Study of finite density lattice QCD by a histogram method

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WHOT-QCD collaboration

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Histogram method

- Problem of Complex Determinant at $\mu \neq 0$
  - Boltzmann weight: complex at $\mu \neq 0$
    - Monte-Carlo method is not applicable.

$$\langle O \rangle_{(m,T,\mu)} = \frac{1}{Z} \int DU \; O \underbrace{\left(\det M(m,\mu) \right)^{N_{\mathrm{f}}}}_{\text{complex}} e^{-S_g}$$

- Distribution function in Density of state method (Histogram method)

X: order parameters, total quark number, average plaquette etc.

$\det M = \left| \det M \right| e^{i\theta}$

$$W(X; m, T, \mu) \equiv \int DU \; \delta\left(X - \hat{X}\right) \left(\det M(m,\mu)\right)^{N_{\mathrm{f}}} e^{-S_g} = W_0 \times \underbrace{\left\langle e^{i\theta} \right\rangle_{X:\mathrm{fixed}}}_{\text{Complex phase factor}}$$

$$W_0(X; T, m, \mu) = \int DU \; \delta\left(X - \hat{X}\right) \left| \det M(m,\mu) \right|^{N_{\mathrm{f}}} e^{-S_g}$$

histogram in phase-quenched simulations

- Expectation values

$$\langle O[X] \rangle_{(m,T,\mu)} = \frac{1}{Z} \int dX \; O[X] W(X, m, T, \mu)$$

$$Z(m,T,\mu) = \int dX \; W(X, m, T, \mu)$$

(β, m, μ)-dependence of the Distribution function

- Distributions of plaquette P (1x1 Wilson loop for the standard action)

$$W(P',\beta,m,\mu) \equiv \int DU \delta(\hat{P}\text{-}P') (\det M(m,\mu))^{N_f} \ e^{6N_{\rm site}\hat{P}}$$

$$R(P,\beta,\beta_0 m,m_0,\mu) \equiv W(P,\beta,m,\mu)/W(P,\beta_0,m_0,0)$$ (Reweight factor)

$$R(P') = e^{6N_{\rm site}(\beta-\beta_0)P'} \frac{\left\langle \delta(\hat{P}\text{-}P') \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_{(\beta_0,\mu=0)}}{\left\langle \delta(\hat{P}\text{-}P') \right\rangle_{(\beta_0,\mu=0)}} \equiv e^{6N_{\rm site}(\beta-\beta_0)P'} \left\langle \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_{P'}$$

Effective potential:

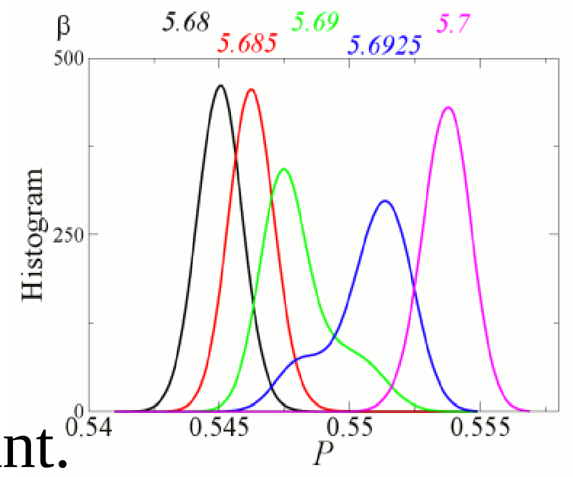
$$V_{\rm eff}(P,\beta,m,\mu) = -\ln[W(P,\beta m,\mu)] = V_{\rm eff}(P,\beta_0,m_0,0) - \ln R(P,\beta,\beta_0 m,m_0,\mu)$$

$$\ln R(P) = \underline{6N_{\rm site}(\beta-\beta_0)P} + \underline{\ln \left\langle \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_P}$$

Overlap problem

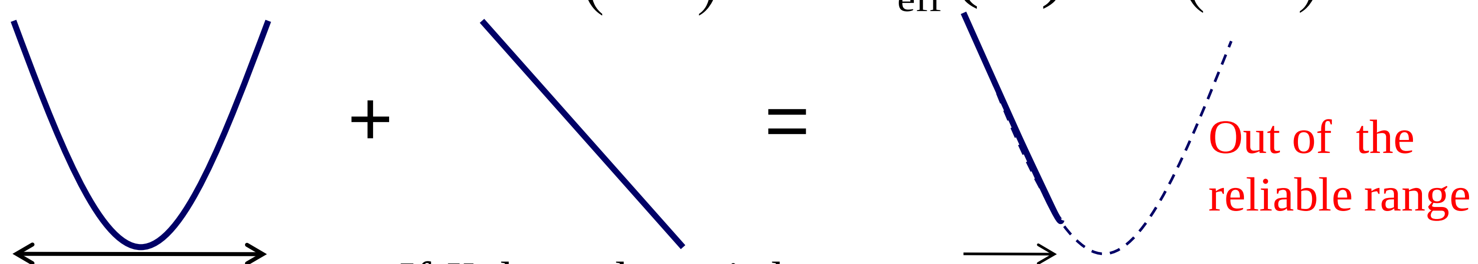
$$\langle OR \rangle = \frac{1}{Z}\int ORW(X)\,dX \;=\frac{1}{Z}\int \exp\left(-V_{\mathrm{eff}}(X)+\ln(OR)\right)dX$$

$$V_{\mathrm{eff}}(X) = -\ln W(X)$$



- W is computed from the histogram.
- Distribution function around X where $V_{\mathrm{eff}}(X) - \ln(OR)$ is minimized: important.
- V_{eff} must be computed in a wide range.

$V_{\mathrm{eff}}(X)$ + $-\ln(OR)$ = $V_{\mathrm{eff}}(X) - \ln(OR)$



reliable range

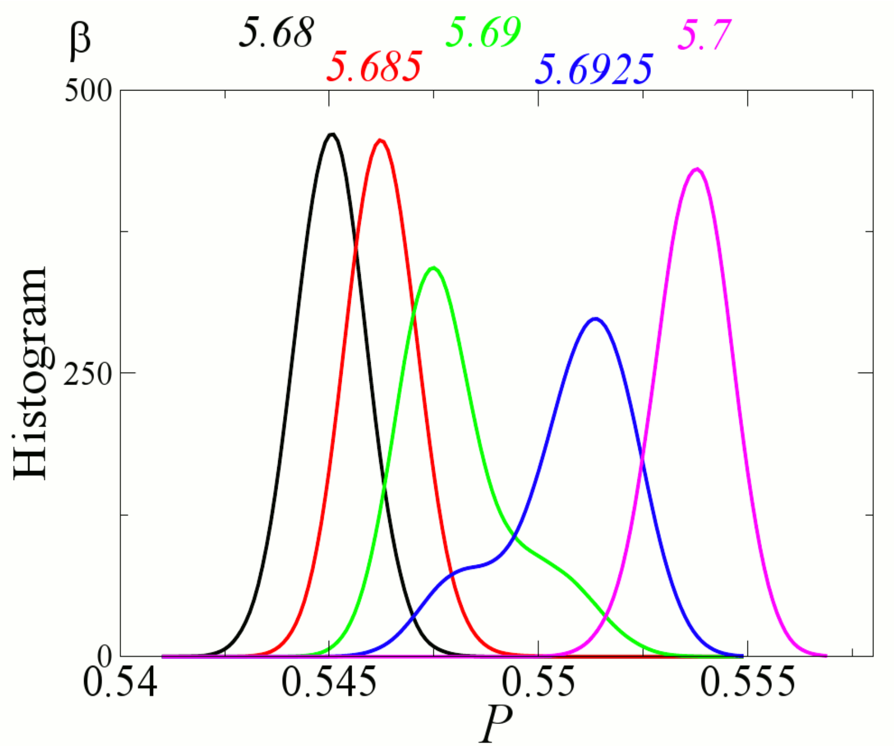
If X-dependence is large.

reliable range

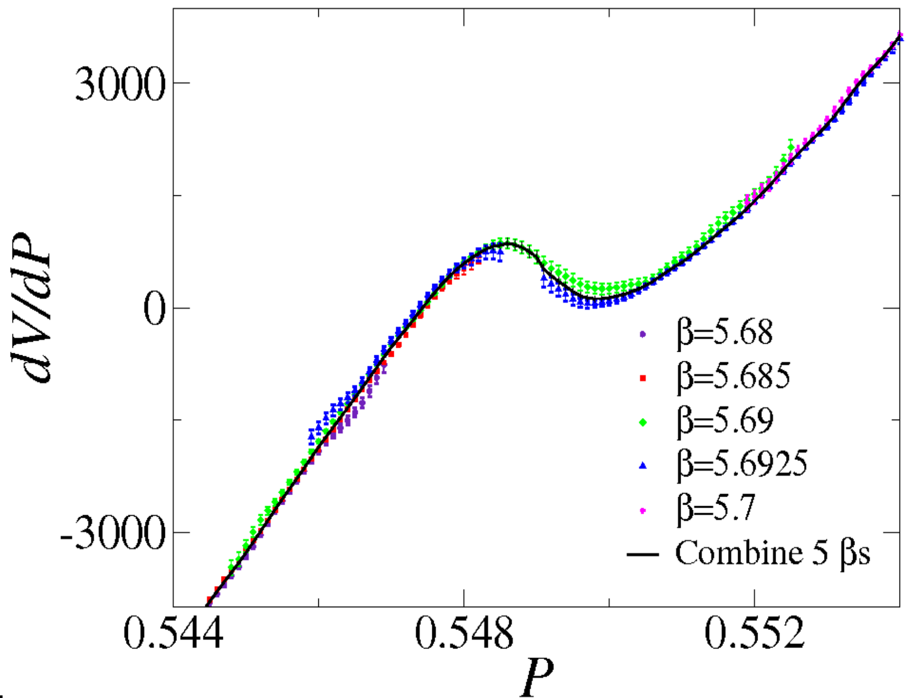
Distribution function in quenched simulations

Effective potential in a wide range of P: required.

Plaquette histogram at K=1/m_q=0.



Derivative of $V_{\rm eff}$ at β=5.69



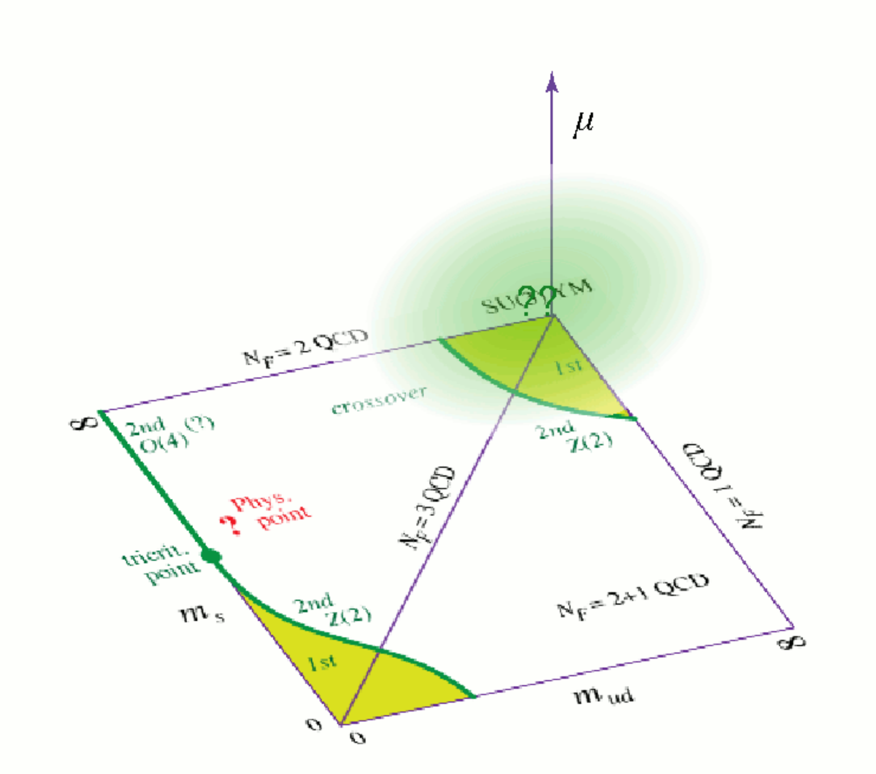
$N_{\rm site} = 24^3 \times 4$, 5 β points, quenched.

$dV_{\rm eff}/dP$ is adjusted to β=5.69, using

$$\frac{dV_{\rm eff}}{dP}(\beta_2) = \frac{dV_{\rm eff}}{dP}(\beta_1) - 6N_{\rm site}(\beta_2 - \beta_1)$$

These data are combined by taking the average.

Distribution function in the heavy quark region



- We study the properties of $W(X)$ in the heavy quark region.
- Performing quenched simulations + Reweighting.
- We find the critical surface.
- Standard Wilson quark action + plaquette gauge action, $S_g = -6N_{\text{site}}\beta P$
- lattice size: $24^3 \times 4$
- 5 simulation points; β=5.68-5.70.

