Dynamical stability of the quantum Lifshitz theory in $2+1$ Dimensions

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Motivation

- There is much interesting in understanding the relation between topological and ordered phases in condensed matter
- Although significant progress has been made in identifying "realistic" models that exhibit topological phases, at present it is still difficult to study the nature of quantum phase transitions into these states
- For these reasons there much interest in simple models with local Hamiltonians that give rise to topological phases
- The simplest examples of such local models are the quantum dimer model and its generalizations.
- Much of the work in the field has focused, justifiably so, in the study of the nature of these phases and, aside for inquiring if these phases are massive or note, relatively little attention is paid to the quantum dynamics implied in these Hamiltonians
- In this talk I will discuss my recent work with Benjamin Hsu on the problem of the stability of the quantum dynamics of the quantum Lifshitz model, the effective field theory of the quantum dimer models and its generalizations.

Outline

- Ordered Phases, Topological Phases and Quantum Criticality in 2D Generalized Quantum Dimer Models
- "Ideal wave functions", topological phases and quantum criticality
- Conformal Quantum Critical Points in $2+1$ dimensions: Quantum Lifshitz Model
- Stability of the quantum Lifshitz model and quantum (multi) criticality
- RG analysis of the perturbed quantum Lifshitz model is $z=2$?
- Consequences of the RG results

Quantum Dimer Models

- Simple local models describing strongly frustrated and ring exchange quantum spin systems with a large spin gap and no long range spin order
- They typically exhibit spin gap phases with different types of valence bond crystal orders
- QDM have special solvable points, the Rokhsar-Kivelson (RK) point, where the exact ground state wave function has the short range RVB form

$$|\Psi_{\rm RVB}\rangle = \sum_{\{\mathcal{C}\}} |\mathcal{C}\rangle, \qquad \{\mathcal{C}\} = \text{ all dimer coverings of the lattice}$$

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  - Bipartite lattices: the RK points are quantum (multi) critical points described by an effective field theory with $z = 2$ and massless deconfined spinons (Kivelson and Rokhsar; Fradkin and Kivelson)
  - Non-bipartite lattices: QDMs have topological $\mathbb{Z}_2$ deconfined phases with massive spinons and a topological 4-fold ground state degeneracy on a torus (Moessner and Sondhi, 1998)

$$H_{\mathrm{RK}} = \sum_i (v\, V_i - J\, F_i), \qquad \text{Rokhsar and Kivelson (1988)}$$

$$V_i = |\,=\,\rangle\langle\,=\,| + |\,\Vert\,\rangle\langle\,\Vert\,| \qquad F_i = |\,=\,\rangle\langle\,\Vert\,| + |\,\Vert\,\rangle\langle\,=\,|$$

Here each bar represents a spin singlet bond = dimer, and i runs over the plaquettes of the lattice.
For $J = v$

$$H_{\mathrm{RK}} = \sum_i Q_i^\dagger Q_i, \qquad Q_i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- The ground state wave function $|\Psi_0\rangle$ has $E = 0$

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z_{\mathrm{cl}}}} \sum_{\mathcal{C}} |\mathcal{C}\rangle \ ,$$

  where Z_{cl} is the sum over all dimer configurations

$$Z_{\mathrm{cl}} = \sum_{\mathcal{C}} 1 = \text{classical dimer partition function}$$

- Equal-*time* correlators in the quantum dimer model at the RK point are given by correlators of the classical dimer model.

