"New quantum states of matter in and out of equilibrium"

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"Quantum Geometrodynamics" of the Fractional Quantum Hall effect

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- The previously-overlooked collective degree of freedom in the FQHE is a dynamical internal geometry associated with incompressibility
- This geometry can be found in the Laughlin state (and other model states), but was hidden for the last 30 years because of misinterpretation of the meaning of the "Laughlin wavefunction"
- A revised description of FQHE edges may reconcile theory and experiment

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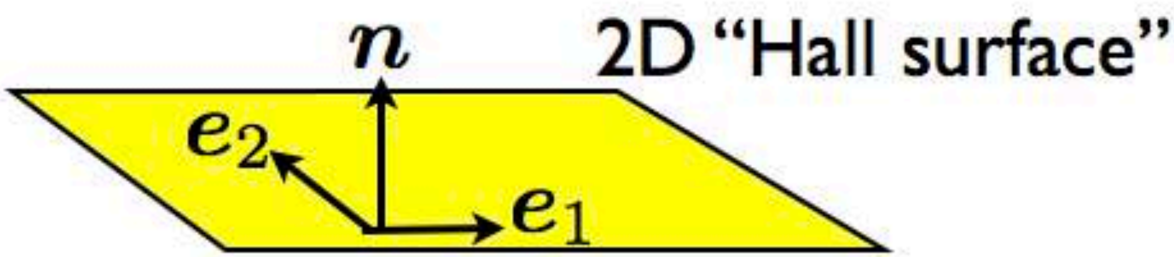
- In the "clean limit" of the **integer** quantum Hall effect, at energies and temperatures below the Landau-level spacing, there are no low-energy electronic excitations

- The electric current on the 2D surface obeys

$$J_e^0 = \sigma_H B \qquad J_e^a = \sigma_H \epsilon^{ab} E_b \qquad \sigma_H = \frac{pe^2}{2\pi\hbar}$$

Surface charge — Surface current

charge conservation ⟷ Faraday Law



$$\boldsymbol{n} \cdot \boldsymbol{\Delta D} = \sigma_H \boldsymbol{n} \cdot \boldsymbol{B}$$
$$\boldsymbol{e}_a \cdot \Delta \boldsymbol{H} = \sigma_H \boldsymbol{e}_a \cdot \boldsymbol{E}$$

"Hall surface" = modified electromagnetic boundary condition (gives magnetic monopole image charges, FDMH and L. Chen PRL1985, X. Qi and S-C Zhang, 2009)

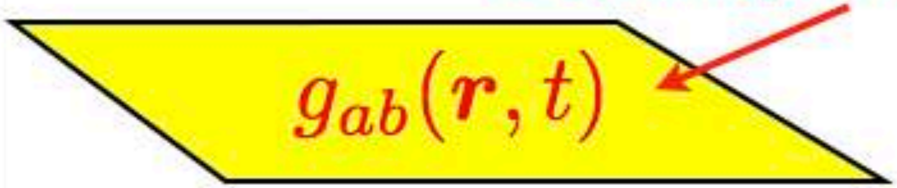
- Simple picture of fractional QHE until recently was similar to integer QHE, except for fractional quantization

$$\sigma_H = \frac{pe^2}{2\pi q\hbar} = \frac{pee^*}{2\pi\hbar}$$

$$e^* = \frac{e}{q}$$ elementary fractional charge

pe = charge of "composite boson" condensate

NEW: dynamical unimodular 2D spatial quantum metric



new fundamental parameter

s = "guiding-center spin" of "composite boson"

(Topologically-quantized by Gauss-Bonnet)

