# Quasiconformal approach to higher spin superalgebras and minimal unitary representations

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- Review of the oscillator method for constructing unitary representations of noncompact groups and supergroups.
- Applications to AdS/Conformal superalgebras in 5/4 and 7/6 dimensions.
- Quasiconformal realizations of noncompact Lie algebras and superalgebras
- Quantization of quasiconformal group actions and minimal unitary representations.
- Minimal unitary representations of spacetime supergroups, their deformations and Joseph ideal
- "Interacting" higher spin superalgebras as universal enveloping algebras of quasiconformal $AdS_{n+1}/CFT_n$ superalgebras and their deformations
- Comments and open problems

# Oscillator construction of the positive energy unitary representations of noncompact groups

Simple Lie groups $G$ that admit positive energy ( lowest weight) unitary representations are in one to one correspondence with the irreducible Hermitian symmetric spaces $G/K \times U(1)$. The Lie algebra $g$ of $G$ has a three graded decomposition with respect to the generator $E$ of $U(1)$

$$g = g^{-1} \oplus g^{0} \oplus g^{+1}$$

$$[g^{(m)}, g^{(n)}] \subseteq g^{(m+n)} \qquad\qquad m, n = \mp 1, 0$$

The generators of $G$ are realized as bilinears of bosonic oscillators transforming in a certain representation of $K$. In the Fock space $\mathcal{F}$ one chooses a set of lowest energy states $|\Omega>$ which transforms irreducibly under $H$ and which are annihilated by the generators in $g^{-1}$ space

$$g^{-1}|\Omega >= 0 \ , \ g^{0}|\Omega >= |\Omega' >$$

Then by acting on the lowest energy irrep $|\Omega>$ repeatedly with the generators in $g^{+1}$ space one obtains an infinite set of states

$$|\Omega >, \quad g^{+1}|\Omega >, \quad g^{+1}g^{+1}|\Omega >, \dots$$

which forms the basis of an irreducible unitary lowest weight representation of $G$. The irreducibility of the representation of $G$ follows from the irreducibility of $|\Omega>$ under $K \times U(1)$.

Table: The complete list of simple non-compact groups G that admit positive energy unitary representations:

| $G$ | $K \times U(1)$ |
|---|---|
| $SU(p,q)$ | $S(U(p) \times U(q))$ |
| $Sp(2n,R)$ | $U(n)$ |
| $SO^*(2n)$ | $U(n)$ |
| $SO(n,2)$ | $SO(n) \times SO(2)$ |
| $E_{6(-14)}$ | $SO(10) \times U(1)$ |
| $E_{7(-25)}$ | $E_6 \times U(1)$ |

Special isomorphisms of conformal groups in 3,4 and 6 dimensions:

$$SO(3,2) \equiv Sp(4,\mathbb{R}) \quad SO(4,2) \equiv SU(2,2) \quad SO(6,2) \equiv SO^*(8)$$

The coordinates in 3, 4 and 6 dimensions can be represented by $2 \times 2$ Hermitian matrices $\sigma_\mu$ over reals $\mathbb{R}$ , complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ :

$$x = x^\mu \sigma_\mu$$

# Oscillator Construction of the Positive Energy Representations of $SU(2,2)$

$$g^0 = SU(2)_L \times SU(2)_R \times U(1)_E \in SU(2,2)$$

The generator $E$ is the AdS energy operator in $d=5$ (or the conformal Hamiltonian in $d=4$ whose eigenvalues give the conformal dimensions). The oscillators satisfy the canonical commutation relations

$$[a_i(\xi), a^j(\eta)] = \delta_i^j \delta_{\xi\eta} \quad i,j = 1,2.$$

$$[b_r(\xi), b^s(\eta)] = \delta_r^s \delta_{\xi\eta} \quad r,s = 1,2$$

Here $\xi, \eta = 1, ..., P$ label different generations ( colors) of oscillators

$$a_i(\xi)|0\rangle = 0 = b_r(\xi)|0\rangle$$

The non-compact generators of $SU(2,2)$ are realized by the following bilinears

$$L_{ir} = \vec{a}_i \cdot \vec{b}_r \quad , \quad L^{ir} = \vec{a}^i \cdot \vec{b}^r$$

where $\vec{a}_i \cdot \vec{b}_r = \sum_{\xi=1}^{P} a_i(\xi) b_r(\xi)$ etc. They close into the generators of the compact subgroup $SU(2)_L \times SU(2)_R \times U(1)$