Generalizations

Local Hamiltonians that for a choice of parameters take the RK form with wave functions with local "Gibbs" weights

$$|\Psi\rangle = \sum_{\mathcal{C}} w[\mathcal{C}] |\mathcal{C}\rangle$$

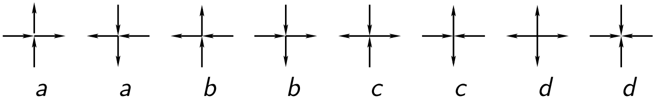
where $w[\mathcal{C}]$ is a local amplitude, *e.g.*

$$w[\mathcal{C}] = \frac{1}{\sqrt{Z_{\rm cl}}} e^{-\frac{u}{2} N_{\|}[\mathcal{C}]}, \qquad Z_{\rm cl} = \sum_{\mathcal{C}} e^{-u N_{\|}[\mathcal{C}]}$$

where $N_{\|}[\mathcal{C}]$ is the number of parallel dimers on all the elementary plaquettes, in configuration $\mathcal{C}$, and u can be interpreted as an inverse temperature of a classical dimer model with attractive dimer interactions ($u > 0$) (Alet et al, Castelnuovo et al, Papanikolaou et al, 2007)

Quantum Eight-Vertex Model

- An interesting model is a quantum eight vertex model (Ardonne et al 2004)
- Its degrees of freedom are arrows placed on a square lattice. Only an even number of arrows can converge or diverge from a lattice site.This defines the configuration space $\{\mathcal{C}\}$.



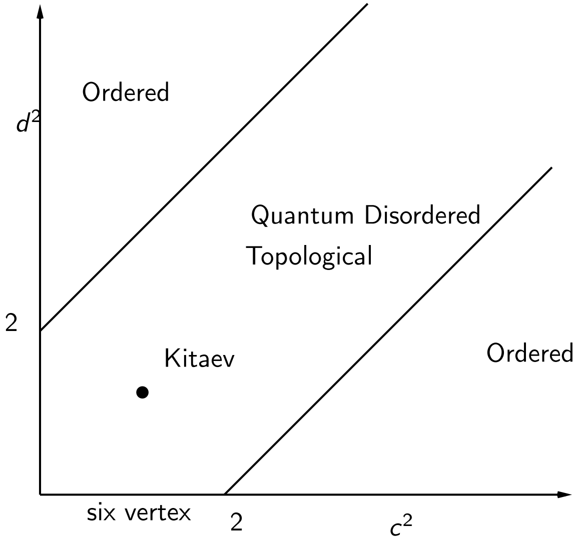
- Ardonne et al (2004) constructed a local Hamiltonian with an RK condition whose ground state wave function is

$$|\Psi\rangle = \sum_{\mathcal{C}} a^{N_a[\mathcal{C}]} b^{N_b[\mathcal{C}]} c^{N_c[\mathcal{C}]} d^{N_d[\mathcal{C}]} |\mathcal{C}\rangle$$

$$||\Psi||^2 = Z = \sum_{\mathcal{C}} a^{2N_a[\mathcal{C}]} b^{2N_b[\mathcal{C}]} c^{2N_c[\mathcal{C}]} d^{2N_d[\mathcal{C}]}$$

which is the partition function of the 2D classical eight vertex (Baxter) model

Phase Diagram



- For $a = b = 1$ it has two ordered phases
- A quantum disordered topological phase
- $c = d = 1$: wave function of the the Kitaev Toric Code state
- The disordered phase is a $\mathbb{Z}_2$ topological fluid with 4-fold degeneracy on the torus.
- Entanglement entropy:

$$S = \alpha L - \gamma_{topo}, \qquad \gamma_{topo} = \ln 2$$

  $\gamma_{topo} = \ln 2$ is constant in the phase (Papanikolaou et al (2008)).
- Phase boundaries: lines of fixed points with continuously varying exponents and Kosterlitz-Thouless multi-critical point: $c^2 = 2$, $d = 0$ (Kadanoff, 1979)
- Along the $d = 0$ ("six vertex") line it has a local conservation law ("charge") which is broken down to $\mathbb{Z}_2$ ("charge parity") for $d > 0$.