FQHE Hall surface has an internal geometric degree of freedom

$$J_e^0 = peJ^0$$

charge density

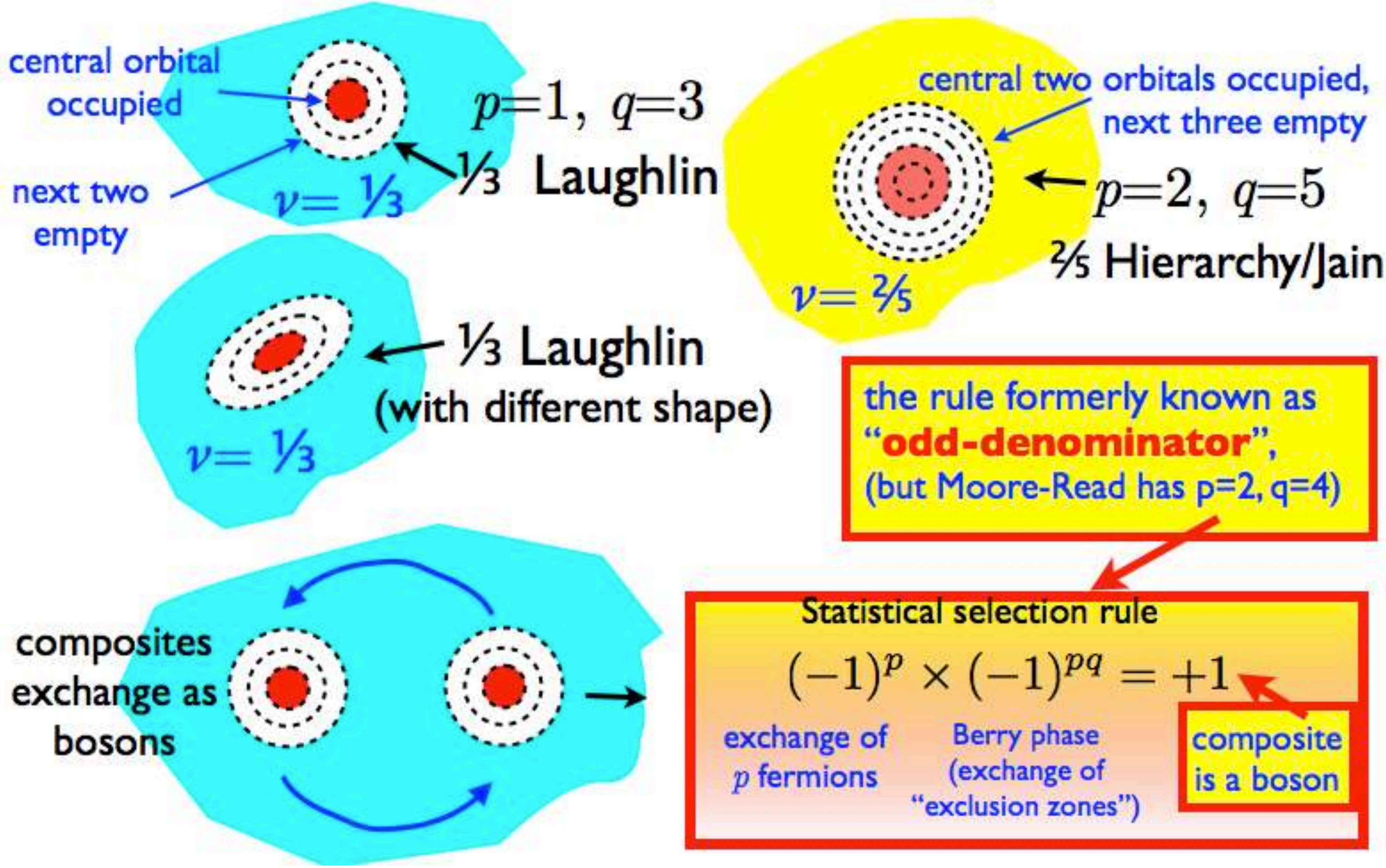
composite boson density

$$J^0 = \frac{1}{2\pi pq\hbar}(peB - \hbar s J_g^0)$$

Gaussian curvature of metric

- New feature is similar to FQH ferromagnet, where electrons couple to a combination of magnetic flux and Berry curvature of the ferromagnetic order parameter(Skyrmions)
- The electron density is no longer rigidly tied to the magnetic flux density, it can deviate from it at the expense of paying the correlation energy cost for geometric distortion.
- Old results of Girvin, Macdonald and Platzman (O(q^4) "guiding-center structure factor") get a simple explanation as zero-point fluctuations of the geometry

- elementary unit of the FQHE fluid with $\nu = p/q$ is a "**composite boson**" of p electrons that <u>**exclude**</u> other electrons from a region with q London (h/e) flux quanta



"unimodular" just means $\det|g| = 1$

- The **shape** of the "composite boson" is a dynamical local variable defined by a **unimodular 2D spatial metric** $g_{ab}(\boldsymbol{r},t)$

- The **orbitals** inside the "composite boson" centered at $\boldsymbol{r}$ are defined by $L(\boldsymbol{r})|\Psi_m(\boldsymbol{r})\rangle = (m + \frac{1}{2})|\Psi_m(\boldsymbol{r})\rangle$

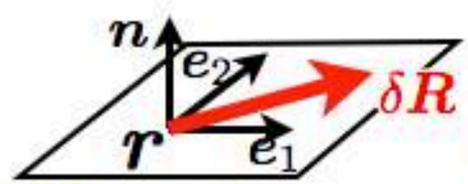
L includes "zero-point" contribution

metric defines rotation operator

$$L(\boldsymbol{r}) = \frac{g_{ab}(\boldsymbol{r})}{2\ell_B^2}\delta R^a \delta R^b$$