$$[L_{ir}, L^{js}] = \delta_r^s L_i^j + \delta_i^j R_r^s + \delta_i^j \delta_r^s E$$

$$L_i^j = \vec{a}^j \cdot \vec{a}_i - \frac{1}{2}\delta_i^j \vec{a}^l \cdot \vec{a}_l \quad R_s^r = \vec{b}^r \cdot \vec{b}_s - \frac{1}{2}\delta_s^r \vec{b}^t \cdot \vec{b}_t$$

$$E = \frac{1}{2}(\vec{a}_i \cdot \vec{a}^i + \vec{b}^r \cdot \vec{b}_r)$$

Defining the number operators

$$N_a = \vec{a}^i \cdot \vec{a}_i = \sum_{i=1}^{2} \sum_{\xi=1}^{P} a^i(\xi) a_i(\xi); \quad N_b = \vec{b}^r \cdot \vec{b}_r; \quad N = N_a + N_b$$

we can write the AdS energy operator E as

$$E = \frac{1}{2}(N_a + N_b + 2P) = \frac{1}{2} N + P$$

The quadratic Casimir operator $C_2$ of $SU(2,2)$ is uniquely defined up to an overall multiplicative constant. We choose this constant such that

$$C_2 = -\frac{1}{2}(L_{ir} L^{ir} + L^{ir} L_{ir}) + \frac{1}{2}(L^i_j L^j_i + R^r_s R^s_r + E^2)$$

This expression can be rewritten in terms of number operators $N_a, N_b$ and $N$ for the case of $P = 1$

$$C_2 = (\frac{N}{2} + 1)(\frac{N}{2} - 3) + N_a(\frac{N_a}{2} + 1) + N_b(\frac{N_b}{2} + 1)$$

The positive energy irreducible unitary representations of $SU(2,2)$ are uniquely defined by a " lowest energy representation" $|\Omega\rangle$ transforming irreducibly under the maximal compact subgroup $S(U(2) \times U(2))$ and that is annihilated by $L_{ir}$

$$L_{ir} |\Omega\rangle = 0$$

Then by acting on $|\Omega\rangle$ repeatedly with the generators $L^{ir}$ one generates an infinite set of states

$$|\Omega\rangle, \quad L^{ir}|\Omega\rangle, \quad L^{ir} L^{js}|\Omega\rangle, \ldots$$

that form the basis of a unitary irreducible representation of $SU(2,2)$.

# Coherent state basis of ULW representations of $SU(2,2)$ and $4D$ conformal fields

The conformal Lie algebra has a non-compact three grading determined by the generator of scale transformations $\mathcal{D}$:

$$su(2,2) = K_\mu \oplus (SL(2,\mathbb{C}) \times \mathcal{D}) \oplus P_\mu$$

The connection between the ULW representations constructed in the compact basis above and the conformal fields in $4d$ transforming covariantly with respect to $SL(2,\mathbb{C})$ with a definite scale dimension ( eigenvalues of $D$ ) is established via the interwining operator $T = e^{\frac{\pi}{4}M_{05}}$ It intertwines between compact $(SU(2)_L \times SU(2)_R)$ and noncompact $SL(2,\mathbb{C})$ pictures. Furthermore $T$ intertwines the momentum generators $P_\mu$ with the "raising" operators $L^{ir}$ and the special conformal generators $K_\mu$ with the "lowering" operators $L_{ir}$.
The oscillators $a_i(a^i)$ and $b_r(b^r)$ that transform in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representation of $SU(2)_L \times SU(2)_R$ get intertwined with the covariant oscillators transforming as Weyl spinors $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of $SL(2,\mathbb{C})$, respectively. Denoting the left and right-handed spinors of $SL(2,\mathbb{C})$ with undotted $(\eta_\alpha,\ \lambda^\alpha)$ and dotted Greek indices $(\tilde{\eta}_{\dot{\alpha}}, \tilde{\lambda}^{\dot{\beta}})$ one finds

$$(\sigma^\mu P_\mu)^{\alpha\dot{\beta}} = \lambda^\alpha \tilde{\lambda}^{\dot{\beta}}, \quad (\bar{\sigma}^\mu K_\mu)_{\alpha\dot{\beta}} = \eta_\alpha \tilde{\eta}_{\dot{\beta}} \ .$$

They satisfy the Lorentz covariant commutation relations

$$[\eta_\alpha, \lambda^\beta] = \delta^\beta_\alpha, \quad [\tilde{\eta}_{\dot{\alpha}}, \tilde{\lambda}^{\dot{\beta}}] = \delta^{\dot{\beta}}_{\dot{\alpha}} \ .$$

