Non-linear perturbations in relativistic cosmology

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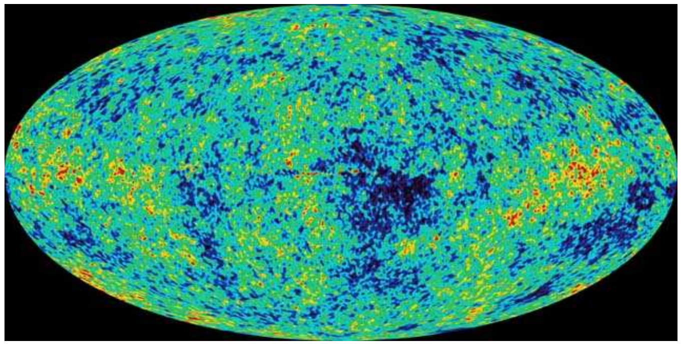
- PRL '05
- PRD '05
- JCAP '06
- astro-ph/0610064

Motivations

- Linear theory of relativistic cosmological perturbations extremely useful $$\frac{\delta T}{T} \sim 10^{-5}$$
- **Non-linear** aspects are needed in some cases:
  - non-gaussianities
  - Universe on very large scales (beyond the Hubble scales)
  - small scales
- **Conservation laws**
  - solve part of the equations of motion
  - useful to relate early universe and ``late cosmology''

Early Universe → geometry



Linear theory

- Perturbed metric (with only scalar perturbations)

$$\begin{aligned} ds^2 &= -(1+2A)dt^2 + 2a(t)\nabla_i B\, dx^i dt \\ &+ a^2(t)\left[(1-2\psi)\gamma_{ij} + 2\nabla_i\nabla_j E\right] dx^i dx^j \end{aligned}$$

- ψ related to the intrinsic curvature of constant time spatial hypersurfaces

$$^{(3)}R = \frac{4}{a^2}\nabla^2\psi \qquad [\kappa = 0]$$

- Change of coordinates, e.g. $t \to t + \delta t$

$$\delta\rho \to \delta\rho - \rho'\delta t$$

$$\psi \to \psi + H\delta t, \qquad H \equiv \frac{a'}{a}$$

Linear theory

- Curvature perturbation on uniform energy density hypersurfaces [Bardeen et al (1983)]

$$-\zeta \equiv \psi + \frac{H}{\rho'}\delta\rho = \psi - \frac{\delta\rho}{3(\rho+p)}$$

gauge-invariant

- The time component of $\nabla_\mu T^{\mu\nu} = 0$ yields

[Wands et al (2000)]

$$\zeta' = -\frac{H}{\rho+P}\delta P_{\mathrm{nad}} - \frac{1}{3}\nabla^2\left(E'+v\right)$$

$$\delta P_{\mathrm{nad}} \equiv \delta p - \frac{p'}{\rho'}\delta\rho = \delta p - c_s^2\delta\rho$$

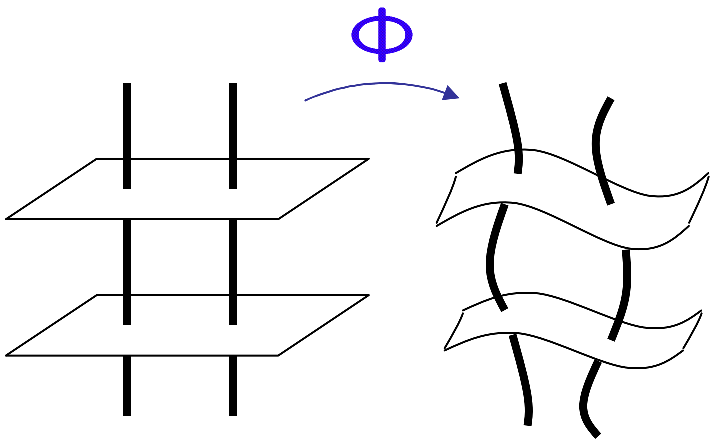
For adiabatic perturbations, ζ conserved on large scales