$\frac{\hbar}{|eB|}$

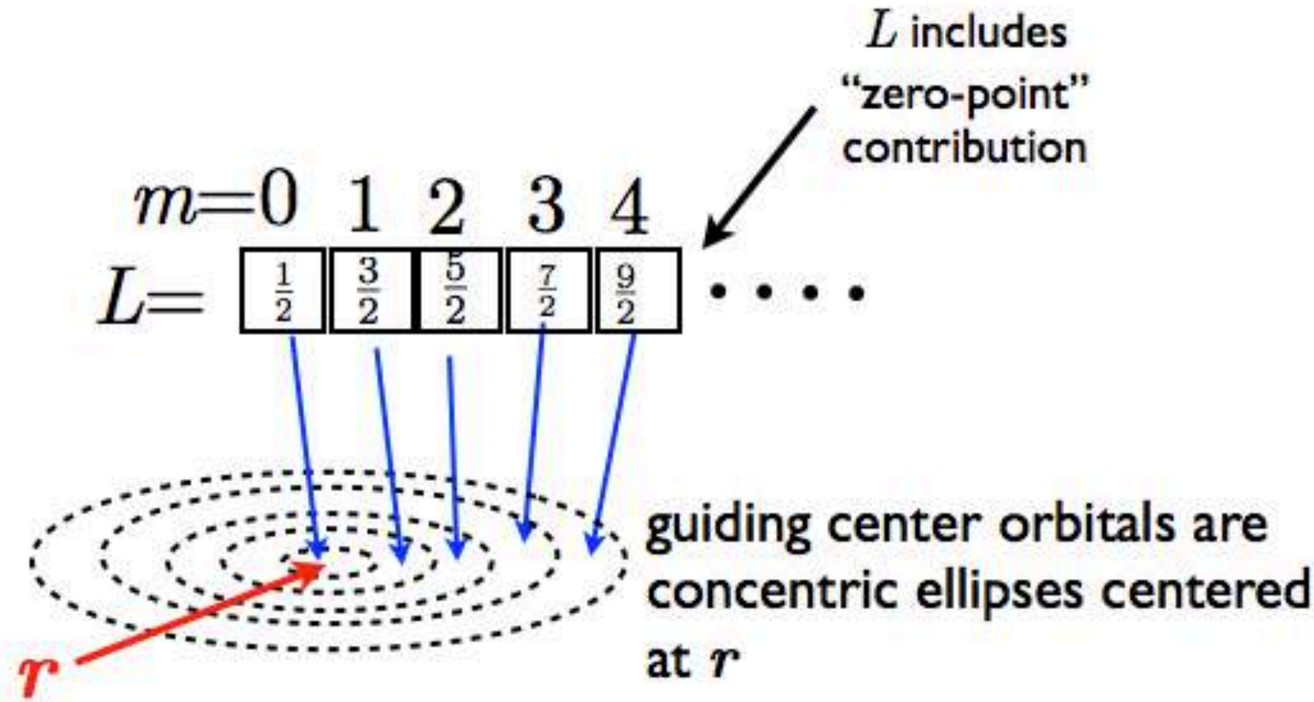
- The **non-commuting** "Landau-orbit guiding center coordinates" obey $[\delta R^a, \delta R^b] = -i\ell_B^2\epsilon^{ab}$



$\delta \boldsymbol{R} = \delta R^a \boldsymbol{e}_a$

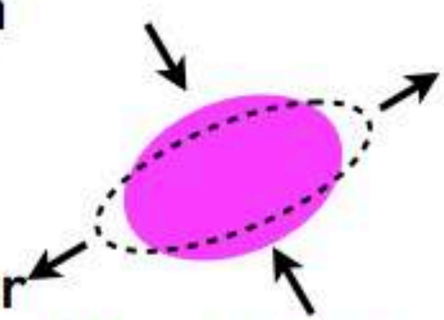
$\delta \boldsymbol{R}$ is the displacement of the "guiding center" from $\boldsymbol{r}$

$$L(\boldsymbol{r}) = \frac{g_{ab}(\boldsymbol{r})}{2\ell_B^2}\delta R^a \delta R^b$$



Shape of ellipse is defined by the metric

- The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape



$\delta E \propto (\text{distortion})^2$

- The zero-point fluctuations of the metric are seen as the $O(q^4)$ behavior of the "guiding-center structure factor" (Girvin et al, (GMP), 1985)

- The metric has a companion "guiding center spin" that is topologically quantized in incompressible states.

⅓ Laughlin

configuration of "elementary droplet" (composite boson)

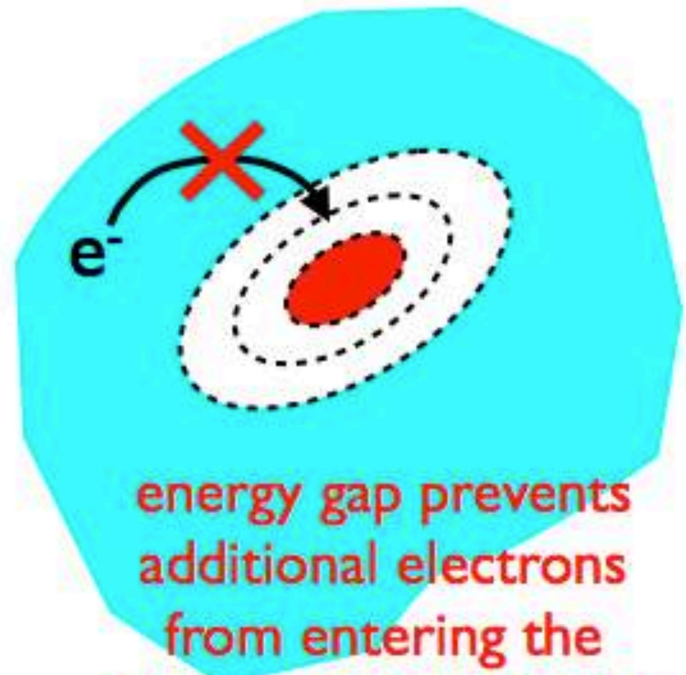
subtract total L (=L_{ref}) of reference configuration (uniform occupation p/q)

