Vector Supersymmetry and Non-linear Realizations[1]

Joaquim Gomis

Departament d'Estructura i Constituents de la Matéria
Universitat de Barcelona

Roberto Casalbuoni Fest
GGI, September 21, 2012

[1] Based on R. Casalbuoni, J. Gomis, K. Kamimura and G. Longhi, arXiv:0801.2702, JHEP (2008) R. Casalbuoni, F. Elmetti, J. Gomis, K. Kamimura, L. Tamassia and A. Van Proeyen, arXiv:0812.1982, JHEP (2009) and unpublished work

Outline

Super Poincare group. Generators

$$M_{\mu\nu}, P_\mu, Q_\alpha$$

Geometrical dynamical objects invariant under this symmetry
Superparticle[2]
Green-Schwarz (GS) string[3]

An alternative formulation
Ramond-Neveu-Schwarz (RNS) string[4]
Spinning particle[5]

Does Exist an space-time supersymmetry group with odd vector Generators associated to these objects?

$$M_{\mu\nu}, P_\mu, G_\mu, G_5$$

[2] Casalbuoni (1976), Brink Schwarz (1981)
[3] Green, Schwarz (84)
[4] Neveu, Schwarz (71), Ramond (71), Gervais, Sakita (1971), Deser, Zumino (76), Brink, Di Vecchia, Howe (1976)
[5] Barducci, Casalbuoni, Lusanna (76), Berezin (77), Brink, Deser, Zumino, Di Vecchia, Howe (77)

Super Space-time

The space-time is identified with the (super)translations of an space-time graded Lie algebra

Ordinary Superspace= IIASuperPoincare/Lorentz

We have the translations P_m, Q_α The flat susyalgebra is, $m = 0, 1, ...9, \alpha = 1, ..., 32$

$$\begin{aligned}
[M_{mn},\ M_{rs}] &= -i\,\eta_{nr}\ M_{ms} + ... \\
[M_{mn},\ P_r] &= -i\,\eta_{[nr}\ P_{m]} \\
[P_m,\ P_r] &= 0 \\
[Q,\ M_{mn}] &= \frac{i}{2}\ Q\Gamma_{mn}, \\
[Q,\ P_m] &= 0, \\
\{Q, Q\} &= 2C\Gamma^m\ P_m
\end{aligned}$$

where C is the charge conjugation matrix,

$$\{\Gamma^m, \Gamma^n\} = +2\eta^{mn} = +2(-; +...+),$$

We parametrize the coset as

$$g = e^{iP_m x^m} e^{iQ_\alpha \theta^\alpha}.$$

The MC form

$$\Omega = -i\, g^{-1}\, d\, g = P_m L^m + \frac{1}{2} M_{mn} L^{mn} + Q_\alpha L^\alpha$$

$$L^m = dX^m + i\bar{\theta}\Gamma^m d\theta,\ L^\alpha = d\theta^\alpha$$

Supersymmetry transformations

$$\delta\theta = \epsilon\,, \qquad \delta X^m = -i\bar{\epsilon}\Gamma^m\theta.$$

Vector Superspace

Vector Superspace= Vector Super Poincare/Lorentz

We have the translations P_μ, G_μ, G_5. The vector susy algebra (VSUSY)[6] is

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\rho}M_{\nu\sigma},$$

$$[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu, \qquad [M_{\mu\nu}, G_\rho] = i\eta_{\mu\rho}G_\nu - i\eta_{\nu\rho}G_\mu,$$

$$[G_\mu, G_\nu]_+ = 0, \quad [G_5, G_5]_+ = 0, \quad [G_\mu, G_5]_+ = -P_\mu.$$

We locally parameterize the coset element as

$$g = e^{iP_\mu x^\mu} e^{iG_5\xi_5} e^{iG_\mu \xi^\mu}.$$

[6] Barducci, Casalbuoni, Lusanna (76)