Action of $T$ intertwines the lowest energy irrep $|\Omega(j_L, j_R, E_0)\rangle$ annihilated by $L_{ir}$ in the compact basis with a state annihilated by the special conformal generators $K_\mu$:

$$L_{ir}|\Omega(j_L, j_R, E_0)\rangle = 0 \quad \Longrightarrow K_\mu T|\Omega(j_L, j_R, E_0)\rangle = 0$$

Thus the state $|\Phi^\ell_{j_M,j_N}(0)\rangle := T|\Omega(j_L, j_R, E_0)\rangle$ transforms irreducibly under the isotropy group $\mathcal{H}$ with $SL(2,\mathbb{C})$ quantum numbers $(j_M, j_N) = (j_L, j_R)$, conformal dimension $\ell = -E_0$ and trivially represented special conformal transformations ( $\kappa_\mu = 0$). Acting on the states $|\Phi^\ell_{j_M,j_N}(0)\rangle$ with the translation operators $e^{-ix^\mu P_\mu}$ one obtains a coherent state labelled by the coordinate $x_\mu$

$$e^{-ix^\mu P_\mu}|\Phi^\ell_{j_M,j_N}(0)\rangle \equiv |\Phi^\ell_{j_M,j_N}(x_\mu)\rangle \ .$$

These coherent states correspond to states created by the action of conformal fields $\Phi^\ell_{j_M,j_N}(x_\mu)$ acting on the vacuum vector $|0\rangle$,

$$\Phi^\ell_{j_M,j_N}(x^\mu)|0\rangle \cong |\Phi^\ell_{j_M,j_N}(x_\mu)\rangle \ ,$$

where the compact $((j_L, j_R, E_0))$ and covariant labels $(j_M, j_N, -l)$ coincide.
The ULW representations of of $SU(2,2)$ with vanishing four-dimensional Poincaré mass $m^2 = P_\mu P^\mu$ are those obtained by taking only one color of oscillators ($a_i$ and $b_r$) and were called doubleton representations.
The coordinates labelling the coherent states can be taken as elements of the Jordan algebra of $2\times 2$ Hermitian matrices over the complex numbers $x = x^\mu \sigma_\mu \ \in J_2^{\mathbb{C}}$ with the symmetric Jordan product taken as $1/2$ the anticommutator.

# Symmetry Groups of Generalized Spacetimes defined by Simple Jordan algebras

| $J$ | $Rot(J)$ | $Lor(J)$ | $Conf(J)$ | $Kor(J)$ |
|---|---|---|---|---|
| $J_2^{\mathbb{C}}$ | $SU(2)$ | $SL(2,\mathbb{C})$ | $SU(2,2)$ | $SU(2)\times SU(2)$ |
| $J_n^{\mathbb{R}}$ | $SO(n)$ | $SL(n,\mathbb{R})$ | $Sp(2n,\mathbb{R})$ | $SU(n)$ |
| $J_n^{\mathbb{C}}$ | $SU(n)$ | $SL(n,\mathbb{C})$ | $SU(n,n)$ | $SU(n)\times SU(n)$ |
| $J_2^{\mathbb{H}}$ | $USp(4)$ | $SU^*(4)$ | $SO^*(8)$ | $SU(4)$ |
| $J_n^{\mathbb{H}}$ | $USp(2n)$ | $SU^*(2n)$ | $SO^*(4n)$ | $SU(2n)$ |
| $J_3^{\mathbb{O}}$ | $F_4$ | $E_{6(-26)}$ | $E_{7(-25)}$ | $E_6$ |
| $\Gamma_{(1,d)}$ | $SO(d)$ | $SO(d,1)$ | $SO(d+1,2)$ | $SO(d+1)$ |

Table: The complete list of simple Euclidean Jordan algebras and their rotation ( automorphism), "Lorentz" ( reduced structure) and "Conformal" ( linear fractional) groups. The last row gives the compact real forms of the Lorentz groups. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. $J_n^{\mathbb{A}}$ denotes a Jordan algebra of $n\times n$ hermitian matrices over $\mathbb{A}$. $\Gamma_{(1,d)}$ denotes the Jordan algebra of Dirac gamma matrices.