Covariant formalism

[Ellis & Bruni (1989)]

- How to define unambiguously

$$\delta\rho = \underbrace{\rho}_{\mathcal{M}} - \underbrace{\bar{\rho}}_{\bar{\mathcal{M}}} \; ?$$



- One needs a map

$$\Phi : \bar{\mathcal{M}} \to \mathcal{M}$$

If Q is such that $\bar{Q} = 0$,

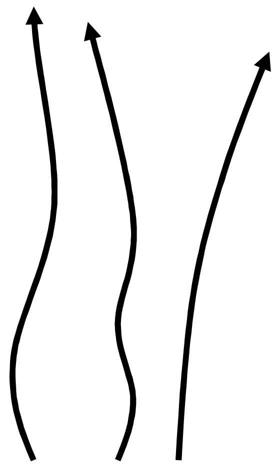
then $\delta Q = Q - \Phi(\bar{Q})$ is Φ−independent

- Idea: instead of ρ , use its spatial gradient

- Perfect fluid: ρ, P, u^a

$$T_{ab} = (\rho + P)u_a u_b + P g_{ab}$$

- Spatial projection: $h_{ab} = g_{ab} + u_a u_b$



- Expansion: $\Theta = \nabla_a u^a$

- **Integrated expansion**: $\alpha = \frac{1}{3}\int d\tau\, \Theta$

  ➡ Local scale factor $S = e^{\alpha}$

- **Spatially projected gradients**:

$$X_a = D_a\rho \equiv h_a{}^b \nabla_b \rho, \quad Y_a = D_a P, \quad Z_a = D_a \Theta$$

New approach

[DL & Vernizzi, PRL ’05; PRD ’05]

- Define $W_a = D_a \alpha$

- Projection of $\nabla_a T^{ab} = 0$ along u^a yields

$$\dot\rho + \Theta(\rho + P) = 0 \qquad [\, \dot\rho \equiv u^a \nabla_a \rho \,]$$

- Spatial gradient

$$D_a\left(\dot\rho\right) + \left(\rho + P\right) D_a \Theta + \Theta\left(D_a \rho + D_a P\right) = 0$$

$$D_a\left(\dot\rho\right) = \mathcal{L}_u\left(D_a \rho\right) - \dot\rho\, a_a, \qquad [a^c = u^b \nabla_b u^c]$$

$$\dot\lambda_a \equiv \mathcal{L}_u \lambda_a = u^b \nabla_b \lambda_a + \lambda_b \nabla_a u^b$$