$L=$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	total L
	1	0	0	$\frac{1}{2}$
	⅓	⅓	⅓	$\frac{3}{2}$

$$s = (\tfrac{1}{2} - \tfrac{3}{2}) = -1$$

- Origin of FQHE incompressibility is analogous to origin of **Mott-Hubbard gap** in lattice systems.
- There is an energy gap for putting an **extra particle** in a quantized region that is **already occupied**

- **On the lattice** the "quantized region" is an atomic orbital with a fixed shape
- **In the FQHE** only the **area** of the "quantized region" is fixed. The **shape** must adjust to minimize the correlation energy.



energy gap prevents additional electrons from entering the region covered by the composite boson

- Geometric action

(after Chern-Simons fields are integrated over)

electromagnetic gauge potentials

spin connection of metric

$$S = \int d^3x \mathcal{L}_0 - \mathcal{H}_0$$

$$\mathcal{L}_0 = \frac{1}{4\pi pq\hbar}\epsilon^{\mu\nu\lambda}\left(peA_\mu - s\Omega^g_\mu\right)\partial_\nu\left(peA_\lambda - \hbar s\Omega^g_\lambda\right)$$

(reduces to electromagnetic Chern-Simons action when s = 0 (integer QHE))

$$\mathcal{H}_0 = J^0 U(J^0 g) \qquad J^0 = \frac{1}{2\pi pq\hbar}\left(peB - \hbar s J^0_g\right)$$

energy function

Gaussian curvature

$$J^\mu_g = \epsilon^{\mu\nu\lambda}\partial_\nu\Omega^g_\lambda$$

correlation energy density

composite-boson density

- Action gives gapped spin-2 (graviton-like) collective mode that coincides at long wavelengths with the "single-mode approximation" of Girvin-MacDonald and Platzman.

- charge fluctuations relative to the back-ground charge density fixed by the magnetic flux are given by the Gaussian curvature

$$J_g^0 = -\tfrac{1}{2}\partial_a\partial_b g^{ab} + \tfrac{1}{8} g_{ac}\epsilon_{bd}\epsilon^{ef}\left(\partial_e g^{ab}\right)\left(\partial_f g^{cd}\right)$$

$$\delta J_e^0 = \frac{e^* s}{2\pi} J_g^0$$

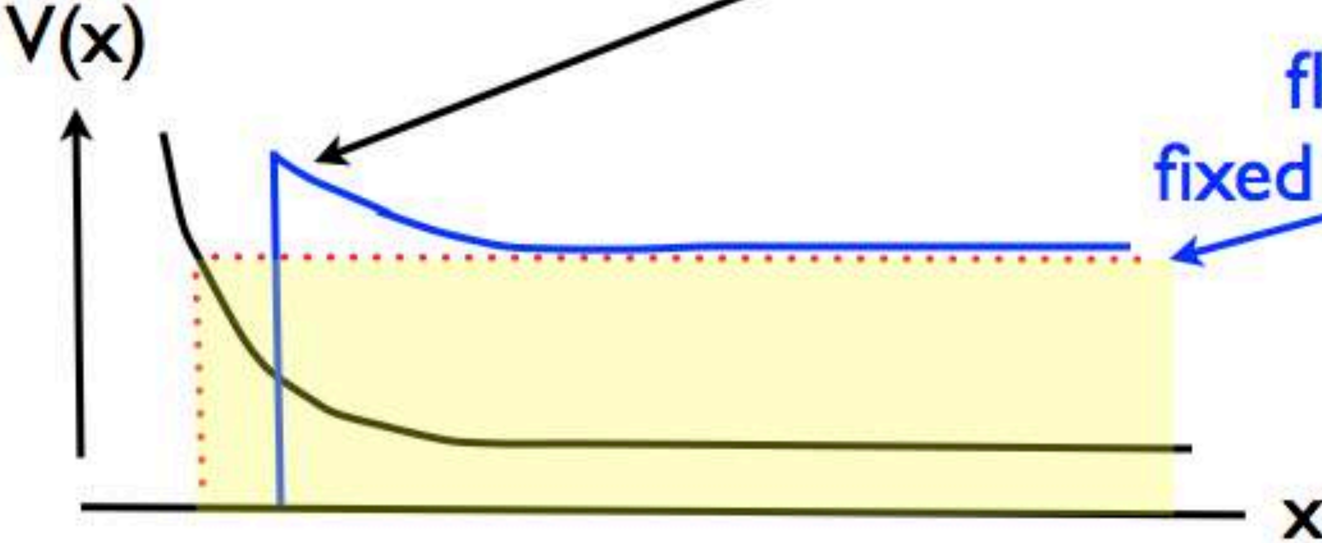
second derivative of metric

zero-point fluctuations of gaussian curvature give quantitatively correct O(q^4) structure factor