# Linear Fractional Groups of Jordan Algebras as Generalized Conformal Groups

On a d-dimensional Euclidean space ($d > 2$) with a non-degenerate positive definite quadratic form $(x,x)$ the conformal transformations leave invariant the following cross-ratio associated with any set of four vectors $x, y, z, w$:

$$\frac{(x-z,x-z)}{(x-w,x-w)}\frac{(y-w,y-w)}{(y-z,y-z)} \tag{1}$$

as well as the quantity $\frac{(x,y)^2}{(x,x)(y,y)}$.

Conformal group $Conf(J)$ of a Euclidean Jordan algebra of degree $p$ with the norm form $N(x) \equiv N(x,x,..,x)$ leaves invariant the cross-ratio

$$\frac{N(x-z)}{N(y-z)}\frac{N(y-w)}{N(x-w)}$$

as well as the measure of $p$-angle defined as

$$\frac{N(x_1,..x_p)^p}{N(x_1)N(x_2)\cdots N(x_p)} \tag{2}$$

$Conf(J)$ leaves invariant light-like separations with respect to "distance function" defined by the norm form:

$$d(x,y) \equiv N(x-y)$$

Coordinates are represented by the elements of $J$ and coherent states corresponding to generalized conformal fields are labelled by $J$.

# $AdS/CFT$: Aspen Summer 1984

- The Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5$ was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet of $N = 8$ $AdS_5$ superalgebra $PSU(2,2\,|\,4)$.
- The CPT self-conjugate doubleton supermultiplet of $PSU(2,2\,|\,4)$ of $AdS_5 \times S^5$ solution of IIB supergravity does not have a Poincaré limit in five dimensions and decouples from the Kaluza-Klein spectrum as gauge modes and the field theory of CPT self-conjugate doubleton supermultiplet of $PSU(2,2\,|\,4)$ lives on the boundary of $AdS_5$, which can be identified with $4D$ Minkowski space on which $SO(4,2)$ acts as a conformal group, and the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that was known to be conformally invariant. MG , Marcus (1984)
- The spectra of $11D$ supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$ were fitted into supermultiplets of the symmetry superalgebras $OSp(8\,|\,4,\mathbb{R})$ and $OSp(8^*|4)$ constructed by oscillator methods. The entire Kaluza-Klein spectra over these two spaces were obtained by tensoring the singleton and doubleton supermultiplets of $OSp(8\,|\,4,\mathbb{R})$ and $OSp(8^*|4)$, respectively.
- The relevant singleton supermultiplet of $OSp(8\,|\,4,\mathbb{R})$ and doubleton supermultiplet of $OSp(8^*|4)$ do not have a Poincaré limit in four and seven dimensions, respectively, and decouple from the respective spectra as gauge modes. Again it was proposed that field theories of the singleton and scalar doubleton supermutiplets live on the boundaries of $AdS_4$ and $AdS_7$ as superconformally invariant theories. MG, Warner (1984), MG, PvN, Warner ( 1984).
- Singletons of $Sp(4,\mathbb{R})$ are the remarkable representations of Dirac (1963). Subsequent important work of Fronsdal and collaborators.

Not all simple Lie algebras admit conformal realizations with a natural 3-graded structure with respect to a subalgebra of maximal rank. The groups $E_8$, $F_4$ and $G_2$ do not admit conformal realizations. However all simple Lie algebra admit a natural 5-graded decomposition with respect to a subalgebra of maximal rank such that grade $\pm 2$ dimensional subspaces are one dimensional.

- QUASICONFORMAL REALIZATION OF $E_{8(8)}$ MG, Koepsell, Nicolai, 2000

$$\begin{aligned} E_{8(8)} &= 1_{-2} \oplus 56_{-1} \oplus E_{7(7)} + SO(1,1) \oplus 56_{+1} \oplus 1_{+2} \\ \mathfrak{g} &= \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K \end{aligned}$$

over a space $\mathcal{T}$ coordinatized by the elements $X$ of the exceptional FTS $\mathcal{F}(J_3^{\mathbb{O}_S})$ plus an extra singlet variable $x$: $56_{+1} \oplus 1_{+2} \Leftrightarrow (X, x) \in \mathcal{T}$:

$$\begin{array}{lll} K(X) = 0 & U_A(X) = A & S_{AB}(X) = (A, B, X) \\ K(x) = 2 & U_A(x) = \langle A, X \rangle & S_{AB}(x) = 2\langle A, B \rangle x \end{array}$$

$$\tilde{U}_A(X) = \frac{1}{2}(X, A, X) - Ax$$

$$\tilde{U}_A(x) = -\frac{1}{6}\langle (X, X, X), A \rangle + \langle X, A \rangle x$$

$$\tilde{K}(X) = -\frac{1}{6}(X, X, X) + Xx$$

$$\tilde{K}(x) = \frac{1}{6}\langle (X, X, X), X \rangle + 2x^2$$

Freudenthal triple product $\Leftrightarrow (X, Y, Z)$
Skew-symmetric invariant form $\Leftrightarrow \langle X, Y \rangle = -\langle Y, X \rangle$
Quartic invariant of $E_{7(7)} \Leftrightarrow \langle (X, X, X), X \rangle$