$$[\, D_a \Theta = 3\left(\mathcal{L}_u D_a \alpha - \dot\alpha\, a_a\right)]$$

- One finally gets

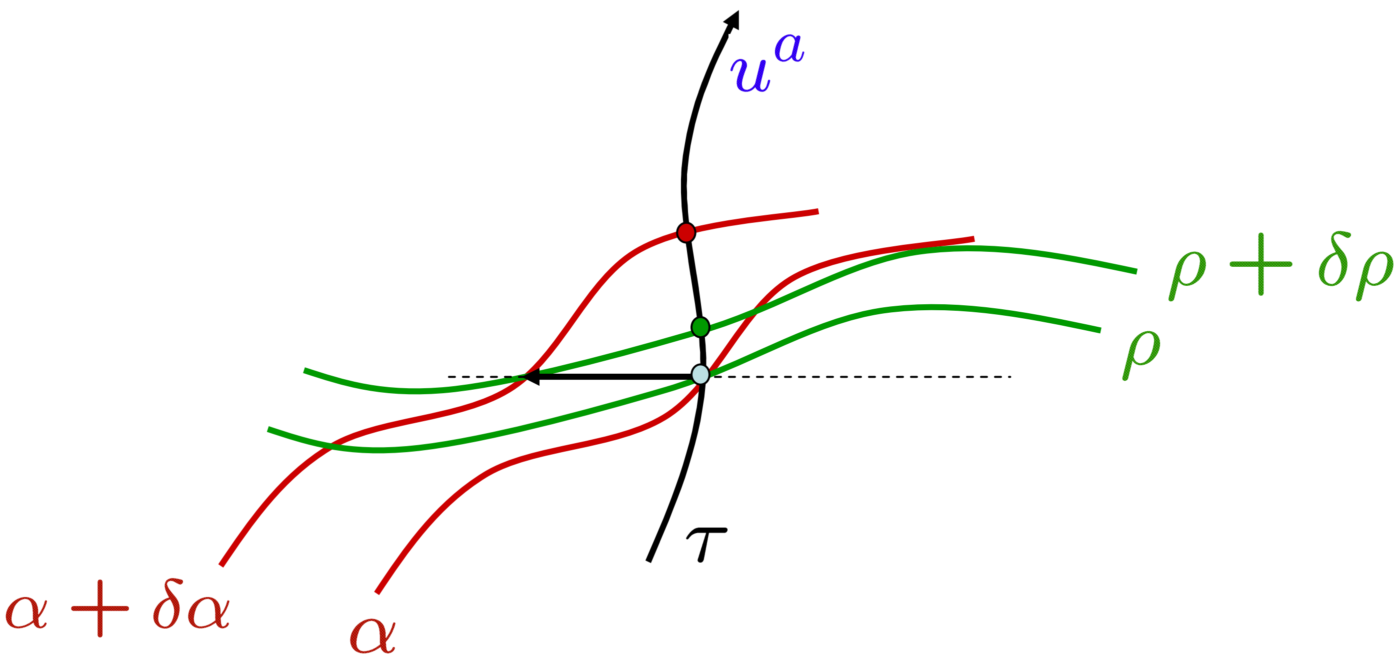
$$\dot{\zeta}_a \equiv \mathcal{L}_u \zeta_a = -\frac{\Theta}{3(\rho + P)} \Gamma_a, \qquad \Gamma_a \equiv D_a P - c_s^2 D_a \rho$$

$$\zeta_a \equiv D_a \alpha + \frac{D_a \rho}{3(\rho + P)} = D_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}} D_a \rho$$

- **This is an exact equation, fully non-linear and valid at all scales !**

- It “mimics” the linear equation $\zeta' \simeq -\frac{\mathcal{H}}{\rho + P}\left(\delta p - c_s^2 \delta \rho\right)$

$$\zeta_a = D_a\alpha - \frac{\dot{\alpha}}{\dot{\rho}} D_a\rho$$



$$\delta\alpha = e^a D_a\alpha, \qquad \delta\rho = e^a D_a\rho$$

$$e^a\zeta_a = \delta\alpha - \frac{\dot{\alpha}}{\dot{\rho}}\delta\rho = \delta\alpha - \delta_\rho\alpha = \dot{\alpha}(\delta\tau_\alpha - \delta\tau_\rho)$$

Comparison with the coordinate based approach

- Choose a coordinate system
- Expand quantities: $X(t, x^i) = \bar{X}(t) + X_{(1)} + X_{(2)} + \ldots$
- **First order**

$$\zeta_i^{(1)} = \partial_i \zeta^{(1)}, \;\; \zeta^{(1)} \equiv \alpha_{(1)} - \frac{\bar{\alpha}'}{\bar{\rho}'}\rho_{(1)}$$

$$\Gamma_i^{(1)} = \partial_i \Gamma^{(1)}, \;\; \Gamma^{(1)} \equiv P_{(1)} - \frac{\bar{P}'}{\bar{\rho}'}\rho_{(1)}$$

$$\zeta_i^{(1)\prime} = -\frac{H}{\rho + P}\Gamma_i^{(1)} \qquad \alpha_{(1)}{}' = -\psi' + \frac{1}{3}\nabla^2(E' + v)$$

➡ **Usual linear equation !**

- Second order perturbations

$$\zeta_a = \partial_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}} \partial_a \rho$$

$$\Rightarrow \quad \zeta_i^{(2)} = \partial_i \left(\alpha_{(2)} - \frac{\bar{\alpha}'}{\bar{\rho}'} \rho_{(2)} \right) - \frac{1}{\bar{\rho}'} \left(\alpha'_{(1)} - \frac{\bar{\alpha}'}{\bar{\rho}'} \rho'_{(1)} \right) \partial_i \rho_{(1)}$$

