theory

- Applies at or close to criticality
- f_b and f_s are non-universal and cannot be obtained
- However corners give rise to an anomaly, and f_c is accessible

Anomaly in corners

- Upper half plane with operator of weight h at the origin: $T(z) \approx \frac{h}{z^2}$
- Conformal mapping $w = z^{1/2}$ provides corner in $w = 0$
- Use transformation law of the stress tensor

$$T(w) = T(z) \left(\frac{\mathrm{d}z}{\mathrm{d}w} \right)^2 + \frac{c}{12} \{z, w\}$$

to obtain

$$T(w) \approx \left(4h - \frac{c}{8} \right) \frac{1}{w^2} .$$

- Even when $h = 0$ there is an anomaly, giving access to c

Relation to f_c

- $T(z)$ is the response of F to a local change in metric
- Cardy and Peschel have worked out the details
- Consider a corner of interior angle γ between boundaries of typical length L

$$\Delta F = -\frac{c\gamma}{24\pi}\left(1 - \left(\frac{\pi}{\gamma}\right)^2\right)\log L$$

- This result should apply also close to criticality, upon replacing L by the correlation length ξ

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Enting's finite-lattice method (FLM)

- Provides series expansions for $f_{M,N}$ of infinite $M \times N$ lattice in terms of finite-lattice data $f_{m,n}$
  - The $f_{m,n}$ are obtained from exact transfer matrix methods
  - Expansion around a trivial (hence non-critical) limit
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- Write $F_{M,N}$ as a sum over sublattice contributions $\tilde{f}_{i,j}$:

$$f_{M,N} = \sum_{[i,j]\subset[M,N]} \tilde{f}_{i,j} = \sum_{i\leq M, j\leq N} (M-i+1)(N-j+1)\tilde{f}_{i,j}$$

- Invert the similar relation for $f_{m,n}$:

$$\tilde{f}_{i,j} = \sum_{m\leq i, n\leq j} f_{m,n}\eta(m,i)\eta(n,j)$$

where

$$\eta(m,i) = \begin{cases} 1 & \text{if } m = i \text{ or } m+2 = i \text{ and } i > 2\,, \\ 2 & \text{if } m+1 = i \text{ and } i > 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

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- Approximate $f_{M,N}$ by restricting the set of finite lattices to:

$$B(k) = \{[m,n]\,, m+n=k\}$$

- This yields series for $f_{\rm b}$, $f_{\rm s}$ and $f_{\rm c}$, correct to an order that depends on k:

$$f_{\rm b} = \sum_{[m,n]\leq B(k)} f_{m,n}\left(\delta_{m,k-n} - 3\delta_{m,k-n-1} + 3\delta_{m,k-n-2} - \delta_{m,k-n-3}\right)\,,$$

$$f_{\rm s} = \sum_{[m,n]\leq B(k)} f_{m,n}\big((1-m)\delta_{m,k-n} + (3m-1)\delta_{m,k-n-1} - (3m+1)\delta_{m,k-n-2} + (m+1)\delta_{m,k-n-3}\big)\,,$$

$$f_{\rm c} = \sum_{[m,n]\leq B(k)} f_{m,n}\big((m-1)(n-1)\delta_{m,k-n} + (1+m+n-3mn)\delta_{m,k-n-1} + (3mn+m+n-1)\delta_{m,k-n-2} - (m+1)(n+1)\delta_{m,k-n-3}\big)\,.$$

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&\quad \left. +(3mn+m+n-1)\delta_{m,k-n-2} - (m+1)(n+1)\delta_{m,k-n-3}\right).
\end{aligned}$$

We have developped several variants of the basic FLM:

- Triangular lattices inscribed in equilateral triangles
  - Gives three $\frac{\pi}{3}$ corners
  - Inscription in rectangle would give two $\frac{\pi}{3}$ corners and two $\frac{2\pi}{3}$ corners
  - Thus we can access the two corner types separately
- Rectangles with particular boundary conditions on one or more sides
  - This gives free and particular f_s
  - And free-free, free-particular and particular-particular f_c
  - Unlike in CFT, no interaction between corners (due to limit order)