Effective field theory: the Quantum Lifshitz Model

Moessner, Sondhi and Fradkin (2001); Henley (1998); Ardonne, Fendley and Fradkin(2004)

- QDM on a square lattice $\Leftrightarrow$ 2D height model
- Coarse-grained slowly varying-height field $\varphi(x)$
- Physical Operators are invariant under $\varphi(x) \to \varphi(x) + 2\pi$
- Quantum Lifshitz Model Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 \left(\nabla^2\varphi\right)^2 \right]$$

  k varies along the fixed lines: marginal operator
- The Ground State Wave Function $\Psi_0[\varphi]$ is scale invariant!

$$\Psi_0[\varphi] \propto e^{-\frac{k}{8\pi}\int d^2x \, \left(\nabla\varphi(\mathbf{x})\right)^2}$$

- $||\Psi_0||^2$ is the partition function of a classical critical conformally invariant system!

$$||\Psi_0||^2 = Z = \int \mathcal{D}\varphi \; e^{-\frac{k}{4\pi}\int d^2x \, (\nabla\varphi(\mathbf{x}))^2}$$

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions
- Euclidean action

$$S = \int d^2x dt \Big[\frac{1}{2}\left(\partial_t \varphi\right)^2 + \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 \left(\nabla^2 \varphi\right)^2\Big]$$

- Same as the free energy for the classical Lifshitz point (Grinstein 1982)!
- Equal-time expectation value of operators are correlators of the massless free boson CFT with central charge $c = 1$.
- Time-dependent correlators: dynamical exponent $z = 2$.
- Matching the correlation functions of the RK QDM and Lifshitz models, one finds $k = 1$
- For the 2D quantum Baxter wave function k varies continuously along the six vertex line
- Entanglement entropy: (Hsu et al (2009, 2010), Stephan et al (2010), Oshikawa (2010)

$$S = \alpha L + \gamma_c, \qquad \gamma_c = \ln R - 1/2, \qquad R = \sqrt{2k}$$

Stability of the Quantum Lifshitz Model

We need to address several questions

- The quantum Lifshitz model describes a line of multi-critical points
- In this sense is a fine-tuned system
- The lattice models are tuned to the RK condition: what is the effect of violations of the RK condition?
- What are the physical perturbations of the effective field theory and what effects they have?
- Is the prediction of $z = 2$ dynamics stable even if the RK constraints are satisfied?
- Recent Monte-Carlo simulations by Isakov, Fendley, Ludwig, Trebst and Troyer (PRB, 2011) found that for the Baxter case $z > 2$ (and varies continuously and non-montonically) along the phase boundary with $d > 0$ but $z = 2$ along the six vertex line
- Can z vary continuosly?

Intermezzo: Classical and Quantum Dynamics

- Microscopic Hamiltonians that satisfy the RK condition are positive semi-definite
- Their ground state wave functions have zero energy
- The amplitudes look like Gibbs weights of an equilibrium statistical mechanical system
- These Hamiltonians can be mapped to the Liouville operators that describe the relaxational dynamics to equilibrium of these classical system. (Henley 1997, Moessner, Sondhi and Fradkin (2001), Castelnuovo, Mudry, Chamon and Pujol (2007)).
- Provided the RK condition is strictly enforced the 2D quantum time evolution (in imaginary time) can be determined from the relaxational dynamics of the classical system
- However the 2D quantum dynamics of perturbed Hamiltonians that do not respect the RK condition generally do not obey the 2D classical dynamics
- There is only so much a wave function alone can do: perturbing the Hamiltonian is not the same as perturbing the wave function.

Intermezzo: Master Equation and RK Hamiltonians

Master Equation for the classical probabilities $p[\mathcal{C}, t]$:

$$\partial_t p[\mathcal{C}, t] = \sum_{\mathcal{C}'} W[\mathcal{C}, \mathcal{C}']\, p[\mathcal{C}', t]$$

$$w[\mathcal{C}] W[\mathcal{C}', \mathcal{C}] = W[\mathcal{C}, \mathcal{C}'] w[\mathcal{C}'], \qquad \text{(detailed balance)}$$

$$\Rightarrow \lim_{t \to +\infty} p[\mathcal{C}, t] = w[\mathcal{C}]$$

Upon a rescaling of the transition rates

$$\tilde{W}[\mathcal{C}, \mathcal{C}'] = w[\mathcal{C}]^{-1/2} W[\mathcal{C}, \mathcal{C}'] w[\mathcal{C}']^{1/2}, \quad \tilde{p}[\mathcal{C}, t] = w[\mathcal{C}]^{-1/2} p[\mathcal{C}, t]$$

$$\partial_t \tilde{p}[\mathcal{C}, t] = \sum_{\mathcal{C}'} \tilde{W}[\mathcal{C}, \mathcal{C}']\, \tilde{p}[\mathcal{C}', t]$$