The Maurer-Cartan form (MC) is[7]

$$\Omega \equiv G_A L^A = P_\mu L_x^\mu + G_5 L^5 + G_\mu L_\xi^\mu + \frac{1}{2} M_{\mu\nu} L^{\mu\nu}$$

with

$$L_x^\mu = dx^\mu - i\xi^\mu d\xi^5, \quad L_\xi^\mu = d\xi^\mu, \quad L^5 = d\xi^5, \quad L^{\mu\nu} = 0.$$

MC equation

$$\begin{aligned} dL_x^\mu &= -L^\mu{}_\nu L_x^\nu - i/2 L_\xi^\mu L_\xi^5, \\ dL^{\mu\nu} &= -L^{\mu\rho} L_\rho{}^\nu, \\ dL_\xi^\mu &= -L^\mu{}_\nu L_\xi^\nu \end{aligned}$$

[7] Casalbuoni, Gomis, Kamimura, Longhi (2007)

We can construct two Lorentz scalar closed invariant 2-forms

$$\Omega_2 = L_\xi^\mu \wedge L_\xi^\nu \eta_{\mu\nu} = d\xi^\mu \wedge d\xi^\nu \eta_{\mu\nu}, \quad \tilde{\Omega}_2 = d\xi^5 \wedge d\xi^5.$$

These forms are exact, since

$$\Omega_2 = d\Omega_1 = d\left(\xi^\mu d\xi^\nu \eta_{\mu\nu}\right), \quad \tilde{\Omega}_2 = d\tilde{\Omega}_1 = d\left(\xi^5 d\xi^5\right).$$

According to Chevalley-Eilenberg cohomology this susy algebra admits two central extensions[8] given by

$$[G_\mu, G_\nu]_+ = \eta_{\mu\nu} Z, \quad [G_5, G_5]_+ = \tilde{Z}.$$

[8] Barducci, Caslabuoni, Lusanna (76)

Relation to N=2 topological algebra[9]

N=2 Susy algebra in 4d has 8 charges. The euclidean formulation has symmetry group

$$SU_L(2) \otimes SU_R(2) \otimes SU_{Rsymmetry}(2)$$

$$\begin{array}{ccc} Q_{i\alpha} & \longrightarrow & (1/2, 0, 1/2) \\ \bar{Q}^{i\dot{\alpha}} & \longrightarrow & (0, 1/2, 1/2) \end{array}$$

if we perform a twisting identifying the R-symmetry index, i, index with the space-time index,α, the new supersymmetric generators are

$$\begin{array}{ccc} Q_{i\alpha} & \longrightarrow & (1, 0) \oplus (0, 0) \longrightarrow H_{\alpha\beta} \oplus Q \\ \bar{Q}^{i\dot{\alpha}} & \longrightarrow & (1/2, 1/2) \longrightarrow G_{\alpha\dot{\beta}} \end{array}$$

[9] Witten (88)

A massive particle breaks spatial translations and boost symmetry. The coset is

$$G_{spacetime}/Rotations$$

$$g = g_L\, U, \quad U = e^{iM_{0i}v^i}$$

where g_L is the coset associated to the space, M_{0i} are the Lorentz boost generators and v^i are the corresponding Goldstone parameters. U represents a finite Lorentz boost. The Maurer-Cartan 1-form is now

$$\Omega = -ig^{-1}dg = U^{-1}\Omega_L U - iU^{-1}dU$$

$$U^{-1}P_\mu U \quad = \quad P_\nu \Lambda^\nu{}_\mu(v),$$

where $\Lambda^n{}_m(v)$ is a finite Lorentz transformation.
The new MC forms associated to the translations are

$$\tilde{L}^\nu = \Lambda^\nu{}_\mu(v) L^\mu$$

The corresponding part of action (NG) is

$$S^{NG} = m \int \left(\tilde{L}^0\right)^* = m \int \left(\Lambda^0{}_m(v) L^\mu\right)^* = \int L^{NG}\, d\tau$$

where * means pull-back on the world line and $\Lambda^0{}_\mu$ is a time-like Lorentz vector, $\Lambda^0{}_\mu \Lambda^0{}_\nu \eta^{\mu\nu} = -1$.
The canonical momentum conjugate to x^μ is

$$p_\mu = \frac{\partial L^{NG}}{\partial \dot{x}^\mu} = -m\Lambda^0{}_\mu(v).$$

Since $\Lambda^0{}_\mu(v)$ is a time-like vector, the momentum verifies the mass-shell constraint $p^2 + m^2 = 0$. If we consider as basic bosonic variables x^μ, p_μ
The action becomes

$$L^{NG} = p_\mu(\dot{x}^\mu - i\xi^\mu \dot{\xi}^5) - \frac{e}{2}(p^2 + m^2),$$

where e is a lagrange multiplier (ein-bein).