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

Hopping parameter expansion

$$N_{\text{f}} \ln\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right) = N_{\text{f}}\left(288 N_{\text{site}} K^4 P + 12\cdot 2^{N_t} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R + \underbrace{i \sinh(\mu/T)\Omega_I}_{\text{phase}}\right) + \cdots\right)$$

P: plaquette, $\Omega=\Omega_R+i\Omega_I$: Polyakov loop $\qquad \det M(0,0)=1$

Distribution function for P and Ω_R

$$\boxed{W(P,\Omega_R,\beta,\kappa)=\int DU\ \delta\left(P-\hat{P}\right)\delta\left(\Omega_R-\hat{\Omega}_R\right)(\det M(\kappa))^{N_f}\ e^{-6N_{\rm site}\hat{P}}}$$

$$\frac{W(\beta,K,\mu)}{W(\beta_0,0,0)}=\frac{\left\langle \delta\left(P\text{-}\hat{P}\right)\delta\left(\Omega_R\text{-}\hat{\Omega}_R\right)e^{6N_{\rm site}(\beta-\beta_0)P}\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right)^{N_f}\right\rangle_{(\beta_0,K=\mu=0)}}{\left\langle \delta\left(P\text{-}\hat{P}\right)\delta\left(\Omega_R\text{-}\hat{\Omega}_R\right)\right\rangle_{(\beta_0,K=\mu=0)}}\equiv\left\langle e^{6N_{\rm site}(\beta-\beta_0)\hat{P}}\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right)^{N_f}\right\rangle_{P,\Omega_R}$$

- Effective potential $\quad V_{\rm eff}(P,\Omega_R;\beta,\kappa)=-\ln W(P,\Omega_R;\beta,\kappa)$
- Hopping parameter expansion

$$V_{\rm eff}(\beta,\kappa)-V_{\rm eff}(\beta_0,0)=-\left(6(\beta-\beta_0)+288N_f K^4\right)N_{\rm site}P-12\times 2^{N_t}N_f N_s^3 K^{N_t}\cosh(\mu/T)\Omega_R-\ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R}$$

$$\equiv V_0(\beta,\kappa)\qquad -\ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R}\qquad \left(\theta=12\cdot 2^{N_t}N_s^3 N_f K^{N_t}\sinh(\mu/T)\hat{\Omega}_I\right)$$

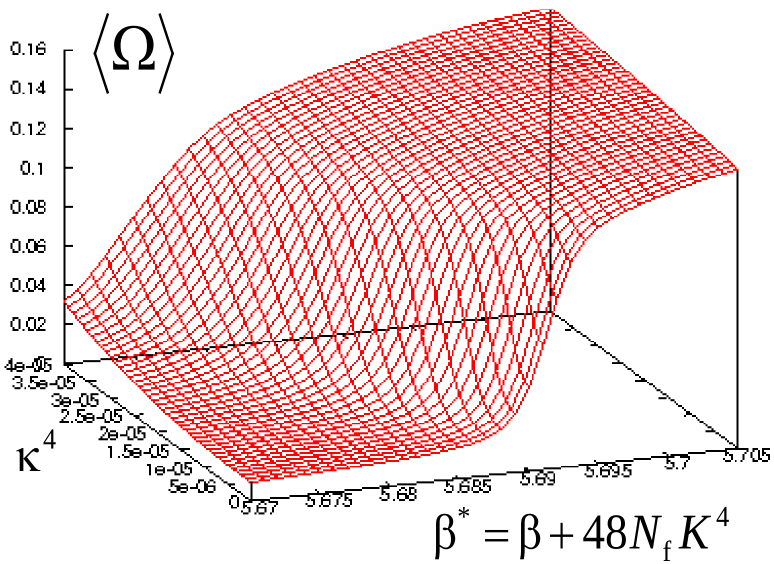
Phase-quenched part

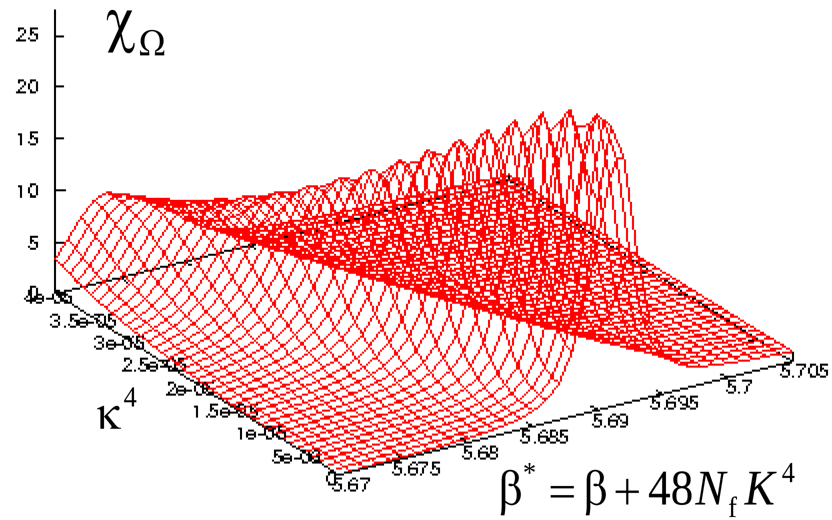
- 2 parameters in V_0: $\boxed{\beta+48N_f K^4\equiv\beta^*,}\quad \boxed{K^{N_t}\cosh(\mu/T)}$
  - V_0 is the same as $V_{\rm eff}$ (μ=0) when $K^{N_t}\Rightarrow K^{N_t}\cosh(\mu/T)$
- 1 parameter in θ: $\boxed{K^{N_t}\sinh(\mu/T)=K^{N_t}\cosh(\mu/T)\tanh(\mu/T)<K^{N_t}\cosh(\mu/T)}$

Distribution function for P and Ω_R

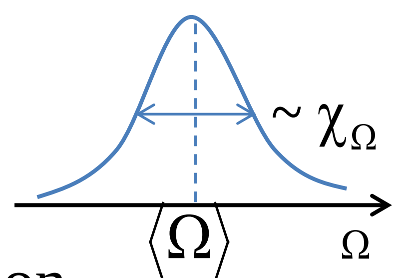
Expectation value of Polyakov loop and its susceptibility by the reweightuing method at μ=0.

$24^3 \times 4$ lattice



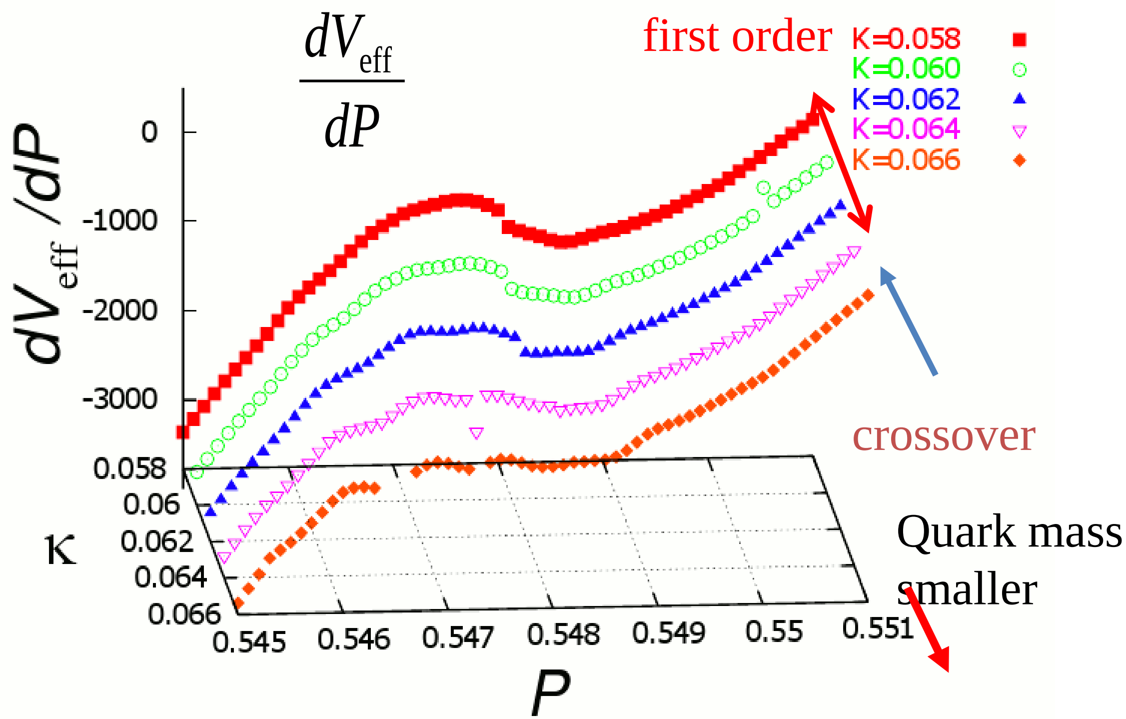


- If $W(P,\Omega)$ is a Gaussian distribution,
  - The peak position of $W(P,\Omega)$ ⟹ $(\langle P \rangle, \langle \Omega \rangle)$
  - The width of $W(P,\Omega)$ ⟹ susceptibilities χ_P, χ_Ω
- If $W(P,\Omega)$ have two peaks, ⟹ first order transition



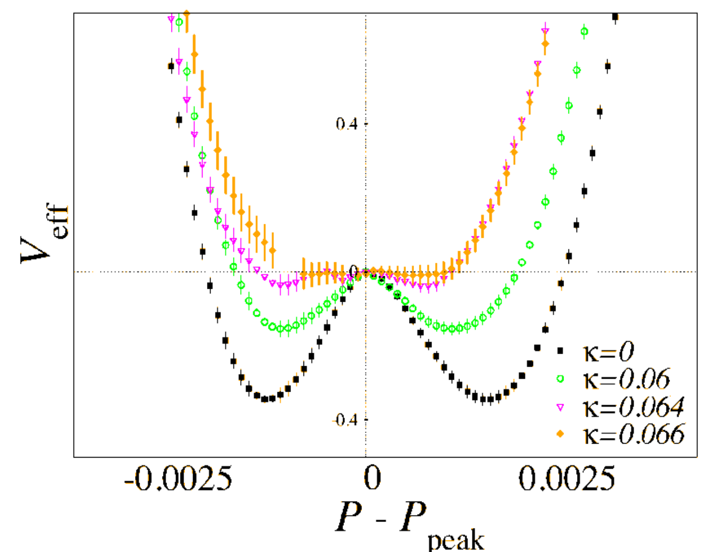
Effective potential near the quenched limit(μ=0)

WHOT-QCD, Phys.Rev.D84, 054502(2011)



Quenched Simulation
(mq=∞, K=0)

K~1/mq for large mq



$24^3 \times 4$ lattice, 5 β points, Nf=2

$$V_{\text{eff}}(P,\beta,m,\mu) = -\ln[W(P,\beta m,\mu)] = V_{\text{eff}}(P,\beta_0,m_0,0) - \ln R(P,\beta,\beta_0 m,m_0,\mu)$$

$$\ln R(P) = 6N_{\text{site}}(\beta-\beta_0)P + \ln\left\langle \left(\frac{\det M(m,\mu)}{\det M(m_0,0)}\right)^{N_f} \right\rangle_P$$

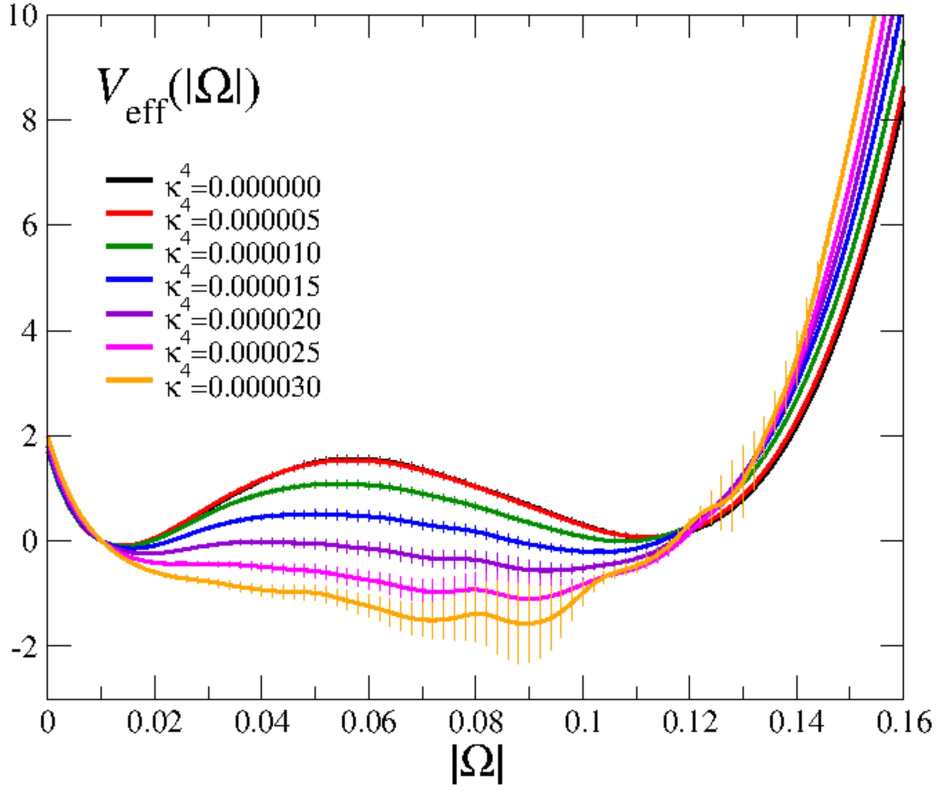
Nf=2: Kcp=0.0658(3)(8)

$$\frac{T_c}{m_\pi} \approx 0.02$$

- detM: Hopping parameter expansion,
- First order transition at K = 0 changes to crossover at K > 0.