After some manipulations, one finds

$$\zeta_i^{(2)} = \partial_i \zeta^{(2)} + \frac{1}{\bar{\rho}'} \rho_{(1)} \partial_i \zeta^{(1)\prime}$$

$$\zeta^{(2)} = \alpha_{(2)} - \frac{\bar{\alpha}'}{\bar{\rho}'} \rho_{(2)} - \frac{1}{\bar{\rho}'} {\alpha_{(1)}}' \rho_{(1)} + \frac{\bar{\alpha}'}{\bar{\rho}'^2} \rho_{(1)} {\rho_{(1)}}' + \frac{1}{2\bar{\rho}'} \left(\frac{\bar{\alpha}'}{\bar{\rho}'} \right)' {\rho_{(1)}}^2$$

in agreement with previous results [Malik & Wands '04]

Gauge-invariance

- Second-order coordinate transformation

Bruni et al. '97

$$x^\mu \to \tilde{x}^\mu = x^\mu - \xi^\mu_{(1)} + \frac{1}{2}\xi^\nu_{(1)}\xi^\mu_{(1),\nu} - \xi^\mu_{(2)}$$

$$\begin{aligned} \delta\mathbf{T}^{(1)} &\to \delta\mathbf{T}^{(1)} + \mathcal{L}_{\xi_{(1)}}\mathbf{T}^{(0)}, \\ \delta\mathbf{T}^{(2)} &\to \delta\mathbf{T}^{(2)} + \mathcal{L}_{\xi_{(2)}}\mathbf{T}^{(0)} + \frac{1}{2}\mathcal{L}^2_{\xi_{(1)}}\mathbf{T}^{(0)} + \mathcal{L}_{\xi_{(1)}}\delta\mathbf{T}^{(1)} \end{aligned}$$

- ζ_a is gauge-invariant at 1st order but not 2nd order

$$\zeta_a^{(2)} \to \zeta_a^{(2)} + \mathcal{L}_{\xi_{(1)}}\zeta_a^{(1)}$$

- But, on large scales, $\mathcal{L}_{\xi_{(1)}}\zeta_i \approx \xi^0_{(1)}\partial_0\zeta_i^{(1)}$
and

$$\frac{\delta\rho}{\rho'} \to \frac{\delta\rho}{\rho'} + \xi^0_{(1)} \quad \Longrightarrow \quad \partial_i\zeta^{(2)} = \zeta_i^{(2)} - \frac{\delta\rho}{\rho'}\partial_i\zeta^{(1)\prime}$$

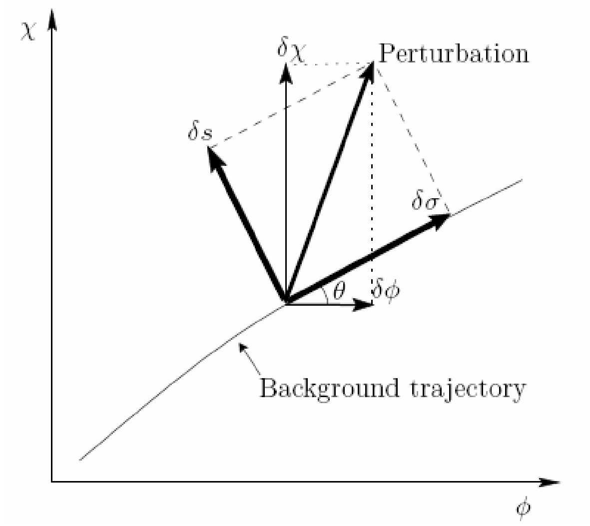
is gauge-invariant on large scales!