We have also two WZ terms There are two Lorentz scalar 1-forms that are quasi-invariant

$$\Omega_1 = \xi^\mu d\xi^\nu \eta_{\mu\nu}, \quad \tilde{\Omega}_1 = \xi^5 d\xi^5.$$

The WZ action in this case

$$S^{WZ} = \beta \int \Omega_1^* + \gamma \int \tilde{\Omega}_1^*$$

The action for an spinning particle is

$$S = S^{NG} + S^{WZ}$$

the lagrangian becomes

$$L^C = p_\mu(\dot{x}^\mu - i\xi^\mu \dot{\xi}^5) - \beta \frac{i}{2}\xi_\mu \dot{\xi}^\mu - \gamma \frac{i}{2}\xi^5 \dot{\xi}^5 - \frac{e}{2}(p^2 + m^2),$$

The Dirac brackets are

$$\{p_\mu, x^\nu\}^* = -\delta_\mu{}^\nu, \qquad \{\xi^\mu, \xi^\nu\}^* = \frac{i}{\beta}\eta^{\mu\nu}, \qquad \{\pi_5, \xi^5\}^* = -1.$$

If $\beta\gamma = -m^2$,there are two first class constraints

$$\phi = \frac{1}{2}(p^2 + m^2), \quad \chi_5 = (\pi_5 - \gamma\frac{i}{2}\xi^5 - ip_\mu\xi^\mu)$$

$$\{\chi_5, \chi_5\}^* = i\gamma - \frac{i}{\beta}p_\mu p_\nu \eta^{\mu\nu} = -\frac{2i}{\beta}\,\phi.$$

χ_5 generates the local kappa variation, world line supersymmetry[10].

[10] Gervais, Sakita (1971)

The generators of the global supersymmetries are

$$
\begin{aligned}
G_\mu &= \pi_\mu + \beta \frac{i}{2}\xi_\mu + i p_\mu \xi^5 = i\beta\xi_\mu + i p_\mu \xi^5, \\
G_5 &= \pi_5 + \gamma \frac{i}{2}\xi^5.
\end{aligned}
$$

They satisfy

$$
\{G_\mu, G_\nu\}^* = -i\beta\eta_{\mu\nu}, \quad \{G_\mu, G_5\}^* = -i p_\mu, \quad \{G_5, G_5\}^* = -i\gamma.
$$

$$
Z = -\beta, \qquad \tilde{Z} = -\gamma, \qquad Z\tilde{Z} = -m^2.
$$

Quantization

We quantize in a covariant manner by requiring the first class constraints to hold on the physical states. The Dirac brackets are replaced by the following graded-commutators,

$$[p_\mu, x^\nu] = -i\delta_\mu{}^\nu, \qquad [\xi^\mu, \xi^\nu]_+ = -\frac{1}{\beta}\eta^{\mu\nu}, \quad [\pi_5, \xi^5]_+ = -i.$$

The odd variables define a Clifford algebra. We define

$$\lambda^\mu = \sqrt{-2\beta}\xi^\mu, \quad \lambda^5 = -i\sqrt{\frac{2}{\gamma}}\left(\pi_5 - \frac{i}{2}\gamma\xi^5\right), \quad \lambda^6 = -i\sqrt{\frac{2}{\gamma}}\left(\pi_5 + \frac{i}{2}\gamma\xi^5\right).$$

Note $(\lambda^A)^* = \lambda^A$.
The λ^A's define a Clifford algebra C_6,

$$[\lambda^A, \lambda^B]_+ = 2\bar{\eta}^{AB}, \quad \bar{\eta}^{AB} = (-,+,+,+,+,-), \quad (A, B = 0, 1, 2, 3, 5, 6).$$