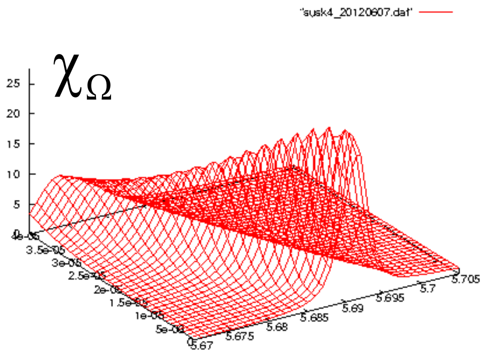
Order of the phase transition
Polyakov loop distribution

Effective potential of $|\Omega|$
on the pseudo-critical line at $\mu=0$



$$\text{Critical point}: \kappa^4 \approx 2.0\times10^{-5}$$

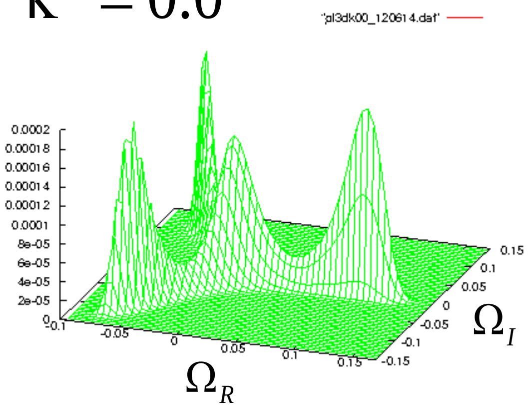
- The pseudo-critical line is determined by χ_Ω peak.



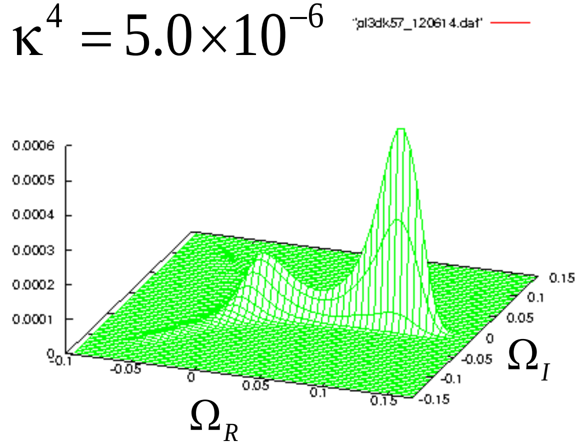
- Double-well at small K
  - First order transition
- Single-well at large K
  - Crossover

Polyakov loop distribution in the complex plane
(μ=0)

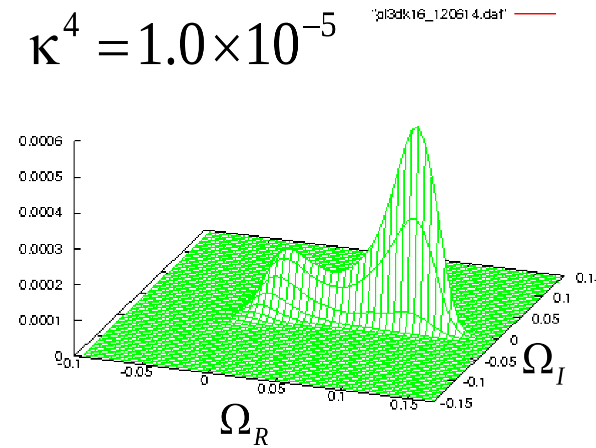
$\kappa^4 = 0.0$



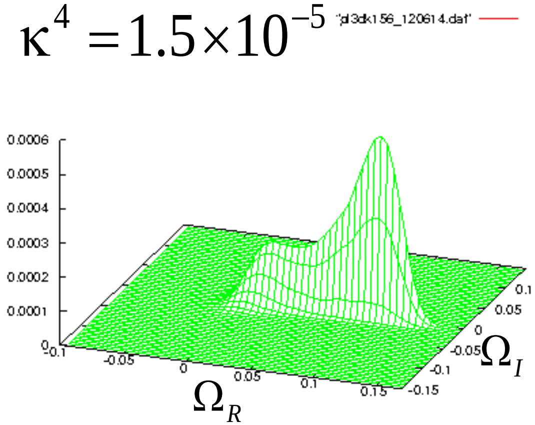
$\kappa^4 = 5.0\times10^{-6}$



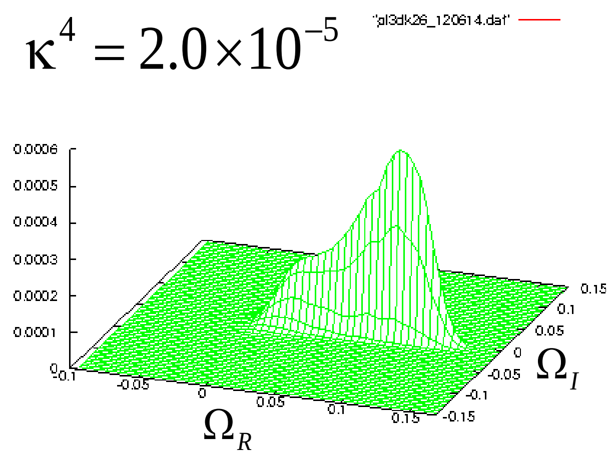
$\kappa^4 = 1.0\times10^{-5}$



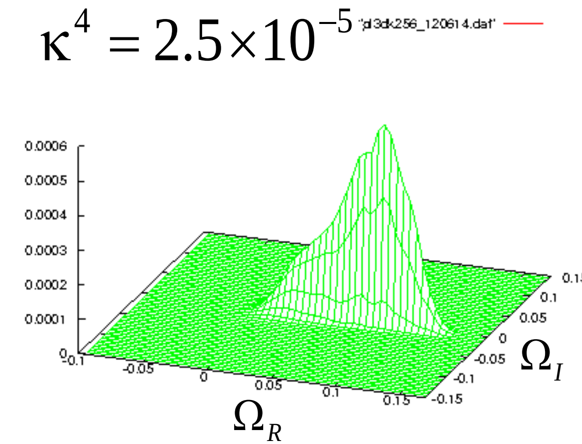
$\kappa^4 = 1.5\times10^{-5}$



$\kappa^4 = 2.0\times10^{-5}$



$\kappa^4 = 2.5\times10^{-5}$



critical point

- on β_{pc} measured by the Polyakov loop susceptibility.

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\theta \equiv \mathrm{Im} \ln \det M \approx 12 \cdot 2^{N_t} N_s^3 N_{\mathrm{f}} K^{N_t} \sinh(\mu/T) \Omega_I$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R \text{ fixed}} << \text{(statistical error)}$$

- Cumulant expansion $<..>_{P,\Omega_R}$: expectation values fixed P and Ω_R.

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R} = \exp\left[\underbrace{\cancel{i\left\langle \theta \right\rangle_C}}_{\to 0} - \frac{1}{2}\left\langle \theta^2 \right\rangle_C - \underbrace{\cancel{\frac{i}{3!}\left\langle \theta^3 \right\rangle_C}}_{\to 0} + \frac{1}{4!}\left\langle \theta^4 \right\rangle_C + \cancel{\cdots} \right]$$

cumulants

$$\left\langle \theta \right\rangle_C = \left\langle \theta \right\rangle_{P,\Omega_R}, \quad \left\langle \theta^2 \right\rangle_C = \left\langle \theta^2 \right\rangle_{P,\Omega_R} - \left\langle \theta \right\rangle_{P,\Omega_R}^2, \quad \left\langle \theta^3 \right\rangle_C = \left\langle \theta^3 \right\rangle_{P,\Omega_R} - 3\left\langle \theta^2 \right\rangle_{P,\Omega_R}\left\langle \theta \right\rangle_{P,\Omega_R} + 2\left\langle \theta \right\rangle_{P,\Omega_R}^3, \quad \left\langle \theta^4 \right\rangle_C = \cdots$$

– Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)

Source of the complex phase

If the cumulant expansion converges, No sign problem.

Convergence in the large volume (V) limit

The cumulant expansion is good in the following situations.

- If the phase is given by $\theta = \sum_x \theta_x$
  - No correlation between θ_x.

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R} = \left\langle e^{i\sum_x \theta_x} \right\rangle_{P,\Omega_R} \approx \prod_x \left\langle e^{i\theta_x} \right\rangle_{P,\Omega_R} = \exp\left[\sum_x \sum_n \frac{i^n}{n!} \left\langle \theta_x^n \right\rangle_C \right]$$

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R} = \exp\left[\sum_n \frac{i^n}{n!} \left\langle \theta^n \right\rangle_C \right] \Rightarrow \left\langle \theta^n \right\rangle_C \approx \sum_x \left\langle \theta_x^n \right\rangle_C \sim O(V)$$

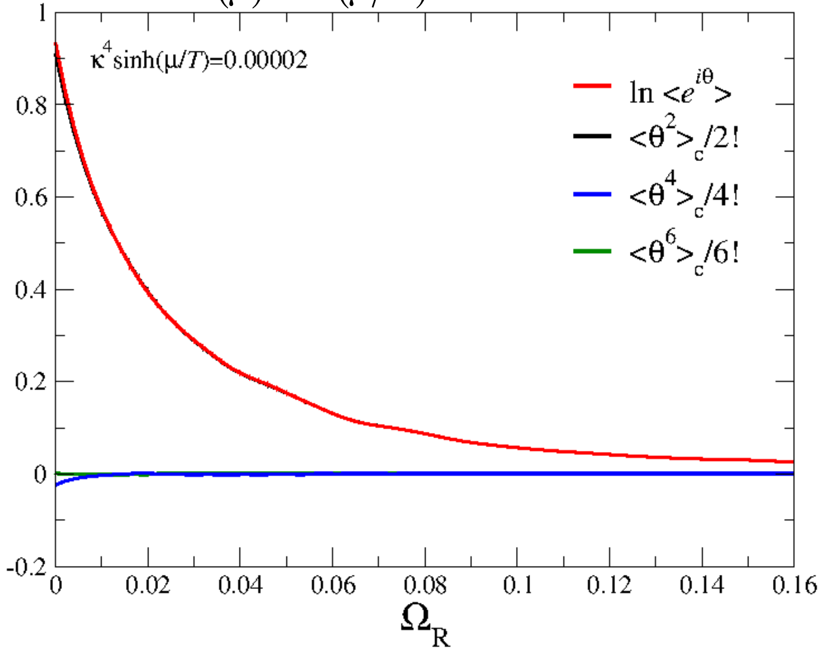
  - Ratios of cumulants do not change in the large V limit.
  - Convergence property is independent of V, although the phase fluctuation becomes larger as V increases.
  - The application range of μ can be measured on a small lattice.
- When the distribution function of θ is perfectly Gaussian, the average of the phase is give by the second order, $\left\langle e^{i\theta} \right\rangle_{P,\Omega_R} = \exp\left[-\frac{1}{2} \left\langle \theta^2 \right\rangle_C \right]$

Cumulant expansion

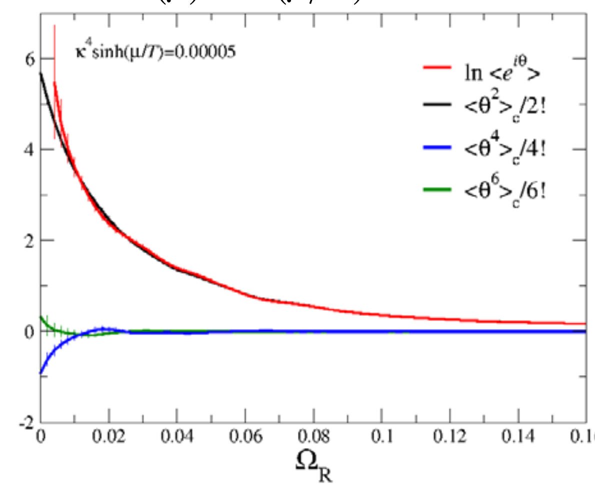
$$\ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R} = -\frac{1}{2}\left\langle\theta^2\right\rangle_C + \frac{1}{4!}\left\langle\theta^4\right\rangle_C - \frac{1}{6!}\left\langle\theta^6\right\rangle_C + \cdots$$

$\beta^* = 5.69$

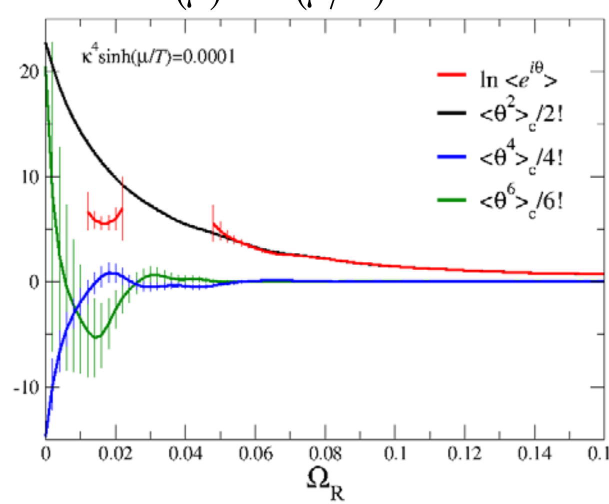
$K^4(\mu)\sinh(\mu/T) = 0.00002$



$K^4(\mu)\sinh(\mu/T) = 0.00005$



$K^4(\mu)\sinh(\mu/T) = 0.0001$



- The effect from higher order terms is small near the critical point.