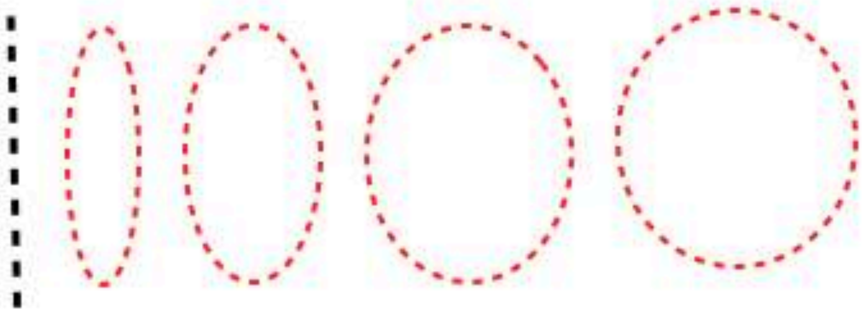
- near edges:

fluid is compressed at edges by creating Gaussian curvature

$$\delta J^0_e = \frac{e^* s}{2\pi} J^0_g$$



fluid density fixed by flux density



$$g = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \frac{1}{\alpha(x)} \end{pmatrix}$$

$$J^0_g = -\frac{1}{2} \frac{d^2}{dx^2} \frac{1}{\alpha(x)}$$

For larger s, fluid becomes more compressible (less distortion needed for a given density change)

- Previous theoretical treatments of FQHE have missed the fundamental geometrical degree of freedom associated with incompressibility
- Indeed, **no** viable theoretical explanation of FQHE incompressibility was previously found. By focussing on topological quantum field theory which ASSUMES the (unexplained) existence of incompressibility, most theorists have avoided the issue.
- In fact, the geometric degree of freedom can already be found in the Laughlin wavefunction, but deeply hidden during 30 years of its misinterpretation. HOW DID THIS HAPPEN?

The "Laughlin wavefunction"

- originally proposed as a "lowest-Landau-level Schrödinger wavefunction" to explain the 1/3 FQHE state

$$\Psi_L^{1/q}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2} z_i^* z_i}$$

$$z_i = \frac{x_i + iy_i}{\sqrt{2}\ell_B} \qquad \ell_B = \left(\frac{\hbar}{|eB|}\right)^{1/2}$$ "magnetic length"

- physical significance of z : n'th Landau orbit has (semiclassical) shape $|z|^2 = n + \frac{1}{2}$

- Landau level raising- and lowering-operators:

$$H = \frac{1}{2m}\left(\pi_x^2 + \pi_y^2\right)$$

↓

$$H = \tfrac{1}{2}\hbar\omega_c\left(a^\dagger a + a a^\dagger\right)$$

$$\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r})$$

$$[\pi_x, \pi_y] = i\hbar e B^z$$

$$a = \frac{(\pi_x + i\pi_y)\ell_B}{\sqrt{2\hbar}}$$

$$[a, a^\dagger] = 1$$

- Schrödinger representation:

$$a = \tfrac{1}{2}z + \partial_{z^*}$$

$$a^\dagger = \tfrac{1}{2}z^* - \partial_z$$

$$\partial_z f(z, z^*) \equiv \frac{\partial}{\partial z} f(z, z^*)\Big|_{z^*}$$

- holomorphic lowest Landau-Level wavefunctions:

$$a\Psi(z, z^*) = \left(\tfrac{1}{2}z + \partial_{z^*}\right)\Psi(z, z^*) = 0$$

- solution :

$$\Psi(z, z^*) = f(z)e^{-\frac{1}{2}z^* z}$$

single-particle LLL wavefunction

↑ holomorphic function

- Physicists seem to have been "mesmerized" by the mathematical beauty of the LLL wavefunctions.........

- The Vandermonde determinant provides a natural "precursor" to the Laughlin state

$$\Psi = \prod_{i<j}(z_i - z_j)\prod_i e^{-\frac{1}{2}z_i^* z_i}$$ Filled N-particle LLL droplet

Laughlin wavefunction

Laughlin's "little" modification!

$$\Psi_L = \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

- In 1983, while "mainstream" Condensed Matter Physicists (like me) were struggling with fancy second-quantized formalisms to try to understand the FQHE, Bob Laughlin arrived from on high carrying the solution......

- 25 years after BCS theory of superconductivity, which had consigned Schrödinger wavefunctions to the "dustbin of (Condensed Matter theory) history", they were back!

- **And** they were providing insight **unimaginable** in second-quantization/Feynman-diagram formulations!

- In retrospect, Laughlin's solution seemed so evidently correct and complete that it appears to have **frozen** the development of any understanding of the origin of FQHE incompressibility for almost 30 years.

- The Laughlin wavefunction for the strongest (1/3) FQHE state works, but why?

- In the attempt to "explain" **<u>why</u>** the Laughlin wavefunction works, theorists have been reduced to mouthing nice-sounding (but meaningless) platitudes such as

"The holomorphic wavefunction cleverly puts its zeroes on top of the other particles to lower the Coulomb energy"

Various attempts to explain FQHE incompressibility:

- Ginzburg-Landau superfluidity with a Higgs-like effect due to a Chern-Simons term
- Filling of "effective Landau levels" by "composite fermions"
- "Hamiltonian theory" of composite fermions
- non-commutative Chern-Simons theory with diffeomorphism invariance

- There is no doubt that **Chern-Simons** theories as Topological Quantum Field theories capture the essential **topological** features of FQHE liquids, but TQFT provides NO information about **energy gaps** and **incompressibility**: because ......