These variables can be identified as a particular combination of the elements of another C_6 algebra with generators Γ^μ's isomorphic to the Dirac matrices

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}, \qquad \Gamma^6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$[\Gamma^A, \Gamma^B]_+ = 2\tilde{\eta}^{AB}, \quad \tilde{\eta}^{AB} = (+,-,-,-,+,-).$$

$$\lambda^A = \begin{array}{ll} \Gamma^A\Gamma^5, & A = 0,1,2,3, \\ \Gamma^5, & A = 5, \\ i\Gamma^5\Gamma^6, & A = 6. \end{array} .$$

both Clifford algebras have the same automorphism group $SO(4,2)$.

Susy generators

$$G^\mu = i\beta\xi^\mu + ip^\mu\xi^5 = \frac{i}{\sqrt{2\gamma}}\left(m\lambda^\mu - p^\mu(\lambda^5 - \lambda^6)\right)$$
$$= \frac{i}{\sqrt{2\gamma}}\Gamma^5\left(m\Gamma^\mu - p^\mu(1 - i\Gamma^6)\right)$$

$$G_5 = \pi_5 + i\frac{\gamma}{2}\xi^5 = i\sqrt{\frac{\gamma}{2}}\lambda^6 = -\sqrt{\frac{\gamma}{2}}\Gamma^5\Gamma^6.$$

The expression for the odd first class constraint

$$\chi_5 = \pi_5 - i\frac{\gamma}{2}\xi^5 - ip_\mu\xi^\mu = \frac{i}{\sqrt{-2\beta}}\left(m\lambda^5 - p_\mu\lambda^\mu\right) = \frac{i}{\sqrt{-2\beta}}\Gamma^5(p_\mu\Gamma^\mu + m).$$

The requirement that the first class constraint χ_5 holds on the physical states is equivalent to require the Dirac equation on an 8-dimensional spinor Ψ

$$(p_\mu\Gamma^\mu + m)\Psi = 0.$$

The other first class constraint, $\phi = \frac{1}{2}(p^2 + m^2) = 0$, is then automatically satisfied.

If we impose a further constraint[11]

$$\pi_5 + i\frac{\gamma}{2}\xi^5 = 0.$$

[11] Barducci, Casalbuoni, Lusanna (1976)

Relation to other models

Consider the lagrangian[12]

$$L^B = p\cdot\dot{x} - \beta\frac{i}{2}\xi_\mu\dot{\xi}^\mu + \gamma\frac{i}{2}\xi^5\dot{\xi}^5 - \frac{e}{2}(p^2+m^2) - i\rho(p\cdot\xi+\gamma\xi^5)$$

where ρ is a Lagrange multiplier and $\beta\gamma = -m^2$ as in L^C. The quantization can be done by using only a C_5 algebra which can be realized in a 4-dimensional space and we recover the 4d Dirac equation.

$$\chi\psi(x) = -\sqrt{\frac{-1}{2\beta}}\gamma^5\,(p_\mu\gamma^\mu - m)\psi(x) = 0,$$

where

$$\xi^\mu = \sqrt{\frac{-1}{2\beta}}\,\gamma^\mu\gamma^5, \qquad \xi^5 = \sqrt{\frac{1}{2\gamma}}\,\gamma^5.$$

[12] Brink, Di Vecchia, Howe (1976)

The lagrangian L^B also has the local fermionic invariance generated by the first class $\chi = 0$,

$$\begin{aligned} \delta x^\mu &= i\xi^\mu \kappa^5(\tau), \quad \delta\xi^\mu = -\frac{1}{\beta} p^\mu \kappa^5(\tau), \quad \delta\xi^5 = \kappa^5(\tau), \\ \delta e &= \frac{2i}{\beta}\rho\kappa^5(\tau), \quad \delta\rho = -\dot{\kappa}^5(\tau). \end{aligned}$$

and the global vector fermionic invariance,

$$\begin{aligned} \delta\xi^\mu &= \epsilon^\mu, \quad \delta\xi^5 = -\frac{1}{\gamma}(p_\mu \epsilon^\mu), \quad \delta x^\mu = i\epsilon^\mu \xi^5, \end{aligned}$$

$$\delta L^B = \frac{d}{d\tau}\left(\frac{i}{2}\epsilon^\mu (p_\mu \xi^5 - \beta\xi_\mu)\right).$$

The generator is, after removing the second class constraints,

$$G_\mu = i(\beta\xi_\mu + p_\mu \xi^5)$$

and satisfies

$$\{G_\mu, G_\nu\}^* = -i\beta(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}).$$

The commutator of the gauge symmetry and the rigid transformation by vanishes.