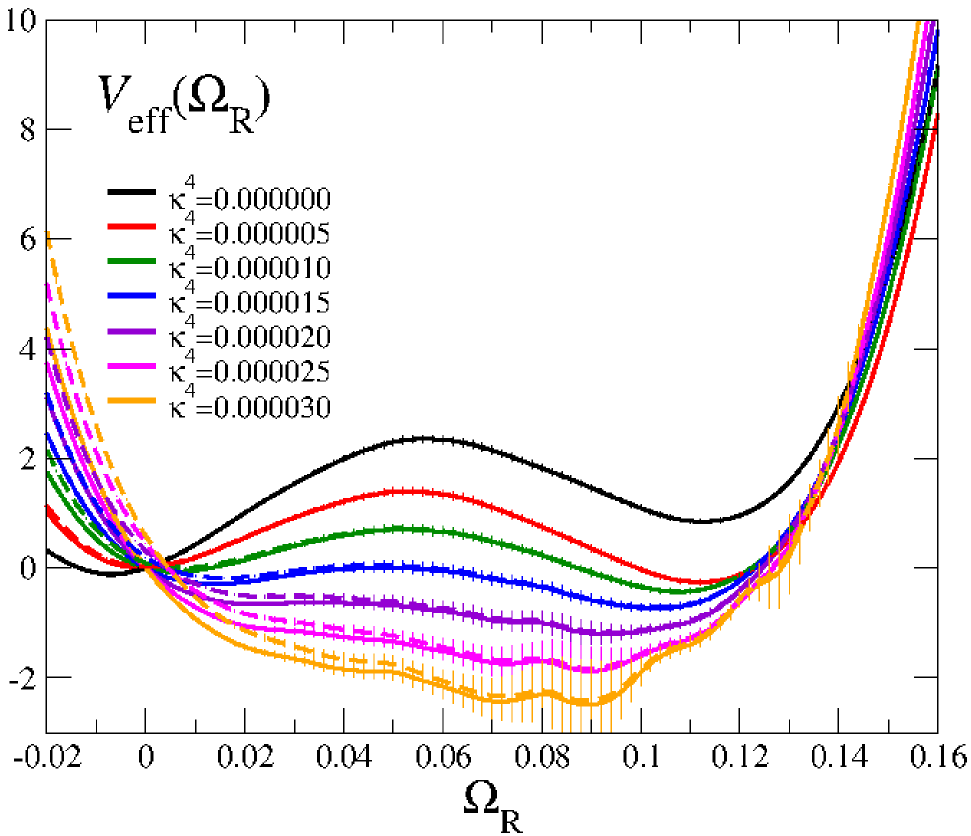
$$K_{\mathrm{cp}}^{N_t}(0) = K_{\mathrm{cp}}^{N_t}(\mu)\cosh(\mu/T) > K_{\mathrm{cp}}^{N_t}(\mu)\sinh(\mu/T)$$

~0.00002

- Phase fluctuations
  - large in the confinement phase
  - small in the deconfinement phase

Effect from the complex phase factor

- Polyakov loop effective potential for each $K^{N_t}\cosh(\mu/T)$ at the pseudo-critical (β, K).
  - Solid lines: complex phase omitted, i.e., $\tanh(\mu/T)=0$
  - Dashed lines: complex phase is estimated from $\langle\theta^2\rangle_c/2$ with $\tanh(\mu/T)=1$



$$V_{\rm eff}(\Omega_R)=V_0(\Omega_R)-\ln\langle e^{i\theta}\rangle_{\Omega_R:\text{fixed}}$$

$$\approx V_0(\Omega_R)+\frac{1}{2}\langle\theta^2\rangle_{\Omega_R:\text{fixed}}$$

The effect from the complex phase factor is very small except near Ω_R=0.

Critical line in 2+1-flavor finite density QCD

- The effect from the complex phase is very small for the determination of K_{cp}.

Nf=2 at μ=0: Kcp=0.0658(3)(8)

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

$$2\ln\left(\frac{\det M(K)}{\det M(0)}\right) = 2\left(288 N_{\rm site} K^4 P + \underline{12\times 2^{N_t} N_s^3 K^{N_t}\Omega_R} + \cdots\right)$$

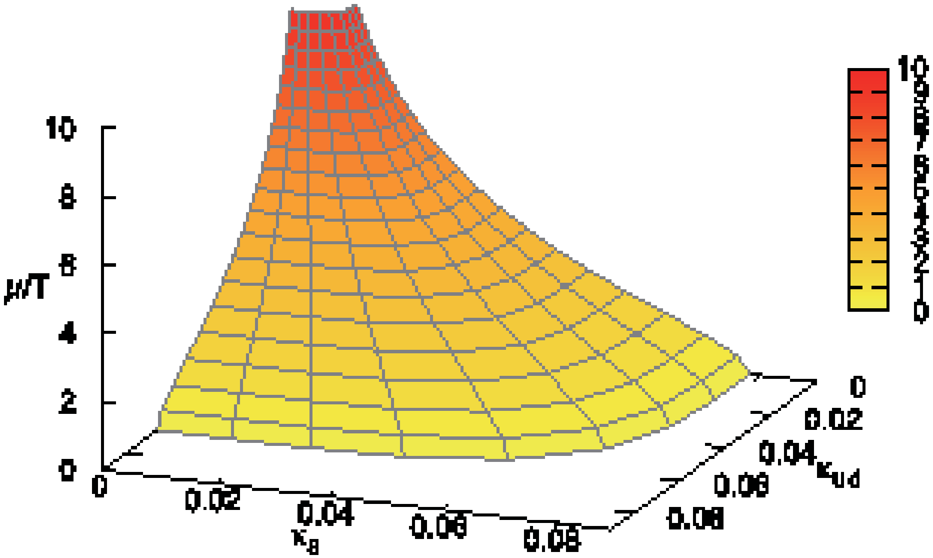
$$K_{\rm cp}^{N_t}(0) = K_{\rm cp}^{N_t}(\mu)\cosh(\mu/T)$$

Nf=2+1

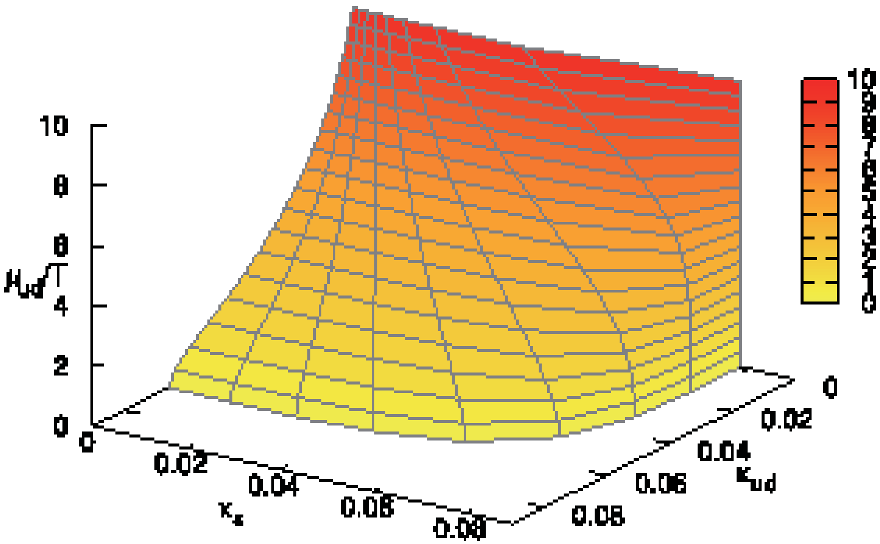
$$\ln\left[\frac{(\det M(K_{ud}))^2 \det M(K_s)}{(\det M(0))^3}\right] = 288 N_{\rm site}\left(2K_{ud}^4 + K_s^4\right)P + \underline{12\times 2^{N_t} N_s^3\left(2K_{ud}^{N_t}\cosh\left(\frac{\mu_{ud}}{T}\right) + K_s^{N_t}\cosh\left(\frac{\mu_s}{T}\right)\right)\Omega_R} + \cdots$$

The critical line is described by $\boxed{2K_{ud}^{N_t}\cosh\left(\frac{\mu_{ud}}{T}\right) + K_s^{N_t}\cosh\left(\frac{\mu_s}{T}\right) = 2K_{{\rm cp}(Nf=2)}^{N_t}}$

Critical line for μu=μd=μs=μ



Critical line for μu=μd=μ, μs=0



Distribution function in the light quark region

WHOT-QCD Collaboration, in preparation,
(Nakagawa et al., arXiv:1111.2116)

- Perform phase quenched simulations
- Add the effect of the complex phase by the reweighting.
- Calculate the probability distribution function.

- Goal
  - The critical point
  - The equation of state

    Pressure, Energy density, Quark number density, Quark number susceptibility, Speed of sound, etc.

Probability distribution function by phase quenched simulation

- We perform phase quenched simulations with the weight:

$$\left|\det M(m,\mu)\right|^{N_{\rm f}} e^{-S_g}$$

$$\boxed{\begin{aligned} W(P',F',\beta,m,\mu) &= \int DU\, \delta(\hat{P}-P')\delta(\hat{F}-F')(\det M(m,\mu))^{N_{\rm f}} e^{-S_g} \\ &= \int DU\, \delta(\hat{P}-P')\delta(\hat{F}-F')e^{i\theta}\left|\det M(m,\mu)\right|^{N_{\rm f}} e^{-S_g} \\ &= \underline{\left\langle e^{i\theta}\right\rangle_{P',F'}} \times \underline{W_0(P',F',\beta,m,\mu)} \end{aligned}}$$

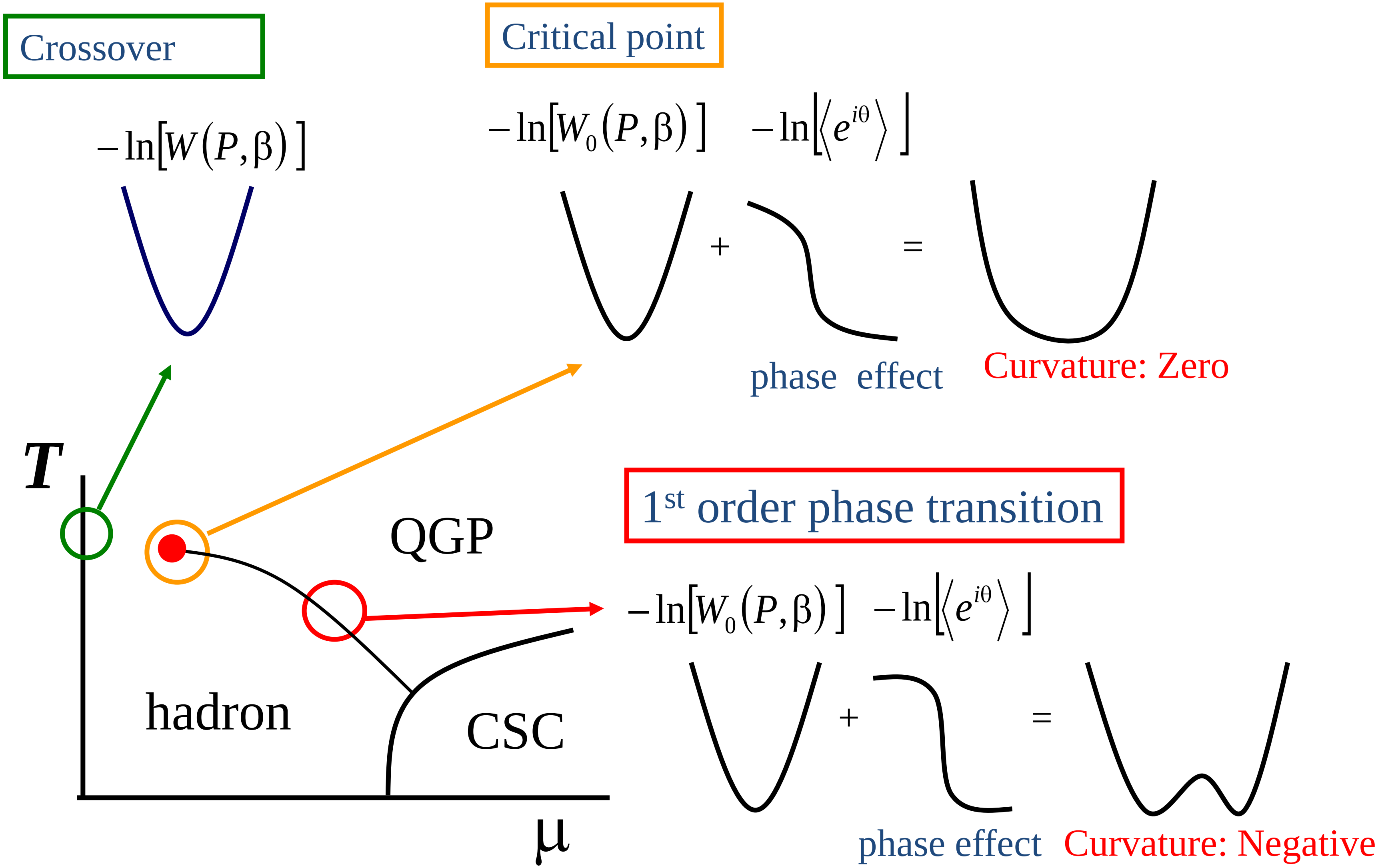
expectation value with fixed *P*,*F* histogram