Cosmological scalar fields: single and multi-field inflation

- Multi-field inflation
  - Generated fluctuations can be richer (adiabatic and isocurvature)
  - Adiabatic and isocurvature perturbations can be correlated (D.L. ’99)
  - Decomposition into adiabatic & isocurvature modes (Gordon et al. ‘01)

E.g. for two fields ϕ and χ, one can write

$$\begin{aligned} \delta\sigma &= \cos\theta\ \delta\phi + \sin\theta\ \delta\chi, \\ \delta s &= -\sin\theta\ \delta\phi + \cos\theta\ \delta\chi \end{aligned}$$



From Gordon et al. ‘01

Cosmological scalar fields: covariant approach

- Several (minimally coupled) scalar fields

$$\mathcal{L} = -\frac{1}{2}\partial_a\varphi_I\partial^a\varphi^I - V(\varphi_K),$$

- Equation of motion

$$\ddot{\varphi}_I + \Theta\dot{\varphi}_I + V_{,I} - D_a D^a \varphi_I - a^a D_a\varphi_I = 0$$

- Energy-momentum tensor

$$T_{ab} = (\rho + P)u_a u_b + q_a u_b + u_a q_b + g_{ab}P + \pi_{ab}$$

$$\begin{aligned}
\rho &= \frac{1}{2}\left(\dot{\varphi}_I\dot{\varphi}^I + D_a\varphi^I D^a\varphi_I\right) + V, \\
P &= \frac{1}{2}\left(\dot{\varphi}_I\dot{\varphi}^I - \frac{1}{3}D_a\varphi_I D^a\varphi^I\right) - V, \\
q_a &= -\dot{\varphi}_I D_a\varphi^I, \qquad \pi_{ab} = D_a\varphi_I D_b\varphi^I - \frac{1}{3}h_{ab}D_c\varphi_I D^c\varphi^I.
\end{aligned}$$

Two scalar fields

- Adiabatic and entropy directions

$$\mathrm{e}_\sigma^I = (\cos\theta, \sin\theta)\,, \qquad \mathrm{e}_s^I = (-\sin\theta, \cos\theta)$$

$$\cos\theta \equiv \frac{\dot\phi}{\dot\sigma}\,, \quad \sin\theta \equiv \frac{\dot\chi}{\dot\sigma}$$

with $\quad \dot\sigma \equiv \sqrt{\dot\phi^2 + \dot\chi^2}$

- Adiabatic and entropy covectors

$$\begin{aligned} \sigma_a &\equiv \mathrm{e}_\sigma^I \nabla_a \varphi_I = \cos\theta\,\nabla_a\phi + \sin\theta\,\nabla_a\chi, \\ s_a &\equiv \mathrm{e}_s^I \nabla_a \varphi_I = -\sin\theta\,\nabla_a\phi + \cos\theta\,\nabla_a\chi \end{aligned}$$

Equations of motion

- "Homogeneous-like" equations

$$\ddot{\sigma}+\Theta\dot{\sigma}+V_{,\sigma} = \nabla^a\sigma^{\perp}_a - Y_{(s)}, \qquad Y_{(s)} \equiv \frac{1}{\dot{\sigma}}(\dot{s}_a+\dot{\theta}\sigma^{\perp}_a)s^a$$

$$\dot{\sigma}\dot{\theta}+V_{,s} = \nabla_a s^a + Y_{(\sigma)}, \qquad Y_{(\sigma)} \equiv \frac{1}{\dot{\sigma}}(\dot{s}_a+\dot{\theta}\sigma^{\perp}_a)\sigma^{\perp a}$$

- FLRW equations

$$\bar{\sigma}'' + 3H\bar{\sigma}' + \bar{V}_{,\sigma} = 0,$$

$$\bar{\sigma}'\bar{\theta}' + \bar{V}_{,s} = 0$$

“Linear-like” equations

1. Evolution of the adiabatic covector

$$\ddot{\sigma}_a + \Theta\dot{\sigma}_a + \dot{\sigma}\nabla_a\Theta + \left(V_{,\sigma\sigma} + \dot{\theta}\frac{V_{,s}}{\dot{\sigma}}\right)\sigma_a - \nabla_a\left(\nabla^c\sigma_c^{\perp}\right) = \left(\dot{\theta} - \frac{V_{,s}}{\dot{\sigma}}\right)\dot{s}_a$$
$$+ \left(\ddot{\theta} - V_{,\sigma s} + \Theta\dot{\theta}\right)s_a - \nabla_a Y_{(s)} \ ,$$