If we consider the model with higher order derivates obtained from L^B by eliminating ξ^5 and quantize the model[13] ones obtains two Dirac equations for mass $\pm m$ like the L^C lagrangian, probably this model has VSUSY symmetry.

[13] Gomis, Paris, Roca (1991)

Casimirs

The central charges Z and $\tilde{Z}$ are trivial Casimirs of VSUSY. It is also easy to see that P^2 is a Casimir for VSUSY. The correct VSUSY generalization of the Pauli-Lubanski vector, W^μ, is[14]

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu (Z M_{\rho\sigma} - G_\rho G_\sigma)\,, \tag{1}$$

whose square $\hat{W}^2$ is a Casimir. By introducing the new vector

$$W_C^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu G_\rho G_\sigma\,, \tag{2}$$

$$Z W^\mu = \hat{W}^\mu + W_C^\mu\,. \tag{3}$$

One can easily prove that W_C^2 is also a Casimir.

$$W_C^2 = Z^2 P^2\,\frac{3}{4}\,. \tag{4}$$

[14] Casalbuoni, Elmetti, Gomis, Kamimura, Tamassia, Van Proeyen (2008)

We are implicitly considering the case of representations with $Z \neq 0$. In that case Z is just a number and can be divided out. Therefore, in the rest frame of the massive states where $P^2 = -m^2$, they satisfy the rotation algebra

$$\left[\frac{W_*^i}{m}, \frac{W_*^j}{m} \right]_- = \epsilon^{ijk} \, \frac{W_{*k}}{m} \,, \tag{5}$$

and define three different spins. The superspin Y labels the eigenvalues $-m^2 Z^2 Y(Y+1)$ of the Casimir $\hat{W}^2$. The spin associated to W_C^2 (C-spin) is fixed to $1/2$, as one can see from (4). Finally, we denote the usual Lorentz spin by s.

On the other hand, only $W_*^\mu = \frac{1}{Z} \hat{W}^\mu$ commutes with G_λ, and thus only the superspin Y characterizes a multiplet. Since

$$\left[\hat{W}^\mu, W_C^\nu \right]_- = 0 \,,$$

one can immediately obtain the particle content of a VSUSY multiplet by using the formal theory of addition of angular momenta applied to (3). A multiplet of superspin Y contains two particles of Lorentz spin $Y \pm 1/2$, for $Y > 0$ integer or half-integer. In the degenerate case of superspin $Y = 0$, the multiplet consists of two spin $1/2$ states.

In particular, we observe that a VSUSY multiplet contains either only particles of half-integer Lorentz spin or only particles of integer Lorentz spin. The spinning particle constructed is a realization of the degenerate case $Y = 0$.

	eigenvalue			vector superp
$\frac{1}{Z^2}\hat{W}^2$	$-m^2\ Y(Y+1)$	$\text{superspin} = Y$	Casimir	$Y = 0$
$\frac{1}{Z^2}W_C^2$	$-m^2\ \frac{3}{4}$	$\text{C spin} = \frac{1}{2}$	Casimir	$C = \frac{1}{2}$
W^2	$-m^2\ s(s+1)$	$\text{Lorentz spin} = s = \lvert Y \pm \frac{1}{2}\rvert$	not Casimir	$s = \frac{1}{2}$

Some field theory representations of VSUSY has been presented[15]

[15] Casalbuoni, Elmetti, Tamassia (2010)

Odd "Casimir"

If

$$Z\tilde{Z} + m^2 = 0\,. \tag{6}$$

$$Q = G \cdot P + G_5 Z\,. \tag{7}$$

Q acts as an odd constant on the states in the previous representations. Unless the model under consideration has a natural odd constant, Q has to annihilate all states in those representations. As a result, the physical role of the odd 'Casimir' is to give a Dirac-type equation for the particle states.