P: plaquette $\quad F(\mu)=\dfrac{N_{\rm f}}{N_{\rm site}}\ln\left|\dfrac{\det M(\mu)}{\det M(0)}\right| \quad \theta\equiv N_{\rm f}\,{\rm Im}\ln\det M$

Distribution function of the phase quenched. $\quad W_0(P',F')=\int DU\,\delta(\hat{P}-P')\delta(\hat{F}-F')\left|\det M\right|^{N_{\rm f}} e^{6N_{\rm site}\beta\hat{P}}$

μ-dependence of the effective potential

Curvature of the effective potential



Curvature of the effective potential

- If the distribution is Gaussian,

$$W_0(P,F) \approx \sqrt{\frac{6N_{\text{site}}}{2\pi\chi_P}} \exp\left[-\frac{6N_{\text{site}}}{2\chi_P}\left(P-\langle P\rangle\right)^2\right] \times \sqrt{\frac{N_{\text{site}}}{2\pi\chi_F}} \exp\left[-\frac{N_{\text{site}}}{2\chi_F}\left(F-\langle F\rangle\right)^2\right]$$

$$\chi_P = 6N_{\text{site}}\left\langle\left(P-\langle P\rangle\right)^2\right\rangle \qquad \chi_F = N_{\text{site}}\left\langle\left(F-\langle F\rangle\right)^2\right\rangle$$

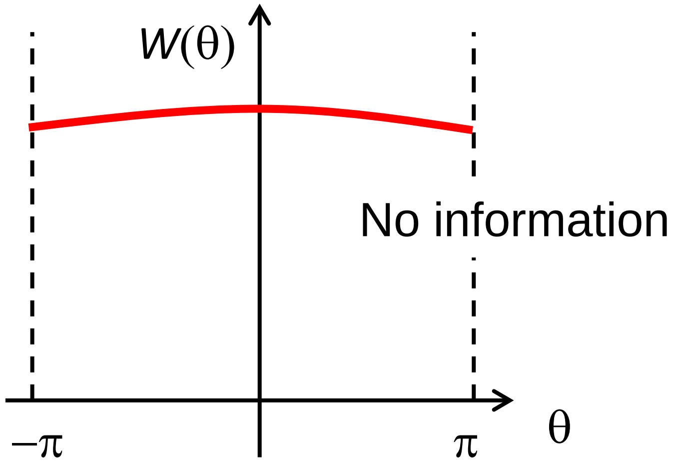
$$\boxed{\frac{\partial^2\left(-\ln W_0\right)}{\partial P^2}\left(\langle P\rangle,\langle F\rangle\right) = \frac{6N_{\text{site}}}{\chi_P}} \qquad \boxed{\frac{\partial^2\left(-\ln W_0\right)}{\partial F^2}\left(\langle P\rangle,\langle F\rangle\right) \approx \frac{N_{\text{site}}}{\chi_F}}$$

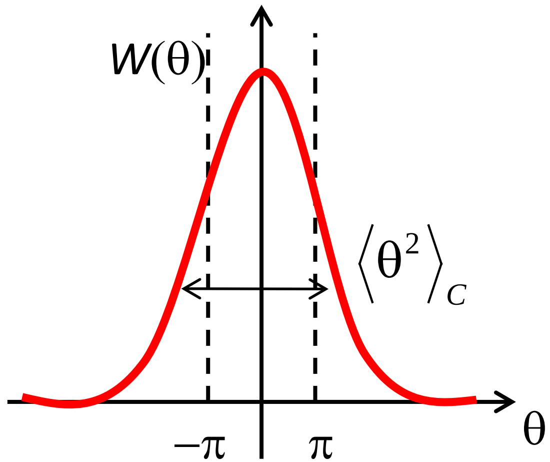
at the peak of the distribution

Complex phase distribution

- We should not define the complex phase in the range from -π to π.
- When the distribution of q is perfectly Gaussian, the average of the complex phase is give by the second order (variance),

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[-\frac{1}{2} \left\langle \theta^2 \right\rangle_C \right]$$



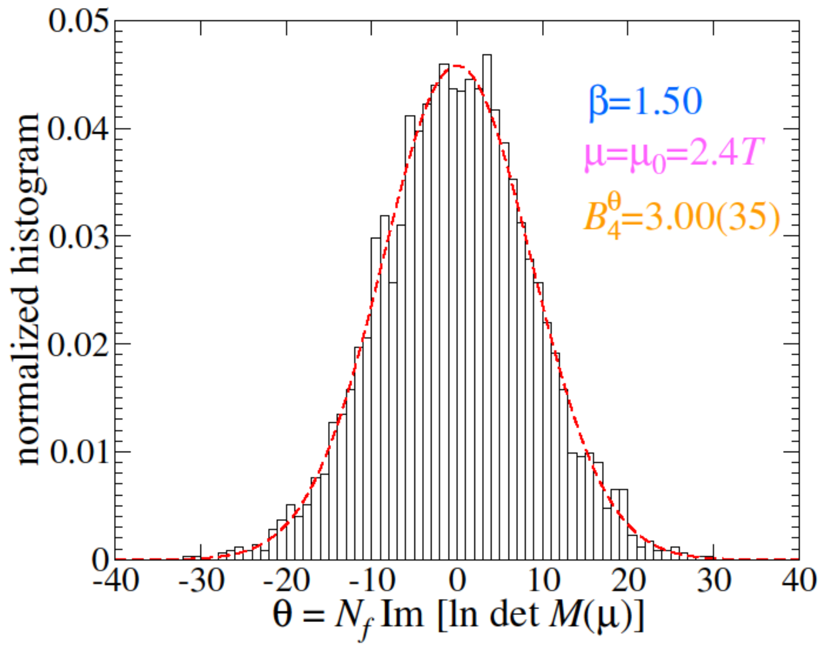


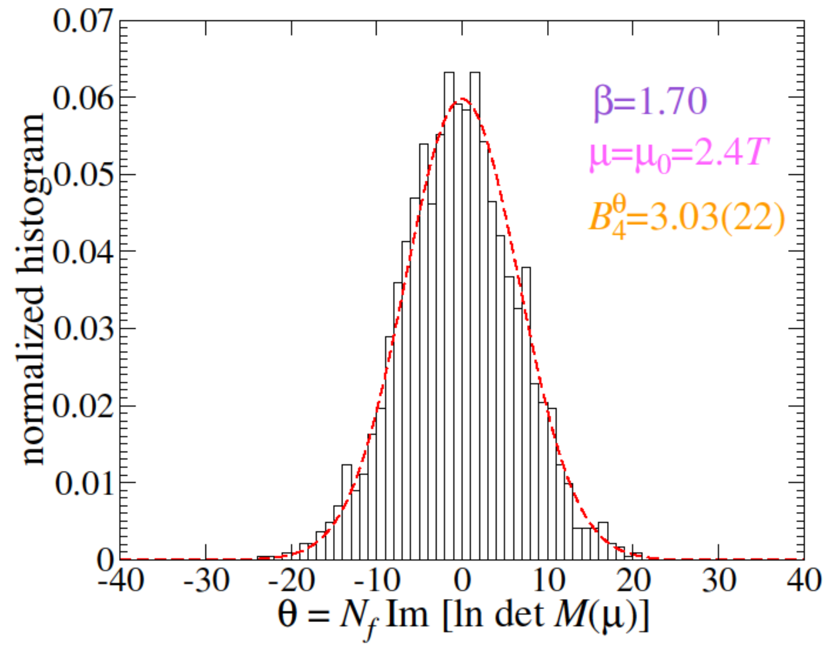
- Gaussian distribution → The cumulant expansion is good.
- We define the phase

$$\theta(\mu) = N_f \,\mathrm{Im}\left(\ln \frac{\det M(\mu)}{\det M(0)} \right) = N_f \int_0^{\mu/T} \mathrm{Im}\left[\frac{\partial \ln \det M}{\partial(\mu/T)} \right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T} \right)$$

  – The range of q is from -∞ to ∞.

Distribution of the complex phase





- Well approximated by a Gaussian function.
- Convergence of the cumulant expansion: good.

$$\left\langle e^{i\theta} \right\rangle_{P,F} \approx \exp\left[-\frac{1}{2} \left\langle \theta^2 \right\rangle \right]$$

$$\frac{1}{2} \left\langle \theta^2 \right\rangle_{\langle P \rangle, \langle F \rangle} \approx \frac{1}{2} \left\langle \theta^2 \right\rangle_{\beta_0, \mu_0}$$

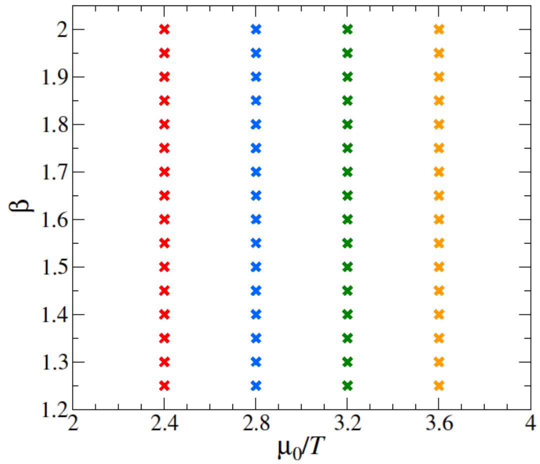
at the peak of W_0 in each simulation

Simulations

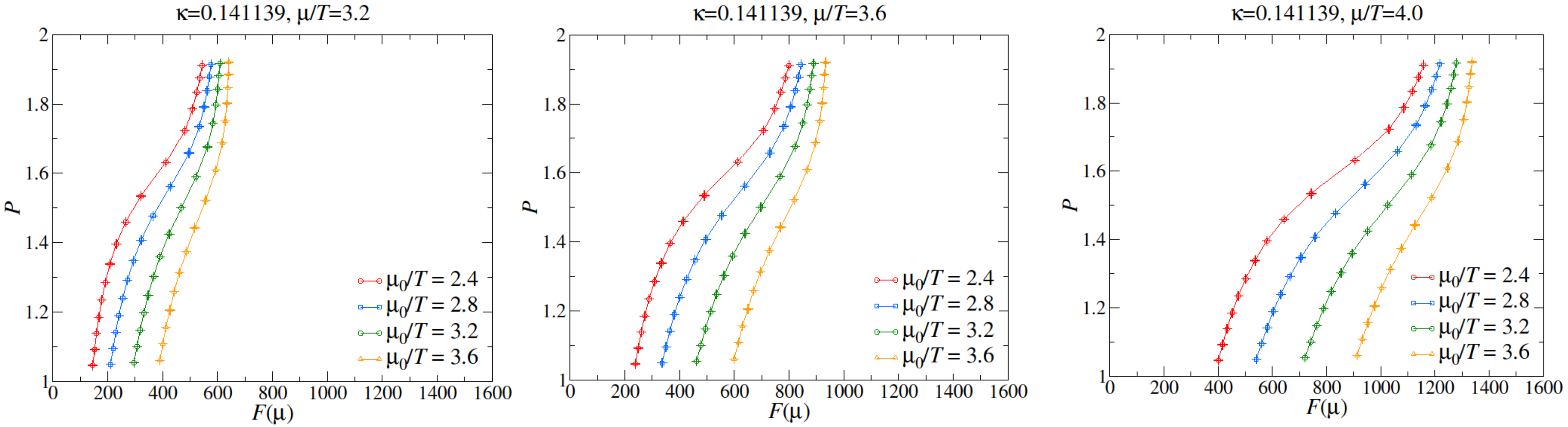
2-flavor QCD Iwasaki gauge
+ clover Wilson quark action
Random noise method is used.