$A, B, .. \in \mathcal{F}(J_3^{\mathbb{O}_S})$

- Geometric meaning of the quasiconformal action of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{T}$ ?
- Define a quartic norm of $\mathcal{X} = (X, x) \in \mathcal{T}$ as $\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2$
  $\mathcal{Q}_4(X)$ is the quartic norm of the underlying Freudenthal system and $X \in \mathcal{F}$.
- Define a quartic "distance" function between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ as

  $$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}))$$

  $\delta(\mathcal{X}, \mathcal{Y})$ is the "symplectic" difference of $\mathcal{X}$ and $\mathcal{Y}$ :

  $$\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y \rangle) = -\delta(\mathcal{Y}, \mathcal{X})$$

- Light-like separations $d(\mathcal{X}, \mathcal{Y}) = 0$ are left invariant under quasiconformal group action.
  $\longrightarrow$ *Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.*
- $E_{8(8)}$ is the invariance group of a quartic light-cone in 57 dimensions!

## Minimal Unitary Representations and Quasiconformal Groups:

- Quantization of the quasiconformal realization of a non-compact Lie group leads directly to its minimal unitary representation $\Rightarrow$ Unitary representation over an Hilbert space of square integrable functions of smallest number of variables possible.
- Minimal unitary representation of $E_{8(8)}$ over $L^2(\mathbb{R}^{29})$ from its geometric realization as a quasiconformal group MG, Koepsell & Nicolai 2000

$$E_{8(8)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(7)} + SO(1,1) \oplus 56_{+1} \oplus 1_{+2}$$

- Minimal unitary representation of $E_{8(-24)}$ over $L^2(\mathbb{R}^{29})$ from its geometric realization as a quasiconformal group MG, Pavlyk, 2004

$$E_{8(-24)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(-25)} + SO(1,1) \oplus 56_{+1} \oplus 1_{+2}$$

56 of $E_7 \Rightarrow$ 28 coordinates and 28 momenta. These 28 coordinates plus the singlet coordinate yield the minimal number ( 29) of variables for $E_8$.

# QCG Approach to Minimal Unitary Representations

GKN & MG, Pavlyk

- Lie algebra $\mathfrak{g}$ of a quasiconformal realization of a group $G$ can be decomposed as :

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{h} \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

$$\mathfrak{g} = E \oplus E^{\alpha} \oplus (J^a + \Delta) \oplus F^{\alpha} \oplus F$$

$\Delta = -\frac{i}{2}(yp + py)$ ($[y,p] = i$) determines the 5-grading and $\Omega^{\alpha\beta}$ is the symplectic invariant tensor of $\mathfrak{h}$ generated by $J^a$ $(\alpha, \beta, .. = 1, 2, ..., 2n)$ and $\left[\xi^{\alpha}, \xi^{\beta}\right] = \Omega^{\alpha\beta}$

$$E = \frac{1}{2} y^2 \qquad E^{\alpha} = y\,\xi^{\alpha}, \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^{\alpha}\xi^{\beta}$$

$$F = \frac{1}{2}p^2 + \frac{\kappa I_4(\xi^{\alpha})}{y^2}, \qquad F^{\alpha} = [E^{\alpha}, F]$$

$I_4(\xi^{\alpha}) = S_{\alpha\beta\gamma\delta}\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \Leftrightarrow$ *quartic invariant of* $\mathfrak{h}$

Choosing a polarization $\xi^{\alpha} = (x^i, p_j)$ one has $[x^i, p_j] = i\delta^i_j$ $(i, j = 1, 2, .., n)$

Gelfand-Kirillov dimension for the minimal unitary representation is $n + 1$ $\leftrightarrow$ $(x^i, y)$.

- $(E, F, \Delta) \Longrightarrow SL(2, \mathbb{R})$ of conformal quantum mechanics with the quartic invariant $I_4$ playing the role of coupling constant.