Is the RNS string VSUSY invariant?

The fields of the RNS sigma model are the target space bosonic coordinate X^μ ($\mu = 1, \cdots, D$), being an even vector from the spacetime point of view and D even scalars from the worldsheet point of view, and the fermions $\psi^\mu_\pm$, being two odd vectors from the spacetime point of view and $2D$ Majorana-Weyl spinors from the worldsheet point of view. The action of RNS string in the conformal gauge is

$$S = \frac{1}{\pi} \int d^2\sigma \left(2\partial_+ X^\mu \partial_- X_\mu + i\psi^\mu_+ \partial_- \psi_{+\mu} + i\psi^\mu_- \partial_+ \psi_{-\mu}\right) \tag{8}$$

and must be supplemented with the super-Virasoro constraints:

$$\begin{aligned} J_+ &= \psi^\mu_+ \partial_+ X_\mu = 0 \\ T_{++} &= \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi^\mu_+ \partial_+ \psi_{+\mu} = 0 \end{aligned} \tag{9}$$

and similar for the $-$ sector.

We can then promote our VSUSY charges to worldsheet spinors,

$$Q^\mu \quad \to \quad \begin{pmatrix} Q_+^\mu \\ Q_-^\mu \end{pmatrix}$$

and construct an $\mathcal{N} = 2$ VSUSY algebra

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\rho}M_{\nu\sigma} \,; \\
&[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu \,; \qquad [M_{\mu\nu}, Q_{\pm\rho}] = i\eta_{\mu\rho}Q_{\pm\nu} - i\eta_{\nu\rho}Q_{\pm\mu} \,; \\
&\{Q_{\pm\mu}, Q_{\pm\nu}\} = Z_\pm\eta_{\mu\nu} \,; \qquad \{Q_{\pm5}, Q_{\pm5}\} = \tilde{Z}_\pm \,; \qquad \{Q_{\pm\mu}, Q_{\pm5}\} = -P_\mu
\end{aligned} \tag{10}$$

The transformations

$$\begin{aligned}
\delta_{V+}X^\mu &= 0 & \delta_{V-}X^\mu &= 0 \\
\delta_{V+}\psi_+^\mu &= \epsilon_+^\mu & \delta_{V-}\psi_+^\mu &= 0 \\
\delta_{V+}\psi_-^\mu &= 0 & \delta_{V-}\psi_-^\mu &= -\epsilon_-^\mu
\end{aligned} \tag{11}$$

$$\begin{aligned}
\delta_{5+}X^\mu &= -i\epsilon_{5+}\psi_+^\mu & \delta_{5-}X^\mu &= i\epsilon_{5-}\psi_-^\mu \\
\delta_{5+}\psi_+^\mu &= 2\epsilon_{5+}\partial_+X^\mu & \delta_{5-}\psi_+^\mu &= 0 \\
\delta_{5+}\psi_-^\mu &= 0 & \delta_{5-}\psi_-^\mu &= -2\epsilon_{5-}\partial_-X^\mu,
\end{aligned} \tag{12}$$

where the parameters $\epsilon_\pm^\mu, \epsilon_\pm^5$ are real odd constants, are realization of N=2VSUSY.

The previous N=2 VSusy transformations leave the action invariant but the super-Virasoro constraints are not Vsusy invariant.

The RNS string model is obtained after gauge-fixing from

$$\mathcal{L}_{\text{Brink}} = e\left[-\frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu g^{\alpha\beta} - \frac{i}{2}\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu \right. \\ \left. + \frac{i}{2}\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu \partial_\beta X_\mu + \frac{1}{8}(\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu)(\bar{\chi}_\beta \psi_\mu)\right] , \qquad (13)$$

which is invariant under local worldsheet supersymmetry and local Weyl transformations (details follow). In this notation α is a “curved” worldsheet vector index, *a* is a “flat” worldsheet vector index and worldsheet spinor indices are not explicitly written.
We have been unable to find a type of VSUSY rigid for this action, unfortunately at this moment we can not prove the non-existence of this type of symmetry.