We can identify the RK Hamiltonian and wave function

$$\Rightarrow H[\mathcal{C}, \mathcal{C}'] = -\tilde{W}[\mathcal{C}, \mathcal{C}'], \quad \text{and} \quad |\Psi(t)\rangle = \sum_{\mathcal{C}} \tilde{p}[\mathcal{C}, t]\, |\mathcal{C}\rangle$$

$$\lim_{t \to +\infty} \tilde{p}[\mathcal{C}, t] = w[\mathcal{C}]^{1/2}, \quad |\Psi_0\rangle = \sum_{\mathcal{C}} w[\mathcal{C}]^{1/2} |\mathcal{C}\rangle, \quad H|\Psi_0\rangle = 0$$

Observables of the Quantum Lifshitz Model

$$\langle \mathrm{vac}|\mathcal{O}[\varphi(\vec{x}_1)]\ldots\mathcal{O}[\varphi(\vec{x}_n)]|\mathrm{vac}\rangle =$$
$$\int [D\varphi]\ \mathcal{O}[\varphi(\vec{x}_1)]\ldots\mathcal{O}[\varphi(\vec{x}_n]\ e^{-\frac{k}{4\pi}\int d^2\vec{x}\ (\nabla\varphi)^2}$$

Charge operators:

$$\mathcal{O}_q[\varphi(x)] = e^{iq\varphi(x)}, \qquad q \in \mathbb{Z}$$

Magnetic operators (vortices):

$$\tilde{\mathcal{O}}_m[\varphi(\vec{x})] = e^{i2m\int d^2z\, \arg(\vec{x}-\vec{z})\Pi(\vec{z})}$$

$$\langle \tilde{\mathcal{O}}_{m_1}(\vec{x}_1)\ldots\tilde{\mathcal{O}}_{m_n}(\vec{x}_n)\rangle = \frac{1}{Z}\int D\varphi\ e^{-\frac{k}{4\pi}\int d^2\vec{x}\ (\nabla\varphi+\vec{A})^2}$$

$$\epsilon^{ij}\nabla_i A_j = 2\pi\sum_{\ell=1}^{n} m_\ell\ \delta^2(\vec{z}-\vec{x}_\ell)\ \Rightarrow\ A_j = \sum_\ell m_\ell\,\partial_j\arg(\vec{z}-\vec{x}_\ell)$$

Perturbed Quantum Lifshitz Models

The analysis is similar to the classical 2D case (Kadanoff, 1979)

- Scaling dimensions: charge operators $\Delta_q = q^2/2k$, magnetic operators $\tilde{\Delta}_m = 2km^2$.
- Since $z = 2$ operators are marginal or relevant if the scaling dimension is $\Delta \leq 2 + z = 4$
- For $k_c = 1/8$, $\mathcal{O}_1$ and $\tilde{\mathcal{O}}_8$ have $\Delta_1 = \tilde{\Delta}_8 = 4$ and both are marginal
- The operator $(\nabla\varphi)^2$ has dimension $\Delta = 2$ and is always relevant: the transition requires to tune this coupling to zero.
- However we also have $(\nabla\varphi)^4$ with dimension 4 (for all k); it turns out to be marginally irrelevant (Grinstein, 1982)
- For the case of the honeycomb lattice there is an allowed relevant operator, cubic in $(\partial\varphi)$, with $\Delta = 3 < 4$, which generally drives the transition 1st order (Fradkin, Huse, Moessner, Oganesyan , Sondhi (2004); Vishwanath, Balents, and Senthil (2004)).