2. Evolution of the entropy covector

$$\ddot{s}_a - \frac{1}{\dot{\sigma}}(\ddot{\sigma} + V_{,\sigma})\dot{s}_a + (V_{,ss} - \dot{\theta}^2)s_a - \nabla_a\left(\nabla_c s^c\right) = -2\dot{\theta}\dot{\sigma}_a$$
$$+ \left[\frac{\dot{\theta}}{\dot{\sigma}}(\ddot{\sigma} + V_{,\sigma}) - \ddot{\theta} - V_{,\sigma s}\right]\sigma_a + \nabla_a Y_{(\sigma)}$$

Linearized equations

- First order spatial components of σ_a and s_a

$$\delta\sigma_i = \cos\bar{\theta}\ \partial_i\delta\phi + \sin\bar{\theta}\ \partial_i\delta\chi \equiv \partial_i\delta\sigma$$

$$\delta s_i = -\sin\bar{\theta}\ \partial_i\delta\phi + \cos\bar{\theta}\ \partial_i\delta\chi \equiv \partial_i\delta s$$

$\Rightarrow$ Second order equations for δσ and δs

- One usually replaces δσ by the gauge-invariant quantity

$$Q_{\mathrm{SM}} \equiv \delta\sigma + \frac{\bar{\sigma}'}{H}\psi$$

Linearized equations

[Gordon et al. '01]

- Adiabatic equation

$$Q''_{\mathrm{SM}} + 3HQ'_{\mathrm{SM}} + \left[\bar{V}_{,\sigma\sigma} - \bar{\theta}'^2 - 2\frac{H'}{H}\left(\frac{\bar{V}_{,\sigma}}{\bar{\sigma}'} + \frac{H'}{H} - \frac{\bar{\sigma}''}{\bar{\sigma}'}\right) - \frac{\vec{\nabla}^2}{a^2}\right] Q_{\mathrm{SM}}$$
$$= 2(\bar{\theta}'\delta s)' - 2\left(\frac{\bar{V}_{,\sigma}}{\bar{\sigma}'} + \frac{H'}{H}\right)\delta s.$$

- Entropy equation

$$\delta s'' + 3H\delta s' + (\bar{V}_{,ss} + 3\bar{\theta}'^2)\delta s - \frac{1}{a^2}\vec{\nabla}^2\delta s = -2\frac{\bar{\theta}'}{\bar{\sigma}'}\delta\epsilon$$

- On large scales,

$$Q'_{\mathrm{SM}} + \left(\frac{H'}{H} - \frac{\sigma''}{\sigma'}\right) Q_{\mathrm{SM}} - 2\bar{\theta}'\delta s \approx 0$$

$$\zeta' \approx -\frac{2H}{\bar{\sigma}'}\bar{\theta}'\delta s$$

Second order perturbations

- Entropy perturbation

$$\delta s_i^{(2)} = \partial_i \delta s^{(2)} + \frac{\delta\sigma}{\bar{\sigma}'}\partial_i \delta s'$$

$$\delta s^{(2)} \equiv -\frac{\bar{\chi}'}{\bar{\sigma}'}\delta\phi^{(2)} + \frac{\bar{\phi}'}{\bar{\sigma}'}\delta\chi^{(2)} - \frac{\delta\sigma}{\bar{\sigma}'}\left(\delta s' + \frac{\bar{\theta}'}{2}\delta\sigma\right)$$

- Adiabatic perturbation

$$\delta\sigma_i^{(2)} = \partial_i \delta\sigma^{(2)} + \frac{\bar{\theta}'}{\bar{\sigma}'}\delta\sigma\partial_i\delta s - \frac{1}{\bar{\sigma}'}V_i$$

$$\delta\sigma^{(2)} \equiv \frac{\bar{\phi}'}{\bar{\sigma}'}\delta\phi^{(2)} + \frac{\bar{\chi}'}{\bar{\sigma}'}\delta\chi^{(2)} + \frac{1}{2\bar{\sigma}'}\delta s\delta s'$$

$$V_i \equiv \frac{1}{2}(\delta s\partial_i\delta s' - \delta s'\partial_i\delta s)$$