$8^3 \times 4$ lattice $\quad m_\pi / m_\rho \approx 0.8$

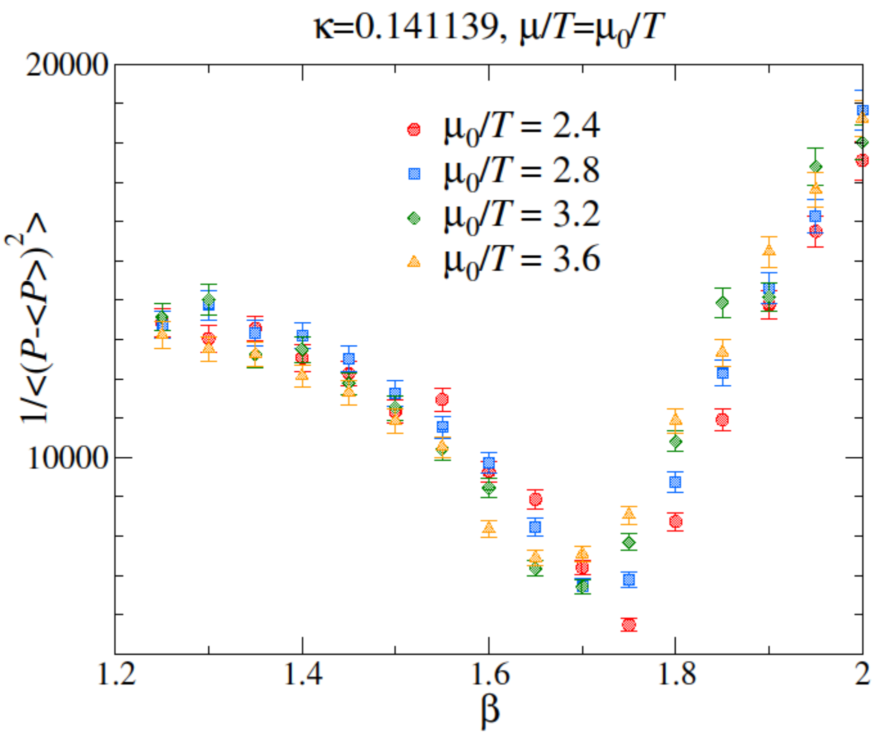
- Simulation point in the (β, μ_0/T)

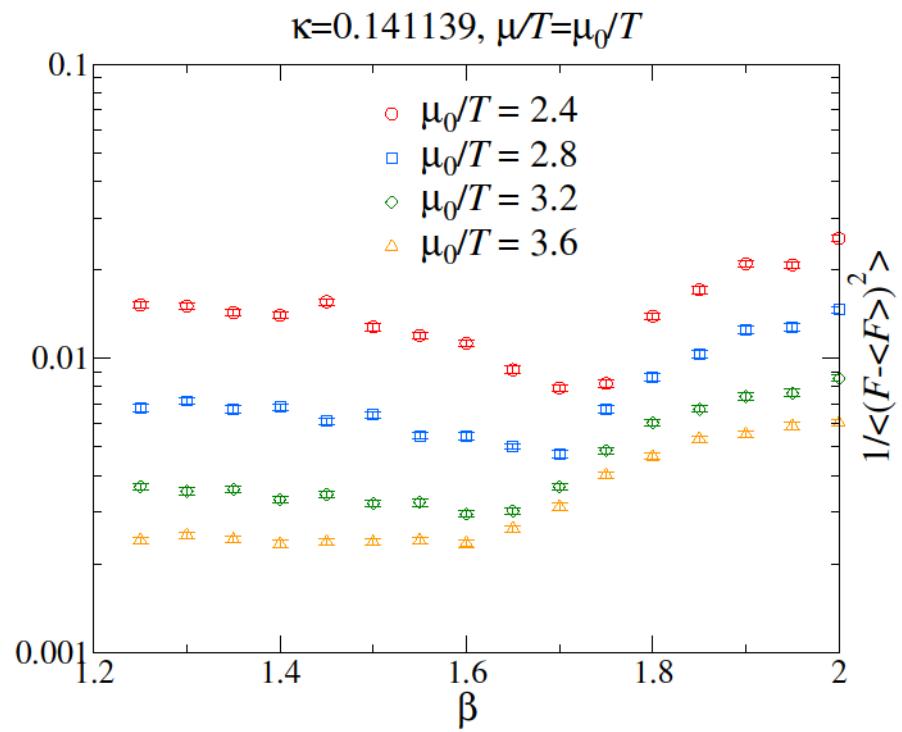


- Peak of $W_0(P,F)$ for each μ



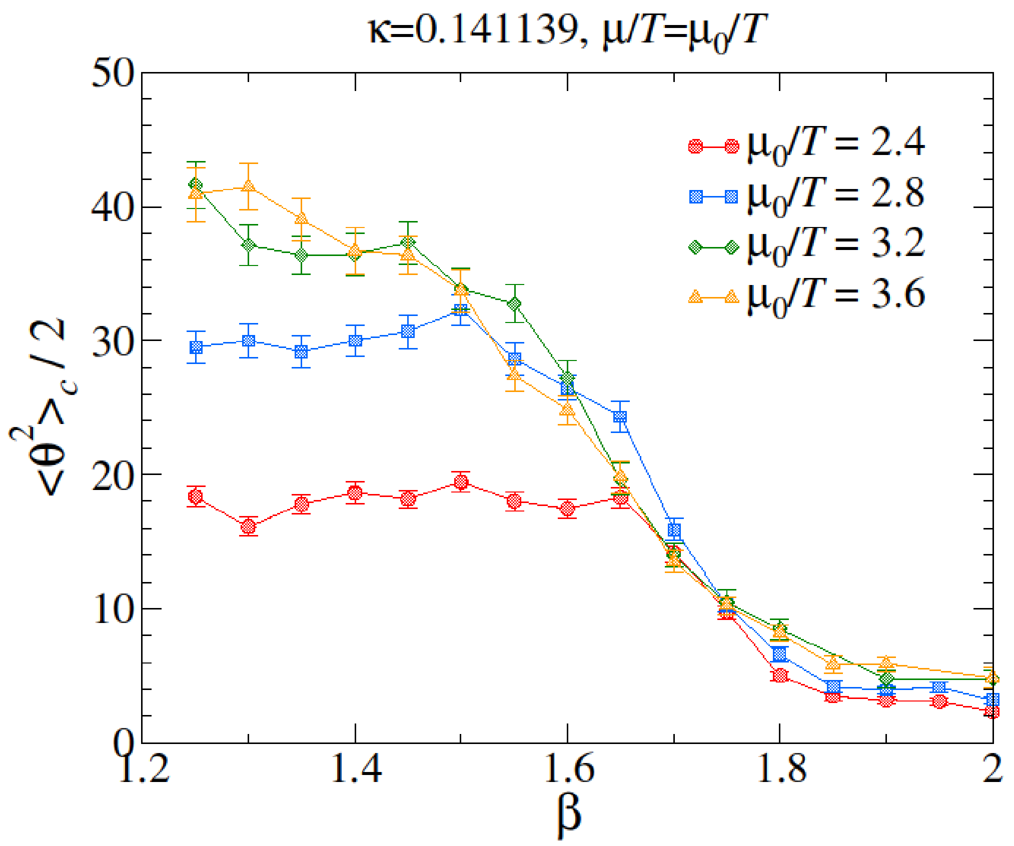
Curvature of the effective potential -lnW_0





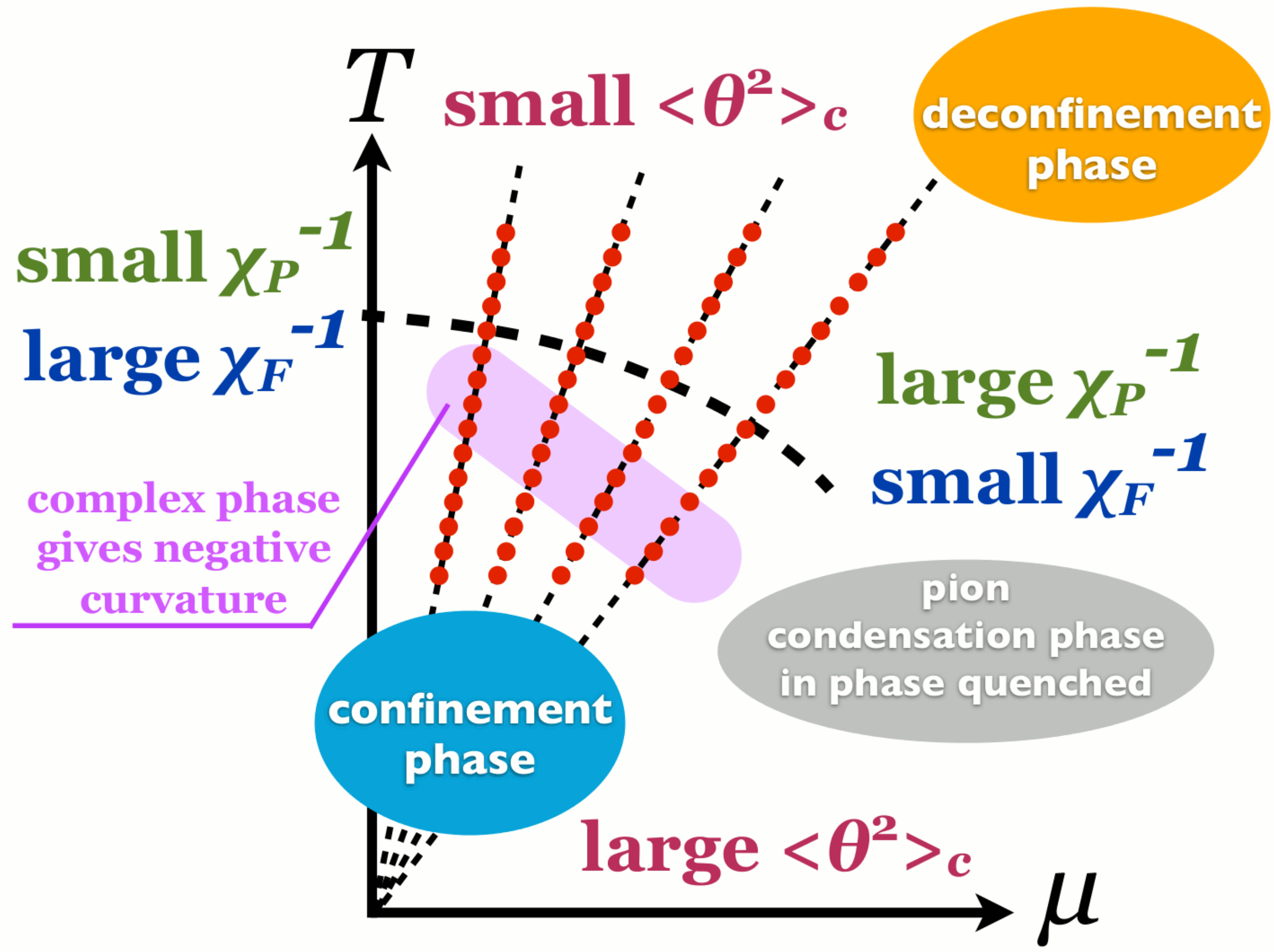
- The curvature for F decreases as μ increases.

Effect from the complex phase



- Rapidly changes around the pseudo-critical point.

Critical point at finite μ

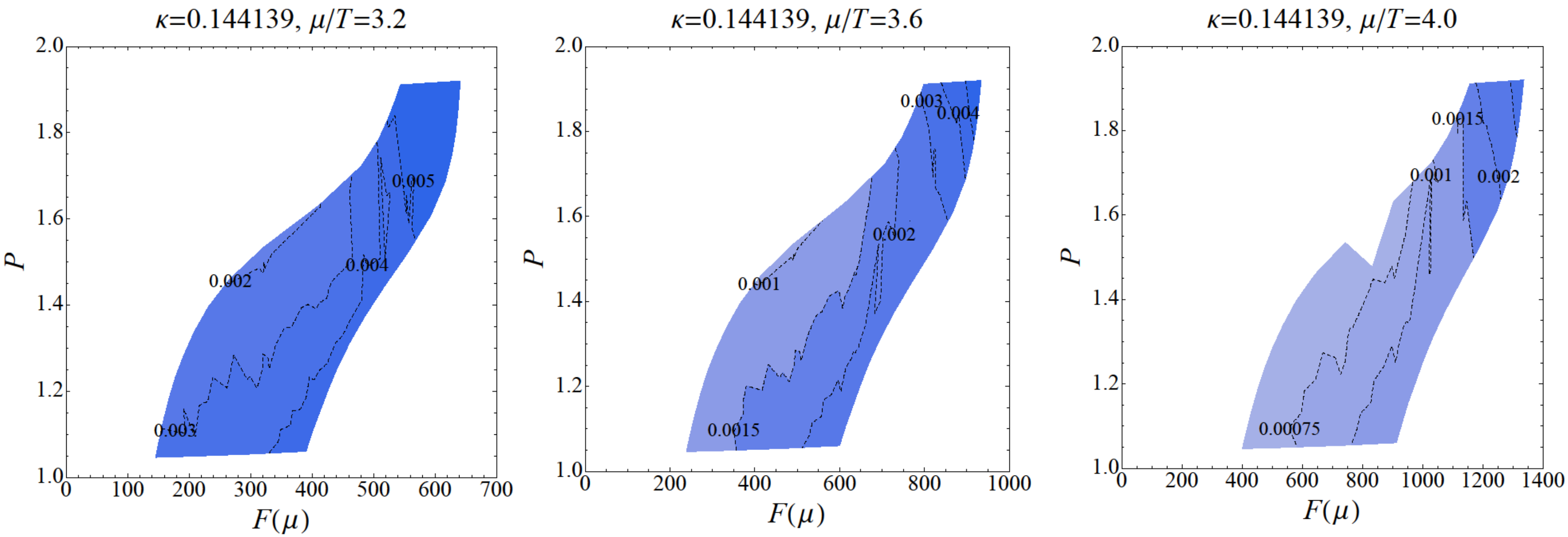


- zero curvature: expected at a large μ.