A N=1 VSUSY string from Non-linear Realizations

Here we are going to use the MC form of VSUSY to construct an VSUSY invariant string action. We consider the coset $G/O(3,1)$ which we locally parameterize the coset as

$$g = e^{iP_\mu x^\mu} e^{iG_5 \xi^5} e^{iG_\mu \xi^\mu} e^{iZc} e^{i\tilde{Z}\tilde{c}},$$

therefore the coordinates of the superspace are $x^\mu,\ \xi^\mu,\ \xi^5$. The MC 1-form is

$$\begin{aligned}\Omega = -ig^{-1}dg &= P_\mu \left(dx^\mu - i\xi^\mu d\xi^5\right) + G_5 d\xi^5 + G_\mu d\xi^\mu + \\ &+ Z\left(dc - \frac{i}{2} d\xi_\mu \xi^\mu\right) + \tilde{Z}\left(d\tilde{c} - \frac{i}{2} d\xi^5 \xi^5\right).\end{aligned}$$

The even MC 1-forms are

$$L_x^\mu = dx^\mu - i\xi^\mu d\xi^5, \quad L_Z = dc + \frac{i}{2}\xi^\mu d\xi_\mu, \quad L_{\tilde{Z}} = d\tilde{c} + \frac{i}{2}\xi^5 d\xi^5$$

whereas the odd ones are given by

$$L_\xi^\mu = d\xi^\mu, \quad L_\xi^5 = d\xi^5.$$

One way to construct a VSUSY string action is to conside the two dimensional Polyakov $g_{\alpha\beta}$ world sheet metric with and the the MC 1-forms of vecorsuperspace.

$$\mathcal{L}^1 = -\frac{\sqrt{-\det g}}{2}\, g^{\alpha\beta}\, \eta_{\mu\nu} v^\mu_\alpha v^\nu_\beta, \qquad v^\mu_\alpha \equiv \partial_\alpha x^\mu - i\xi^\mu \partial_\alpha \xi^5.$$

Upon integration over $g_{\alpha\beta}$ we obtain the NG forms,

$$\mathcal{L}^1 = -\sqrt{-\det G_{\alpha\beta}}, \qquad G_{\alpha\beta} \equiv \eta_{\mu\nu} v^\mu_\alpha v^\nu_\beta.$$

A possible WZ term for N=1 is given

$$L^{WZ} = i\,\epsilon^{\alpha\beta}\,\partial_\alpha x^\mu\,\xi_\mu\,\partial_\beta \xi^5 - 2i\,c\,\epsilon^{\alpha\beta}\,\partial_\alpha \xi^5\,\partial_\beta \xi^5.$$

The WZ term verifies

$$dL^{WZ} = -i\,L^\mu_P d\xi_\mu d\xi^5 - 2i\,L_c\,d\xi^5\,d\xi^5,$$

an invariant three form.

We consider as complete lagrangian

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^1 + \mathcal{L}^{WZ} \\ &= -\sqrt{-\det G} + i\,\epsilon^{\alpha\beta}\,\partial_\alpha x^\mu\,\xi_\mu\,\partial_\beta \xi^5 - 2i\,c\,\epsilon^{\alpha\beta}\,\partial_\alpha \xi^5\,\partial_\beta \xi^5. \end{aligned}$$

If we perform the analysis of constraints and we work in the reduced space $(x^\mu, p_\nu, \xi^5, \pi_5)$. The canonical lagrangian is

$$\begin{aligned} L_c &= p\dot{x} + \pi_5 \dot{\xi}^5 - \mathcal{H} \\ &= p\dot{x} + \pi_5(\partial_0 - \lambda_+ \partial_1)\xi^5 - \frac{\lambda_+}{4}(p + x')^2 + \frac{\lambda_-}{4}(p - x')^2. \end{aligned}$$

where

$$\begin{aligned} H^+ &= \frac{1}{4}(p + x')^2 + \pi_5\,\xi^{5'}, \\ H^- &= -\frac{1}{4}(p - x')^2. \end{aligned}$$

and λ_-, λ_+ are arbitrary functions.
Notice that we have the ordinary bosonic lagragian of the string plus a "bc" like system.