TQFT has an effective Hamiltonian: H=0!

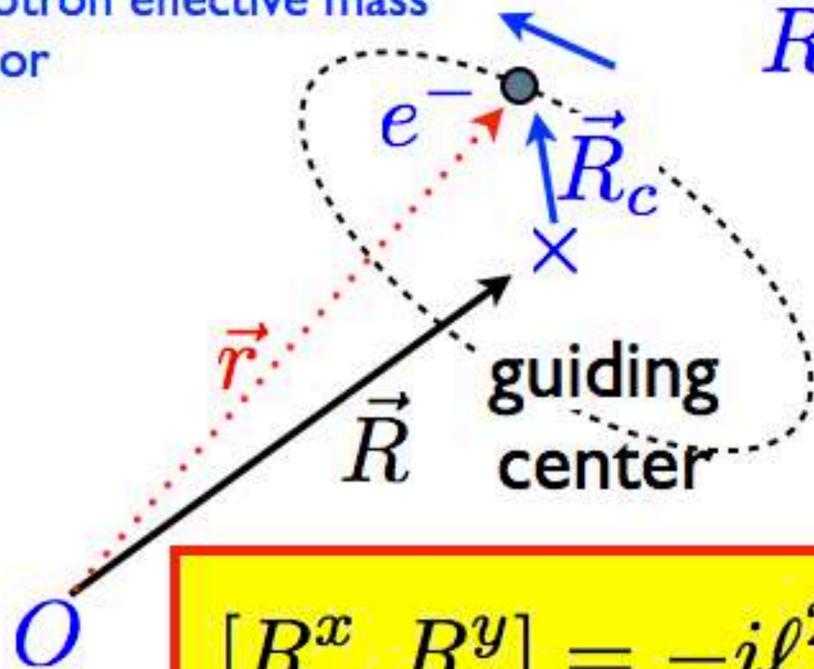
- None of these essentially “verbal” explanations has provided a viable quantitative theory of FQHE incompressibility.
- They attempt in various ways to provide an after-the-fact explanation of the success of the Laughlin wavefunction

Why all these attempts fail*, is because they did not incorporate what turns out to have been the **only** fundamental post-Laughlin result on FQHE incompressibility, the 1985 results on the “guiding-center structure factor” of Girvin, MacDonald, and Platzman

*The CF picture due to Jain provides good model wavefunctions and a description of the compressible Fermi-liquid-like state at $\nu = 1/2$ but no non-verbal explanation of incompressibility.

- Non-commutative geometry of Landau-orbit guiding centers

shape of orbit around guiding center is fixed by the cyclotron effective mass tensor



$\vec{r}$ displacement of electron from origin

$\vec{R}$ displacement of guiding center from origin

$\vec{R}_c$ displacement of electron relative to guiding center of Landau orbit

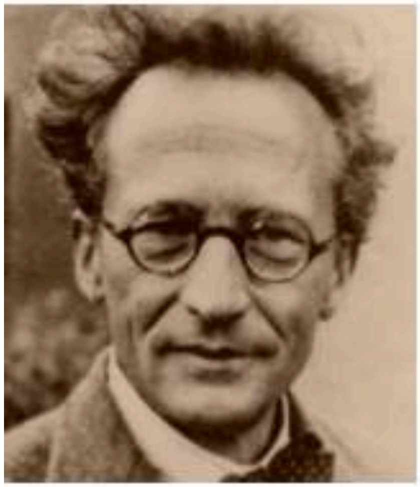
$$\vec{r} = \vec{R} + \vec{R}_c$$

$$[r^x, r^y] = 0$$
classical geometry

$$[R^x, R^y] = -i\ell_B^2$$
quantum geometry

$$[R_c^x, R_c^y] = +i\ell_B^2$$
Landau orbit (harmonic oscillator)

guiding centers commute with Landau radii $[R^a, R_c^b] = 0 \quad (a, b \in \{x, y\})$



Schrödinger vs Heisenberg

- Schrödinger's picture describes the system by a **wavefunction** $\psi(r)$ in real space
- Heisenberg's picture describes the system by a **state** $|\psi\rangle$ in Hilbert space
- They are only equivalent if the basis $|r\rangle$ of states in real-space are orthogonal:

$$\psi(\boldsymbol{r}) = \langle \boldsymbol{r}|\psi\rangle$$

requires

$$\langle \boldsymbol{r}|\boldsymbol{r}'\rangle = 0 \quad (\boldsymbol{r} \neq \boldsymbol{r}')$$

← this fails in a quantum geometry

classical electron coordinate $\vec{r} = \vec{R} + \vec{R}_c$

The one-particle Hilbert-space factorizes

$$\mathscr{H} = \bar{\mathscr{H}}_{\mathrm{GC}} \otimes \mathscr{H}_c$$

space isomorphic to phase space in which the guiding-centers act

$$[R^x, R^y] = -i\ell_B^2$$

space isomorphic to phase space in which the Landau orbit radii act

$$[R_c^x, R_c^y] = +i\ell_B^2$$

- FQHE physics is *COMPLETELY* defined in the many-particle generalization (coproduct) of $\mathscr{H}_{\mathrm{GC}}$

Once $\mathscr{H}_{\mathrm{C}}$ is discarded, the Schrödinger picture is no longer valid!