Curvature of the effective potential

- Without the complex phase effect

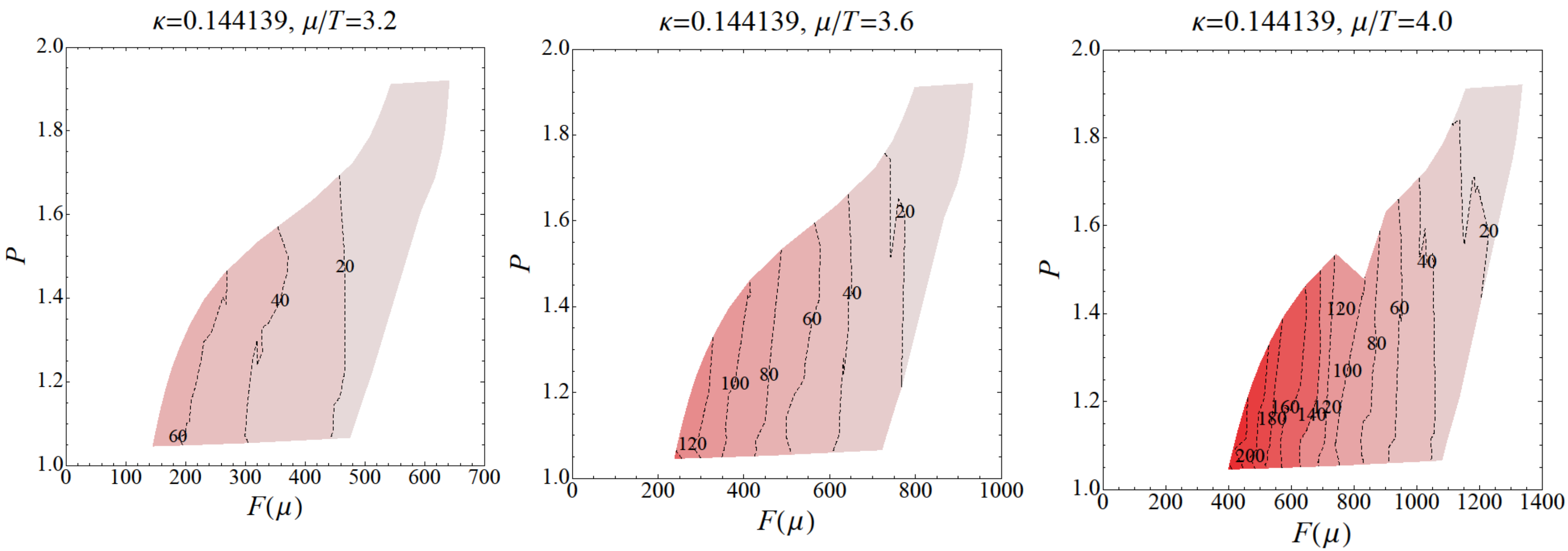
$$\boxed{\frac{\partial^2(-\ln W_0)}{\partial F^2}(\langle P\rangle,\langle F\rangle)\approx\frac{N_{\rm site}}{\chi_F}}\qquad \chi_F=N_{\rm site}\left\langle\left(F-\langle F\rangle\right)^2\right\rangle$$



Phase average

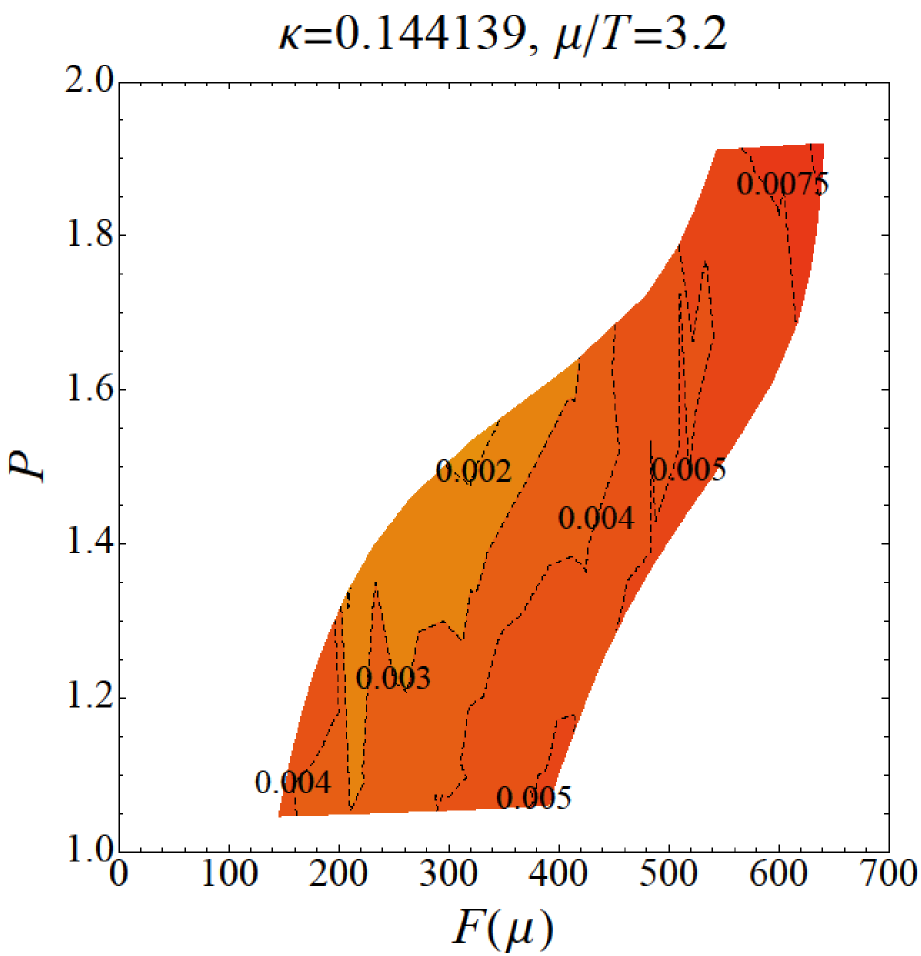
- 2nd order cumulant

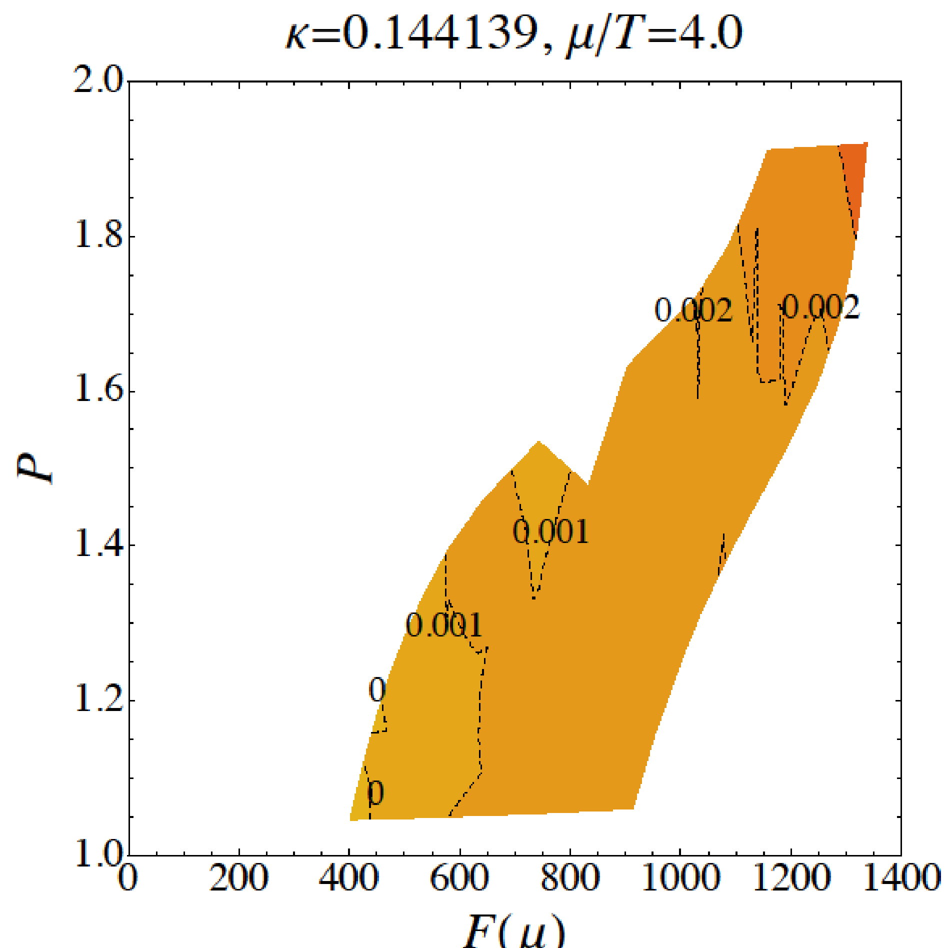
$$\ln\left\langle e^{i\theta}\right\rangle_{P,F} \approx -\frac{1}{2}\left\langle\theta^2\right\rangle_{P,F} \qquad \frac{1}{2}\left\langle\theta^2\right\rangle_{\langle P\rangle,\langle F\rangle} \approx \frac{1}{2}\left\langle\theta^2\right\rangle_{\beta_0,\mu_0}$$



Curvature of the effective potential

- The effect of the phase incruded.

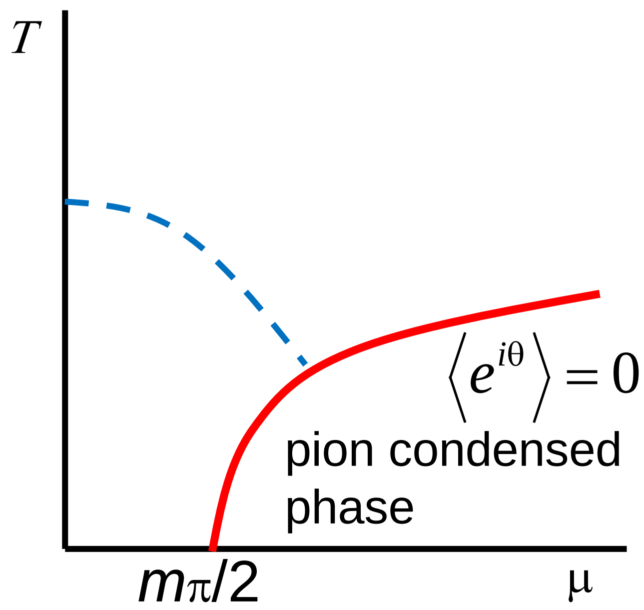


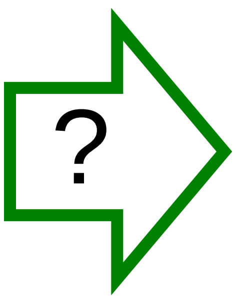


QCD phase diagram

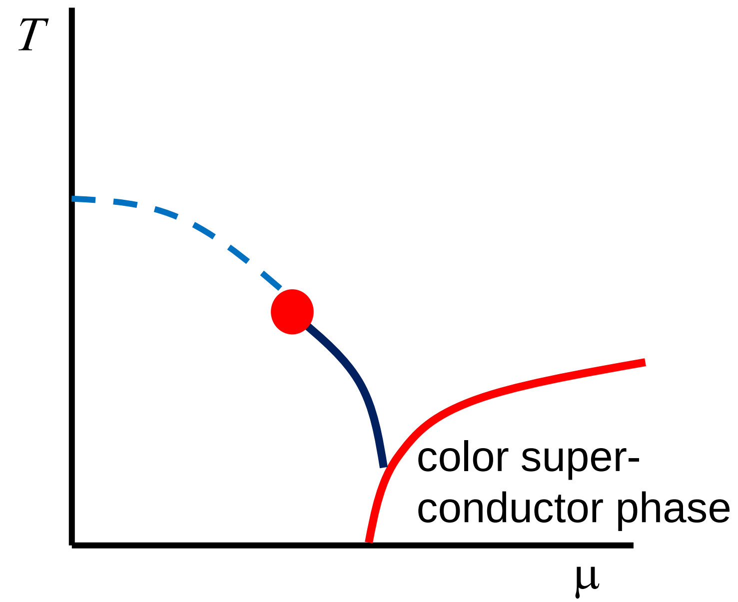
$$\boxed{W_0(P,F,\beta,m,\mu)\times\left\langle e^{i\theta}\right\rangle_{P,F}=W(P,F,\beta,m,\mu)}$$

phase-quenched QCD





finite-density QCD



Summary

- We studied the quark mass and chemical potential dependence of the nature of QCD phase transition.

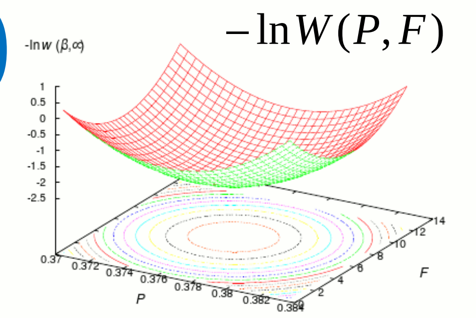
- The shape of the probability distribution function changes as a function of the quark mass and chemical potential.

- To avoid the sign problem, the method based on the cumulant expansion of θ is useful.

- To find the critical point at finite density, further studies in light quark region are important applying this method.

Backup

Peak position of $W(P,F)$



- The slopes are zero at the peak of $W(P,F)$. $\quad \dfrac{\partial \ln W}{\partial P}=0,\quad \dfrac{\partial \ln W}{\partial F}=0$

$$\left(R(P,F,\mu,\mu_0)=\frac{W_0(P,F,\beta,\mu)}{W_0(P,F,\beta,\mu_0)} \right)$$

$$\frac{\partial \ln W}{\partial P}(P,F,\beta,\mu)=\frac{\partial \ln W_0}{\partial P}(P,F,\beta,\mu)+\frac{\partial \ln\left\langle e^{i\theta}\right\rangle_{P,F}}{\partial P}$$

$$=\frac{\partial \ln W_0}{\partial P}(P,F,\beta_0,\mu_0)+\underbrace{6N_{site}(\beta-\beta_0)+\frac{\partial \ln R}{\partial P}(P,F,\mu,\mu_0)+\frac{\partial \ln\left\langle e^{i\theta}\right\rangle_{P,F}}{\partial P}}_{=0}$$

$$\frac{\partial \ln W}{\partial F}(P,F,\beta,\mu)=\frac{\partial \ln W_0}{\partial F}(P,F,\beta,\mu)+\frac{\partial \ln\left\langle e^{i\theta}\right\rangle_{P,F}}{\partial F}$$

$$=\frac{\partial \ln W_0}{\partial F}(P,F,\beta_0,\mu_0)+\underbrace{\frac{\partial \ln R}{\partial F}(P,F,\mu,\mu_0)+\frac{\partial \ln\left\langle e^{i\theta}\right\rangle_{P,F}}{\partial F}}_{=0}$$

If these terms are canceled,

$$\boxed{W(P,F,\beta,\mu)\approx W_0(P,F,\beta_0,\mu_0)\times(\text{const.})}$$

- $W(\beta,\mu)$ can be computed by simulations around (β_0,μ_0).