Therefeore in the conformal gauge

$$\lambda_+ = -\lambda_- = 1$$

we have a conformal theory

$$\begin{aligned} L_{con} &= p\dot{x} - \frac{1}{2}(p^2 + x'^2) + \pi_5(\partial_0 - \partial_1)\xi^5. \\ &= \frac{1}{2}(\dot{x}^2 - x'^2) + \pi_5(\partial_0 - \partial_1)\xi^5. \end{aligned}$$

Notice that this model is chiral, we have only right movers fermions

The VSUSY generators in the reduced space are

$$
\begin{aligned}
G_\mu &= \int d\sigma \left(-i\xi^5(p_\mu + x_\mu{}')\right), \\
G_5 &= \int d\sigma\,(\pi_5)\,.
\end{aligned}
$$

They verify

$$
\begin{aligned}
\{G_5, G_5\} &= 0, \\
\{G_\mu, G_5\} &= \int d\sigma\,(-i(p_\mu + x'_\mu)) = -i\int d\sigma\, p_\mu, \\
\{G_\mu, G_\nu\} &= \eta_{\mu\nu}\int d\sigma \left(-2\,\xi^5\xi^{5\prime}\right) \equiv \eta_{\mu\nu}\, iZ.
\end{aligned}
$$

where

$$
Z = \int d\sigma \left(2i\,\xi^5\xi^{5\prime}\right), \qquad \tilde{Z} = 0.
$$

The VSUSY transformations are

$$
\delta x^\mu = -i\,\xi^5\,\epsilon^\mu, \quad \delta p^\mu = -i\,\xi^{5\prime}\,\epsilon^\mu, \quad \delta\pi_5 = -i\,(p + x')_\mu\,\epsilon^\mu, \quad \delta\xi^5 = \epsilon^5.
$$

If consider a non-chiral model

$$
\begin{aligned}
L_{conc} &= p\dot{x} - \frac{1}{2}(p^2 + x'^2) + \pi(\partial_0 - \partial_1)\xi + \underline{\pi}(\partial_0 + \partial_1)\underline{\xi} \\
&= \frac{1}{2}(\dot{x}^2 - x'^2) + \pi(\partial_0 - \partial_1)\xi + \underline{\pi}(\partial_0 + \partial_1)\underline{\xi}.
\end{aligned}
$$

It is invariant under N=2 VSUSY,

$$
\begin{aligned}
\delta x^\mu &= -i\,\xi\,\epsilon^\mu - i\,\underline{\xi}\,\underline{\epsilon}^\mu, \quad \delta\pi = -i\,(\dot{x} + x')_\mu\,\epsilon^\mu, \quad \delta\underline{\pi} = -i\,(\dot{x} - x')_\mu\,\underline{\epsilon}^\mu, \\
\delta\xi &= \epsilon^5, \quad \delta\underline{\xi} = \underline{\epsilon}^5.
\end{aligned}
$$

If we introduce a 2D notaion

$$\Psi = \begin{pmatrix} \xi \\ \underline{\xi} \end{pmatrix}, \quad \Pi = \begin{pmatrix} \pi \\ \underline{\pi} \end{pmatrix}$$

$$L_{conc} \quad = \quad -\frac{1}{2}(\partial_\alpha x^\mu \, \partial^\alpha x_\mu) + i \, \overline{\Pi} \, \gamma^\alpha \, \partial_\alpha \, \Psi, \qquad \overline{\Pi} \equiv \Pi \, C.$$

Note this action is similar to the NSR string but the Ψ does not have Lorentz index μ. In addition Π and Ψ are independent and can transform differently under VSUSY.

Conclusions

- Vector supersymmetry appears naturally in the spinning particle model of[16]
- It seems RNS string is not VSUSY invariant
- We can construct VSUSY string theories, whose physical meaning should be further analyzed.

Grazie Roberto

[16] Barducci, Casalbuoni, Lusanna (1976)