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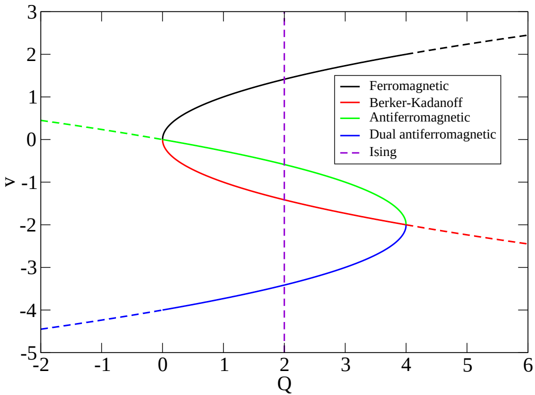
Square-lattice Potts model

- Fortuin-Kasteleyn representation $Z = \sum_{A \subseteq E} Q^{k(A)} v^{|A|}$ with $v = e^{J/k_B T} - 1$ and $k(A) = \#$ connected components



- Consider first the ferromagnetic transition curve $v = \sqrt{Q}$

Ground state ($Q \to \infty$) and one of the first excitations (of relative weight $\frac{v^3}{Q^3} = Q^{-3/2}$), with the smallest finite sublattice $[i, j]$ containing it.

- Write $\sqrt{Q} = q + \frac{1}{q}$. The "good" expansion parameter $q \ll 1$ is linked to the quantum group symmetry $U_q(sl_2)$.
- Cutoff set $B(k)$ gives series correct to order q^k. Numerics feasible with $k = 31$.

Conjectures [square-lattice ferromagnetic Potts model]

$$e^{f_b} = \frac{q^2+1}{q^2(q-1)^2} \prod_{k=1}^{\infty} \left(\frac{1-q^{4k-1}}{1-q^{4k+1}} \right)^4 , \quad \text{[Baxter]}$$

$$e^{f_s} = (1-q) \prod_{k=1}^{\infty} \left(\frac{1-q^{8k-1}}{1-q^{8k-5}} \right)^2 ,$$

$$e^{f_c} = \prod_{k=1}^{\infty} \frac{1}{(1-q^{8k-6})(1-q^{8k-4})^4(1-q^{8k-2})} .$$

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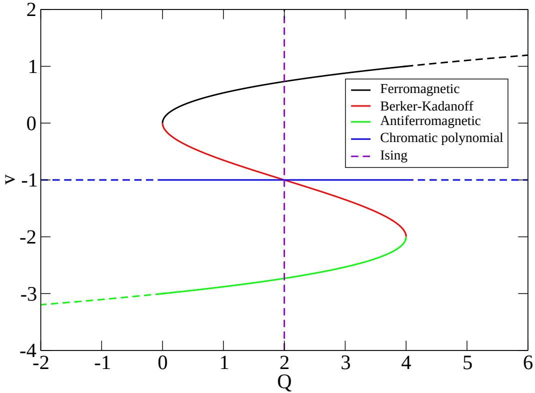
TRIANGULAR-LATTICE POTTS MODEL

- Integrable along the curve $v^3 + 3v^2 = Q$



- Parameterize $\sqrt{Q} = t^{3/2} + t^{-3/2}$ and $v = -1 + t + t^{-1}$
- The "good" expansion parameter is $t = q^{2/3}$

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$$e^{f_b} = \frac{1}{q^2}\frac{1-q^4}{1-q^2}\prod_{k=1}^{\infty}\left(\frac{(1-q^{4k-\frac{4}{3}})(1-q^{4k-\frac{2}{3}})}{(1-q^{4k-\frac{8}{3}})(1-q^{4k+\frac{2}{3}})}\right)^3, \quad \text{[Baxter]}$$

$$e^{f_s} = \frac{(1-q^{\frac{4}{3}})^2}{(1-q^{\frac{2}{3}})^2}\prod_{k=1}^{\infty}\left(\frac{(1-q^{8k-\frac{8}{3}})(1-q^{8k-\frac{22}{3}})}{(1-q^{8k-\frac{11}{3}})(1-q^{8k-\frac{14}{3}})}\right)^2.$$

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- Series expansion of e^{f_c} is more complicated
  - It appears to be of the form $e^{f_c} = \prod_{k=1}^{\infty} (1 - q^k)^{\alpha_k^{(1)} + \alpha_k^{(2)}}$, with $\alpha_k^{(i)} = \beta_k^{(i)} k + \gamma_k^{(i)}$ for corner of type $i = 1, 2$
  - Presumably $\beta_k^{(i)}$ and $\gamma_k^{(i)}$ are 8-periodic in q
- Using instead expansion for equilateral triangles