Q: **When is a "wavefunction" NOT a wavefunction?**

A: **When it describes a "quantum geometry"**

- In this case space is "fuzzy"(non-commuting components of the coordinates), and the Schrödinger description in real space (i.e., in "classical geometry") fails, though the Heisenberg description in Hilbert space survives
- The closest description to the classical-geometry Schrödinger description is in a non-orthogonal overcomplete **coherent-state** basis of the quantum geometry.

Previous hints that the Laughlin "wavefunction" should not be interpreted as a wavefunction:

- Laughlin states also occur in the second Landau level, and in graphene, and more recently in simulations of "flat-band" Chern insulators

These don't fit into the original paradigm of the Galileian-invariant Landau level

- First, translate Laughlin to the Heisenberg picture:

$$a^\dagger = \tfrac{1}{2}z^* - \partial_z$$
$$a = \tfrac{1}{2}z + \partial_{z^*}$$

Landau-level ladder operators

$$z \leftrightarrow \bar{z}$$

$$\bar{a}^\dagger = \tfrac{1}{2}\bar{z}^* - \partial_{\bar{z}}$$
$$\bar{a} = \tfrac{1}{2}\bar{z} + \partial_{\bar{z}^*}$$

Guiding-center ladder operators

usual identification is
$$\bar{z} = z^*$$

Gaussian lowest-weight state
$$\psi_0(z, z^*) = e^{-\frac{1}{2}z^* z}$$

$$\bar{a}\psi_0(z, z^*) = 0$$

$$a\psi_0(z, z^*) = 0$$

action of guiding-center raising operators on LLL states

$$\bar{a}^\dagger = \tfrac{1}{2}z - \partial_{z^*} \qquad \bar{a} = \tfrac{1}{2}z^* + \partial_z$$

$$\bar{a}^\dagger f(z)\Psi_0(z, z^*) = z f(z)\Psi_0(z, z^*)$$

- Heisenberg form of Laughlin state (not "wavefunction")

$$\bar{a}_i|\bar{\Psi}_0\rangle = 0 \qquad a_i|\Psi_0\rangle = 0$$

$$|\Psi_L^{1/q}\rangle = \left(\prod_{i<j} (\bar{a}_i^\dagger - \bar{a}_j^\dagger)^q |\bar{\Psi}_0\rangle \right) \otimes (|\Psi_0\rangle)$$

$$\in \bar{\mathscr{H}}_{\mathrm{GC}} \equiv \bar{\mathscr{H}} \qquad \in \mathscr{H}_c \equiv \mathscr{H}$$

Guiding-center factor (keep)

Landau-orbit factor (discard)

- At this point we discard the Landau-orbit Hilbert space.
- The only "memory" of the shape of the Landau orbits is "hidden" in the definition of $\bar{a}$

The "purified" Laughlin state

$$|\Psi_L^{1/q}\rangle = \prod_{i<j}(\bar{a}_i^\dagger - \bar{a}_j^\dagger)^q|\bar{\Psi}_0\rangle \qquad \bar{a}_i|\bar{\Psi}_0\rangle = 0$$

- This is now defined in the many-particle guiding-center Hilbert space, without reference to any Landau-level structure
- What defines $\bar{a}_i^\dagger$?

$$[L(g), \bar{a}_i^\dagger(g)] = \bar{a}_i^\dagger(g)$$

$$L(g) = \frac{g_{ab}}{2\ell_B^2}\sum_i R_i^a R_i^b$$

It is the raising operator for the "guiding-center spin" $L(g)$ of particle i

g_{ab} **is a 2x2 positive-definite unimodular (det = 1) 2D spatial metric tensor**

- The Laughlin state has suddenly revealed its well-kept secret- a hidden geometric degree of freedom! It is parameterized by a unimodular metric g_{ab}!

$$|\Psi_L^{1/q}(g)\rangle = \prod_{i<j}(\bar{a}_i^\dagger(g) - \bar{a}_j^\dagger(g))^q|\bar{\Psi}_0(g)\rangle$$

$$\bar{a}_i(g)|\bar{\Psi}_0(g)\rangle = 0$$

- In the naive LLL wavefunction picture, the unimodular metric g_{ab} is fixed to be proportional to the cyclotron effective mass tensor $m^*{}_{ab}$.
- In the reinterpretation it is a **free parameter**.