Large scale evolution

- Alternative adiabatic variable

$$Q_{\mathsf{SM}}^{(2)} \equiv \delta\sigma^{(2)} + \frac{\bar{\sigma}'}{H}(\psi^{(2)} + \psi^2) + \frac{\psi}{H}\left[Q_{\mathsf{SM}}^{(1)\prime} - \frac{1}{2}\left(\frac{\bar{\sigma}'}{H}\right)'\psi - \bar{\theta}'\delta s\right]$$

- Using the 2nd order energy and moment constraints,

$$\begin{aligned} Q_{\mathsf{SM}}^{(2)\prime} &+ \left(\frac{H'}{H} - \frac{\bar{\sigma}''}{\bar{\sigma}'}\right) Q_{\mathsf{SM}}^{(2)} - 2\bar{\theta}'\delta s^{(2)} \\ &\approx -\frac{1}{\bar{\sigma}'}\left[-2\frac{H'}{H}\frac{\bar{V}_{,\sigma}}{\bar{\sigma}'} + \frac{1}{2}\left(\frac{\bar{\sigma}''}{\bar{\sigma}'} - 3\frac{H'}{H}\right)\left(3H + \frac{H'}{H}\right)\right] Q_{\mathsf{SM}}^2 - \frac{\widehat{\Delta}\rho}{\bar{\sigma}'} \\ &- 3\frac{\bar{\theta}'}{\bar{\sigma}'}\left(H + \frac{H'}{H}\right) Q_{\mathsf{SM}}\delta s + \frac{1}{\bar{\sigma}'}\left(3H + \frac{H'}{H}\right)\partial^{-2}\partial^i V_i \end{aligned}$$

Large scale evolution

- The entropy evolution on large scales is given by

$$\delta s^{(2)\prime\prime} + 3H\delta s^{(2)\prime} + \left(\bar{V}_{,ss} + 3\bar{\theta}'^2\right)\delta s^{(2)} \approx -\frac{\bar{\theta}'}{\bar{\sigma}'}\delta s'^2$$

$$-\frac{2}{\bar{\sigma}'}\left(\bar{\theta}'' + \bar{\theta}'\frac{\bar{V}_{,\sigma}}{\bar{\sigma}'} - \frac{3}{2}H\bar{\theta}'\right)\delta s\delta s'$$

$$-\left(\frac{1}{2}\bar{V}_{,sss} - 5\frac{\bar{\theta}'}{\bar{\sigma}'}\bar{V}_{,ss} - 9\frac{\bar{\theta}'^3}{\bar{\sigma}'}\right)\delta s^2 - 6H\frac{\bar{\theta}'}{\bar{\sigma}'}\partial^{-2}\partial^i V_i$$

- Evolution for $\zeta^{(2)}$

$$\zeta^{(2)\prime} \approx -\frac{H}{\bar{\sigma}'^2}\left[2\bar{\theta}'\bar{\sigma}'\delta s^{(2)} - \left(\bar{V}_{,ss} + 4\bar{\theta}'^2\right)\delta s^2 - \frac{2\bar{V}_{,\sigma}}{\bar{\sigma}'}\partial^{-2}\partial^i V_i\right]$$

- Non-local term

$$V_i' + 3HV_i = \mathcal{O}(\partial^3)$$

Conclusions

- New approach to study cosmological perturbations
  - **Non linear**
  - Purely **geometric** formulation (extension of the **covariant** formalism)
  - ``Mimics'' the linear theory equations
  - Get easily the second order results
  - Exact equations: no approximation

- Can be extended to scalar fields
  - Covariant and fully non-linear generalizations of the adiabatic and entropy components
  - Evolution, on large scales, of the 2^{nd} order adiabatic and entropy components