Conjecture [triangular-lattice ferromagn Potts model, $\frac{\pi}{3}$ corner]

$$e^{f_c} = \prod_{k=1}^{\infty} \frac{1}{(1-q^{8k-6})(1-q^{8k-2})(1-q^{8k-14/3})^3(1-q^{8k-10/3})^3}$$

- We need 24+ terms to fix $\frac{2\pi}{3}$ corner, but only have 22

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Square-lattice Ising model

- Low-temperature expansion in $x = \mathrm{e}^{-J/k_B T}$
- "Good" expansion variable is q in the parameterization $x^2 = q^{1/2} \prod_{k=1}^{\infty} \frac{(1-q^{8k-7})(1-q^{8k-1})}{(1-q^{8k-5})(1-q^{8k-3})}$ [Baxter, Sykes, Watts]
- Critical point $J/k_B T = \frac{1}{2} \log(1+\sqrt{2})$ occurs when $q \to 1^-$

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Conjectures [square-lattice Ising model]

$$e^{f_b} = \frac{1}{q^{1/2}} \prod_{k=1}^{\infty} \frac{(1-q^{8k-1})^{8k-1}(1-q^{8k-5})^{8k-5}}{(1-q^{8k-7})^{8k-7}(1-q^{8k-3})^{8k-3}} \frac{(1-q^{8k-4})^2}{(1-q^{8k-6})(1-q^{8k-2})}, \quad \text{[Baxter]}$$

$$e^{f_s} = x^{-1} \prod_{k=1}^{\infty} \left(\frac{1-q^{\frac{8k-3}{2}}}{1-q^{\frac{8k-5}{2}}}\right)^{4k-2} \left(\frac{1-q^{\frac{8k-1}{2}}}{1-q^{\frac{8k+1}{2}}}\right)^{4k} \left(\frac{1-q^{8k-5}}{1-q^{8k-3}}\right)^{2k-1} \left(\frac{1-q^{8k+1}}{1-q^{8k-1}}\right)^{2k},$$

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- Critical point $J/k_B T = \frac{1}{4}\log 3$ corresponds to $q \to 1^-$

Conjecture [triangular-lattice Ising model]

$$e^{f_b} = x^{-3} \prod_{k=1}^{\infty} \frac{(1-q^{8k-4})^2}{(1-q^{8k-6})(1-q^{8k-2})} \prod_{k=1}^{\infty} \left(\frac{1-q^{8k-\frac{14}{3}}}{1-q^{8k-\frac{10}{3}}} \right)^{6k-1} \prod_{k=1}^{\infty} \left(\frac{(1-q^{8k-\frac{4}{3}})(1-q^{8k-\frac{2}{3}})(1-q^{8k+\frac{8}{3}})}{(1-q^{8k-\frac{8}{3}})(1-q^{8k+\frac{2}{3}})(1-q^{8k+\frac{4}{3}})} \right)^{6k}$$ [Baxter]

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- For e^{f_c} problems similar to those of the Potts model on the triangular lattice
- But for three $\frac{\pi}{3}$ angles on an equilateral triangle we find

Conjecture [triangular-lattice Ising model, $\frac{\pi}{3}$ corner]

$$\mathrm{e}^{f_c} = 2\prod_{k=1}^{\infty}\frac{1}{1-q^{8k-4}}\prod_{k=1}^{\infty}\left(\frac{1-q^{8k-\frac{14}{3}}}{1-q^{8k-\frac{22}{3}}}\right)^{12k-9}\left(\frac{1-q^{8k-\frac{2}{3}}}{1-q^{8k-\frac{10}{3}}}\right)^{12k-3}$$

$$\prod_{k=1}^{\infty}\frac{\left(1-q^{8k-\frac{16}{3}}\right)^{15k-9}}{\left(1-q^{8k-\frac{8}{3}}\right)^{15k-6}}\prod_{k=1}^{\infty}\frac{\left(1-q^{8k-\frac{4}{3}}\right)^{9k}}{\left(1-q^{8k-\frac{20}{3}}\right)^{9k-9}}$$

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We have similar results for the following other models:

- Antiferromagnetic square-lattice Potts model
- Triangular-lattice chromatic polynomial
- Fully-packed two-color loop model FPL^2

Periodicities of the exponents in the product formulae:

Model	Lattice	e^{f_b}	e^{f_s}	e^{f_c}
Potts ferromagnet	Square	4	8	8
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Potts antiferromagnet	Square	8	16	16
Chromatic polynomial	Triangular	6	12	12
FPL^2	Square	8	16	16
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Critical limits

- The limit $q \to 1^-$ (and sometimes $q \to -1^+$) gives a CFT
- Hence can compare with results of [Cardy, Peschel]

- Asymptotic analysis based on properties of $\eta(\tau)$ and some ad hoc methods
- Desirable to have general results on $\prod_{k=1}^{\infty}(1-q^k)^{\alpha_k}$, with $\alpha_k = \beta_k k + \gamma_k$ [Refs. needed!]

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- Desirable to have general results on $\prod_{k=1}^{\infty}(1-q^k)^{\alpha_k}$, with $\alpha_k = \beta_k k + \gamma_k$ [Refs. needed!]

- $\mathrm{e}^{f_\mathrm{b}}$ and $\mathrm{e}^{f_\mathrm{s}}$ have finite limits
- Focus henceforth on asymptotic divergence of $\mathrm{e}^{f_\mathrm{c}}$

ASYMPTOTIC DIVERGENCE OF e^{f_c}

Square lattice (four $\frac{\pi}{2}$ corners)

- Potts: $e^{f_c} \underset{q\to 1^-}{\sim} 2^{-5/2} e^{\frac{\pi^2}{8}\left(\frac{1}{1-q}-\frac{1}{2}\right)}$
- Ising: $e^{f_c} \underset{q\to 1^-}{\propto} e^{\frac{\pi^2}{16(1-q)}}$

Triangular lattice (three $\frac{\pi}{3}$ corners)

- Potts: $e^{f_c} \underset{q\to 1^-}{\sim} \frac{3\sqrt{3}-5}{2} e^{\frac{\pi^2}{6}\left(\frac{1}{1-q}-\frac{1}{2}\right)}$
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Comparison with Cardy-Peschel formula

$$\Delta F = -\frac{c\gamma}{24\pi}\left(1-(\pi/\gamma)^2\right)\ln\xi = \begin{cases} \frac{c}{16}\ln\xi & \text{for } \gamma=\frac{\pi}{2}\,, \\ \frac{c}{9}\ln\xi & \text{for } \gamma=\frac{\pi}{3}\,. \end{cases}$$

- Our expressions have the correct angular dependence, thus confirming lattice universality

Potts model ($c = 1$)

$$\xi_{\rm sq} \underset{q\to 1^-}{\sim} \frac{1}{2^{10}{\rm e}^{\pi^2/4}}\,{\rm e}^{\frac{\pi^2}{2(1-q)}} \quad \text{and} \quad \xi_{\rm tri} \underset{q\to 1^-}{\sim} \frac{(3\sqrt{3}-5)^3}{8{\rm e}^{\pi^2/4}}\,{\rm e}^{\frac{\pi^2}{2(1-q)}}$$

Agrees with $\xi_{\rm BA} \underset{q\to 1^-}{\sim} \frac{1}{8\sqrt{2}}\,{\rm e}^{\frac{\pi^2}{2(1-q)}}$ [Wallon, Buffenoir]

Predicts $\lim_{q\to 1^-} \frac{\xi_{\rm tri}}{\xi_{\rm sq}} = 128(3\sqrt{3}-5)^3 \simeq 0.966031$

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$\ln \xi(q) \underset{q \to 1^-}{\propto} \frac{\pi^2}{2(1-q)}$ for both lattices

- Looks strange until one realizes that $T_c - T \underset{T \to T_c^-}{\sim} e^{-\frac{\pi^2}{2(1-q)}}$
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- How to prove our conjectures? Any link to the corner transfer matrix?
- Any link to the boundary state for free boundary conditions $|B\rangle = \lim_{n\to\infty} \mathrm{e}^{-\frac{1}{2^n-1}L_{-2^n}} \cdots \mathrm{e}^{-\frac{1}{2}L_{-4}} \mathrm{e}^{-L_{-2}} |0\rangle$ [Bondesan, Dubail, Jacobsen, Saleur] ?
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