Complex phase

- Gaussian distribution $\rightarrow$ The cumulant expansion is good.
- We define the phase

$$\theta(\mu) = N_f \,\mathrm{Im} \ln \frac{\det M(\mu)}{\det M(0)} = N_f \int_0^{\mu/T} \mathrm{Im}\left[\frac{\partial \ln \det M}{\partial(\mu/T)}\right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right)$$

  – The range of θ is from -∞ to ∞.

- At the same time, we calculate *F* as a function of μ,

$$F(\mu) = N_f \ln\left|\frac{\det M(\mu)}{\det M(0)}\right| = N_f \int_0^{\mu/T} \mathrm{Re}\left[\frac{\partial \ln \det M}{\partial(\mu/T)}\right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right)$$

- The reweighting factor is also computed,

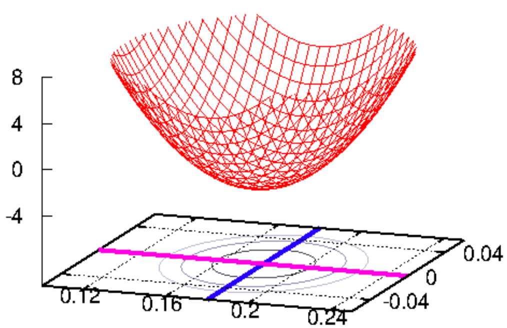
$$C(\mu) = N_f \ln\left|\frac{\det M(\mu)}{\det M(\mu_0)}\right| = N_f \int_{\mu_0/T}^{\mu/T} \mathrm{Re}\left[\frac{\partial \ln \det M}{\partial(\mu/T)}\right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right)$$

Order of phase transitions and Distribution function

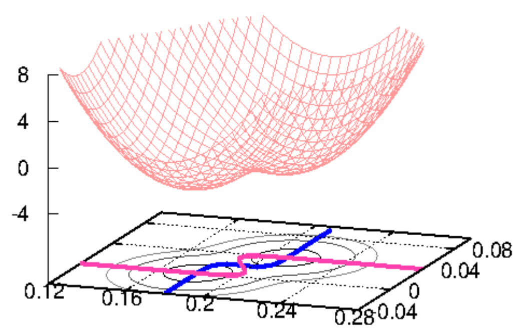
$$\boxed{W(P,\Omega_R,\beta,\kappa) = \int DU\ \delta\left(P-\hat{P}\right)\delta\left(\Omega_R-\hat{\Omega}_R\right)\left(\det M(\kappa)\right)^{N_\mathrm{f}}\ e^{-6N_\mathrm{site}P}}$$

$$V_\mathrm{eff}(P,\Omega_R;\beta,\kappa) = -\ln W(P,\Omega_R;\beta,\kappa)$$

- Peak position of W: $\dfrac{dV_\mathrm{eff}}{dP} = \dfrac{dV_\mathrm{eff}}{d\Omega_R} = 0$

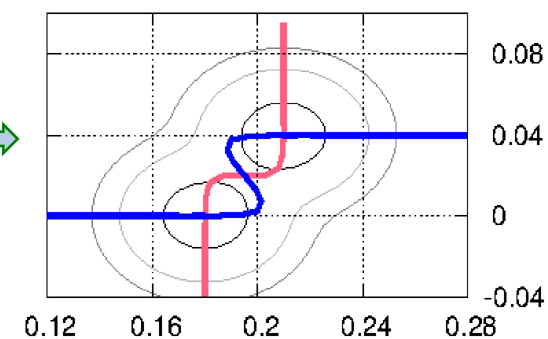


crossover
1 intersection



first order transition
3 intersections

Lines of zero derivatives
for first order



Derivatives of $V_{\rm eff}$ in terms of P and Ω_R

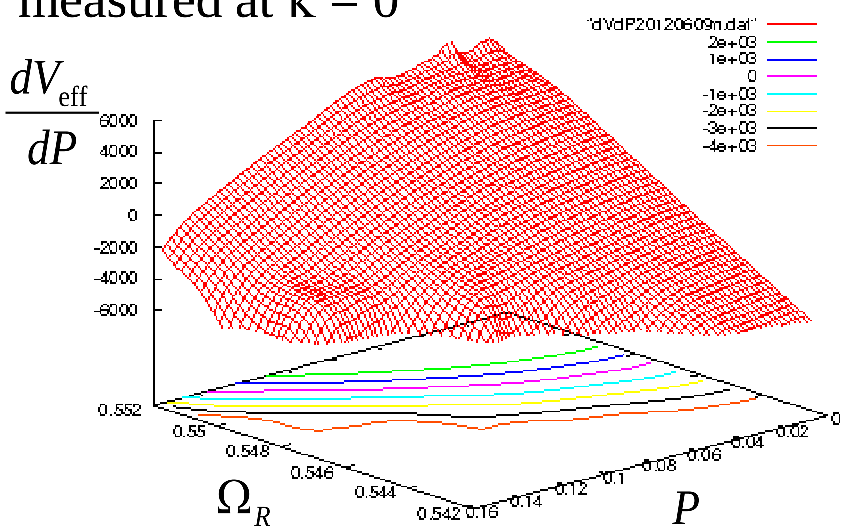
Phase-quenched part: when $\ln\langle e^{i\theta}\rangle$ is neglected,

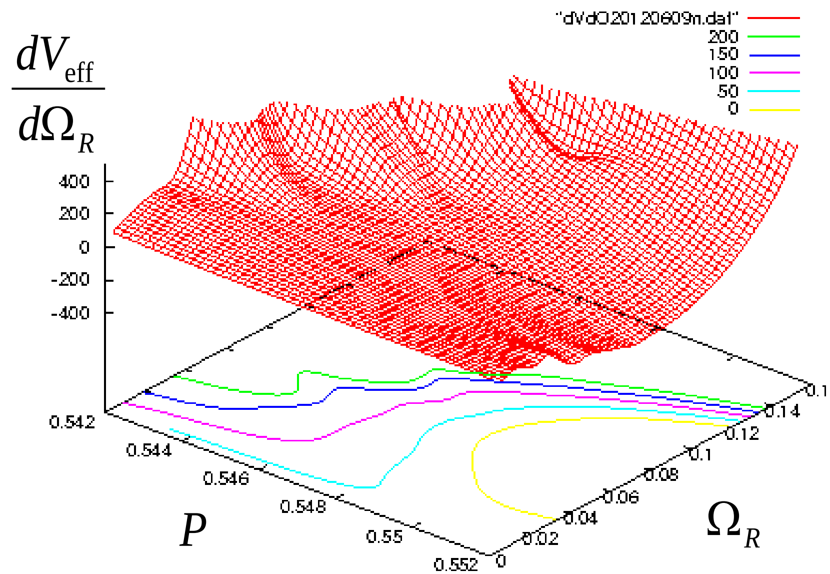
$$V_{\rm eff}(\beta,\kappa)-V_{\rm eff}(\beta_0,0)=-\left(6(\beta-\beta_0)+288N_{\rm f}K^4\right)N_{\rm site}P-12\times 2^{N_t}N_{\rm f}N_s^3K^{N_t}\cosh(\mu/T)\Omega_R$$

$$\frac{dV_{\rm eff}(\beta,K)}{dP}-\frac{dV_{\rm eff}(\beta_0,0)}{dP}=\underbrace{-\left(6(\beta-\beta_0)+288N_{\rm f}K^4\right)N_{\rm site}}_{\text{constant shift}}$$

$$\frac{dV_{\rm eff}(\beta,K)}{d\Omega_R}-\frac{dV_{\rm eff}(\beta_0,0)}{d\Omega_R}=\underbrace{-12\times 2^{N_t}N_{\rm f}N_s^3K^{N_t}\cosh\left(\frac{\mu}{T}\right)}_{\text{constant shift}}$$

measured at κ = 0



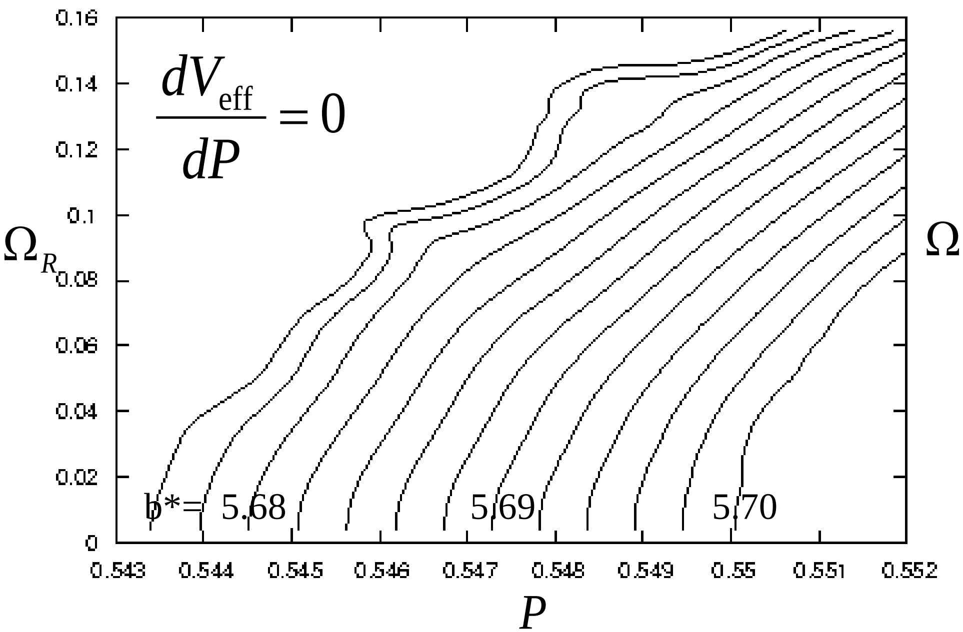


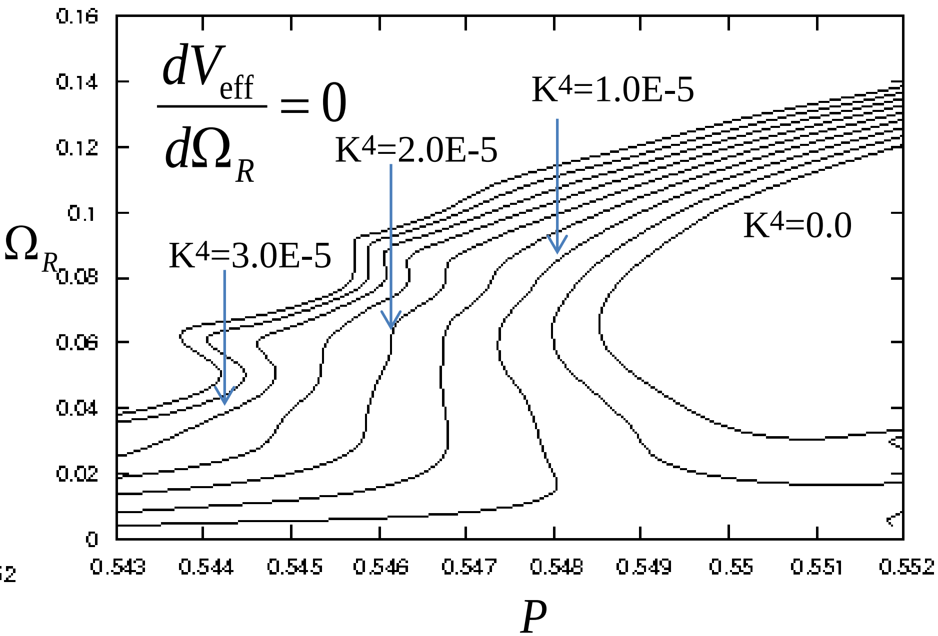
- Contour lines of $\frac{dV_{\rm eff}}{dP}$ and $\frac{dV_{\rm eff}}{d\Omega_R}$ at (β,κ) = (β0,0) correspond to the lines of the zero derivatives at (β,κ).

lines of $\frac{dV_{\text{eff}}}{dP}=0$ and $\frac{dV_{\text{eff}}}{d\Omega_R}=0$ in the (P,Ω) plane

$$\frac{dV_{\text{eff}}(\beta_0,0)}{dP}=6N_{\text{site}}\left(\beta-\beta_0+48N_{\text{f}}K^4\right)=6N_{\text{site}}\left(\beta^*-\beta_0\right)$$

$$\frac{dV_{\text{eff}}(\beta_0,0)}{d\Omega_R}=12\times 2^{N_t}N_{\text{f}}N_s^3K^{N_t}\cosh\left(\frac{\mu}{T}\right)$$





- Small K: lines of $\frac{dV_{\text{eff}}}{d\Omega_R}=0$: S-shape ➡ first order
- Large K: lines of $\frac{dV_{\text{eff}}}{d\Omega_R}=0$: straight line ➡ crossover