- As a variational parameter of the Laughlin state, g_{ab} must be chosen to minimize the correlation energy.
- Unless there is rotational symmetry around the normal to the 2D "Hall surface", it will not be congruent to the cyclotron effective mass tensor, and $\bar{z} \neq z^*$

$$\begin{pmatrix} \bar{z} \\ \bar{z}^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} z^* \\ z \end{pmatrix}$$

$$\alpha^* \alpha - \beta^* \beta = 1$$

A Bogoliubov transformation

- If we reconstruct a wavefunction in the full Hilbert space, the naive statement about the zeroes of the wavefunction (as a function of any one particle coordinate) coinciding with the positions of the other particles is only true if

$$\bar{z} = z^*$$

- guiding-center Coherent states (single particle)

$$\bar{a}(g)|\Psi_g(0)\rangle = 0$$

$$|\Psi_g(\bar{z})\rangle = e^{\bar{z}a^\dagger(g) - \bar{z}^* a(g)}|\Psi_g(0)\rangle$$

- This is a non-orthogonal overcomplete basis

$$S(\bar{z}, \bar{z}') = \langle \Psi_g(\bar{z})|\Psi_g(\bar{z}')\rangle$$

- non-zero eigenvalues of the positive Hermitian overlap function are holomorphic!

$$\int \frac{d\bar{z}' d\bar{z}'^*}{2\pi} S(\bar{z}, \bar{z}')\Psi(\bar{z}', \bar{z}'^*) = \Psi(\bar{z}, \bar{z}^*)$$

$$\Psi(\bar{z}, \bar{z}^*) = f(\bar{z}^*) e^{-\frac{1}{2}\bar{z}^* \bar{z}}$$

- The "wavefunction" reappears as a guiding-center coherent-state representation of the Laughlin state!

$$|\Psi_L(g)\rangle = \prod_i \int \frac{d^2 \boldsymbol{r}_i}{2\pi \ell_B^2} \left(\prod_{i<j} (\bar{z}_i^* - \bar{z}_j^*)^q \prod_i e^{-\frac{1}{2}\bar{z}_i^* \bar{z}_i} \right) |\Psi_g(\{\boldsymbol{r}_i\})\rangle$$

same as "Laughlin wavefunction", but with $z_i \to \bar{z}_i^*$

Many-particle coherent state, parametrized by g

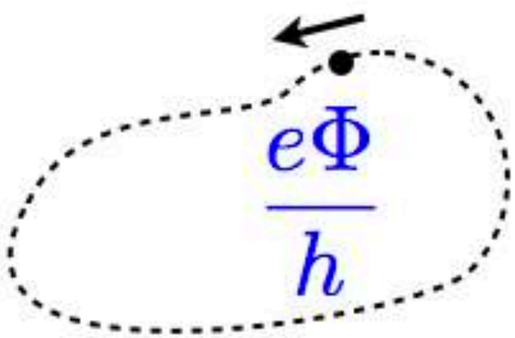
- The "wavefunction" has reappeared, but is not a wavefunction! This will apply in general to model states (Moore-Read, etc,obtained as CFT correlators)

Quantum Geometry

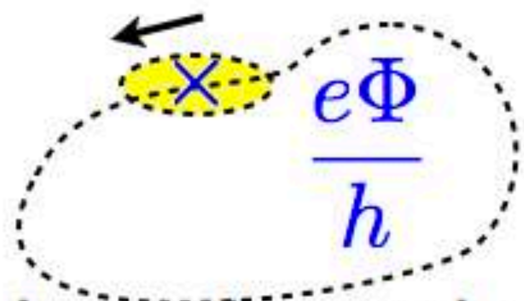
- A much more powerful result follows from the realization that the metric g_{ab} is not just a "variational parameter of the Laughlin state" but the fundamental dynamical degree of freedom of the FQHE.
- Once the connection to quantum geometry is made many things follow.

"Bohm-Aharonov becomes Berry"

- How does a "fuzzy electron" experience a Bohm-Aharonov phase?



point particle going around a loop picks up the Bohm-Aharonov phase

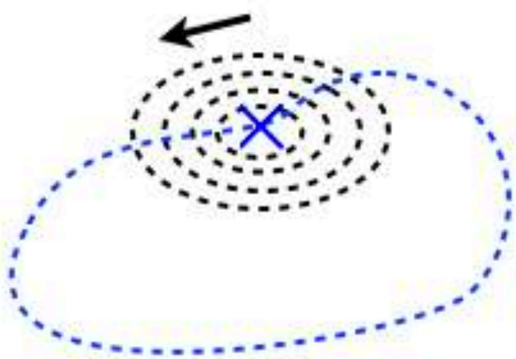


"fuzzy" charged particle is transported as a guiding-center coherent state with a center that traces out a closed path

Berry phase = BA phase of path

- The state that is transported doesn't have to be the central coherent state of the basis

$$L(g)|\Psi_{g,m}(0)\rangle = s_m|\Psi_{g,m}(0)\rangle$$
$$s_m = m + \tfrac{1}{2}, \quad m = 0, 1, 2, \ldots$$



Provided the coherent-state metric g_{ab} does not change during the adiabatic transportation process, the Berry phase equals the BA phase of the path, independent of m

If the metric does vary, the Berry phase turns out to be

$$\frac{e\Phi_B}{\hbar} + s_m \times (\text{geodesic curvature of metric } g_{ab} \text{ around path})$$

Which leads directly to the guiding-center spin coupling to Gaussian curvature!

SUMMARY

- New collective geometric degree of freedom leads to a description of the origin of incompressibility in FQHE in a continuum "geometric field theory"

- many new relations: guiding-center spin characterizes coupling to Gaussian curvature of intrinsic metric, stress in fluid, guiding-center structure-factors, etc.

http://wwwphy.princeton.edu/~haldane

Can be also be accessed through Princeton University Physics Dept home page
(look for Research:condensed matter theory)

also see arXiv (search for author=haldane)