Perturbed Lagrangian in $2+1$ dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}\left(\frac{k_c}{4\pi}\right)^2 (\nabla^2\varphi)^2 + A(\nabla\varphi)^2 + \delta(\nabla^2\varphi)^2 + u(\nabla\varphi)^4$$
$$+ \frac{\alpha}{2a^4}(\mathcal{O}_q + \mathcal{O}_{-q}) + \frac{\tilde{\alpha}}{2a^4}(\tilde{\mathcal{O}}_m + \tilde{\mathcal{O}}_{-m})$$

- $A > 0$ uniform phase, b) $A < 0$ "tilted" phase (Lifshitz transition) $\Rightarrow$ we must fix $A = 0$.
- We used the OPE to determine the form of the effective perturbed action
- In the absence of magnetic (vortex) perturbations we find $z = 2$ (already found by Grinstein)

One-loop RG

We find (Hsu and Fradkin 2012) (using OPEs) ($J_1 = 0.290\ldots$, $J_2 = 0.111\ldots$)

$$\beta(u) = -\frac{41}{32\pi}u^2 - \frac{\pi e^{4\gamma}}{4^7}J_2\left(\alpha^2 - \tilde{\alpha}^2\right),$$

$$\beta(\alpha) = -4\delta\alpha - \frac{16}{\pi}\log\left(\frac{32}{27}\right)\alpha u,$$

$$\beta(\tilde{\alpha}) = 4\delta\tilde{\alpha} + \frac{16}{\pi}\log\left(\frac{32}{27}\right)\tilde{\alpha}u$$

$$\beta(\delta) = -\frac{e^{4\gamma}}{4^7}J_1\left(\alpha^2 - \tilde{\alpha}^2\right) - \frac{1}{2\pi^2}\left[\log\left(\frac{4}{3}\right) - \frac{5}{12}\right]u^2.$$

Consequences of the RG analysis

- u flows to zero (marginally irrelevant)
- if $\alpha \neq 0$ and $\tilde{\alpha} = 0 \Rightarrow$ KT transition to an ordered phase
- If $\alpha = 0$ and $\tilde{\alpha} \neq 0$, the system flows into the topological phase
- If $\tilde{\alpha} > 0$, both α and $\tilde{\alpha}$ flow to strong coupling
- Two options:
  - the transition between the topological and ordered phase is first order
  - The transition is continuous but controlled by a fixed point not accessible in perturbation theory, presumably with $z = 1$ (as in the Ising case simulated by Isakov et al)

Consequences of the RG analysis

- The RG equations we derived for the $2+1$ dimensional perturbed quantum Lifshitz model do not see a continuously varying z
- Why is the analysis of the perturbed effective action different from the perturbed wave function?
- Important difference: The wave function has a self-duality inherited from the 2D classical Baxter model
- The RK Hamiltonian and the perturbed $2+1$ action are not self-dual
- As a result the dynamics along the topological phase-ordered phase phase boundary is not the same as in the six vertex line
- The six vertex line has a conservation law and the classical dynamics is Model B (Halperin, Hohenberg, 1975) while for $d > 0$ the continuous symmetry, and its conservation law are broken to a $\mathbb{Z}_2$ symmetry and charge parity conservation (the dynamics should be Model A)

Consequences of the RG analysis

- Is it possible to have a continuously varying z in classical critical dynamics?
- At the level of a continuum theory, the dynamics is formulated as a Langevin equation for the relaxational dynamics (with and without a continuous symmetry and a conservation law)
- The effective field theory is a supersymmetric version of Martin-Siggia-Rose
- In the equilibrium theory the existence of a marginal operator leads to the electric and magnetic "vertex" operators to acquire anomalous dimensions while the effective field theory remains local
- However to find a continuously varying dynamical exponent z requires that the effective field theory of the dynamics to be non-local
- It is possible (but so far unproven) that this happens since the coarse-grained height field φ relaxes locally, the relax ion of magnetic operators is non-local: the dynamics is not-self-dual.