# Minimal unitary representation of $4d$ conformal group $SU(2,2)$

MG, Pavlyk (2006), MG, Fernando (2009/10)

The Lie algebra $\mathfrak{su}(2,2)$ admits a 5-grading:

$$\mathfrak{su}(2,2) = \mathbf{1}^{(-2)} \oplus \mathbf{4}^{(-1)} \oplus [\mathfrak{su}(1,1) \oplus \mathfrak{u}(1) \oplus \Delta] \oplus \mathbf{4}^{(+1)} \oplus \mathbf{1}^{(+2)}$$

where $J^n_m$, $U$ and $\Delta$ are the $SU(1,1)$, $U(1)$ and $SO(1,1)$ generators, respectively.

$$\mathfrak{su}(2,2) = E \oplus (E^1, E^2, E_1, E_2) \oplus [J^n_m, U, \Delta] \oplus (F^1, F^2, F_1, F_2) \oplus F\,.$$

$$J^2_1 = d\,g \quad J^1_2 = -d^\dagger g^\dagger \quad J^1_1 = -J^2_2 = \frac{1}{2}(N_d + N_g + 1) \quad \Delta = \frac{1}{2}\,(xp + px)$$

where $N_d = d^\dagger d$ and $N_g = g^\dagger g$ and $U = N_d - N_g$.

$$E^1 = x\,d^\dagger \qquad E^2 = x\,g \qquad E_1 = x\,d \qquad E_2 = -x\,g^\dagger$$

$$E = \frac{1}{2}x^2, \qquad F = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left[(N_d - N_g)^2 - \frac{1}{4}\right]$$

$$F^1 = d^\dagger\left[p + \frac{i}{x}\left(N_d - N_g + \frac{1}{2}\right)\right] \qquad F^2 = g\left[p + \frac{i}{x}\left(N_d - N_g + \frac{1}{2}\right)\right]$$

$$F_1 = d\left[p - \frac{i}{x}\left(N_d - N_g - \frac{1}{2}\right)\right] \qquad F_2 = -g^\dagger\left[p - \frac{i}{x}\left(N_d - N_g - \frac{1}{2}\right)\right]$$

where $x$ is again the singlet coordinate of the quasiconformal realization and $p$ is its conjugate momentum.

By going to the $SU(2) \times SU(2) \times U(1)$ basis one can easily show that the minimal unitary representation of $SU(2,2)$ is simply the scalar doubleton representation.
The above minrep admits a one parameter family of deformations obtained by shifting the quartic invariant of grade zero algebra $SU(1,1)$ : One parameter deformations of the minimal unitary representation are obtained by replacing the quartic invariant $\mathcal{I}_4$ by

$$\mathcal{I}_4\left(\zeta\right) = \left(N_d - N_g + \zeta\right)^2 - 1\,.$$

Then the grade $+2$ generator becomes

$$F\left(\zeta\right) = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left[\left(N_d - N_g + \zeta\right)^2 - \frac{1}{4}\right]$$

while the negative grade generators $E$, $E^m$ and $E_m$ remaining as in the undeformed case.
The grade $+1$ generators are modified by $\zeta$ dependent terms and are given by

$$\begin{aligned} F^1(\zeta) &= d^\dagger\left[p + \frac{i}{x}\left(N_d - N_g + \zeta + \frac{1}{2}\right)\right] \\ F_1(\zeta) &= d\left[p - \frac{i}{x}\left(N_d - N_g + \zeta - \frac{1}{2}\right)\right] \end{aligned}$$

The only bosonic generator in $\mathfrak{g}^{(0)}$ subspace that changes under this deformation is $U$ which becomes

$$U(\zeta) = N_d - N_g + \frac{\zeta}{2}\,.$$

The quadratic Casimir of $SU(2,2)$ takes on the value

$$\mathcal{C}_2(\zeta) = -\frac{1}{2}\left(\frac{\zeta}{2} - 1\right)\left(\frac{\zeta}{2} + 1\right)\,.$$

The Poincaré mass operator $P_\mu P^\mu$ vanishes identically for the minrep and its deformations. For integer values of $\zeta$ they are simply the doubleton irreps of $SU(2,2)$ corresponding to conformal fields in the representation $(\zeta/2, 0)$ or $(0, -\zeta/2)$ of the Lorentz group. These representations remain irreducible under the restriction to the Poincaré subgroup and describe massless particles of helicity $\zeta/2$.

## Minimal Unitary Supermultiplets of $SU(2,2|p+q)$ and their deformations

MG, Fernando (2009)

The Lie superalgebra $\mathfrak{su}\,(2,2\,|\,\mathfrak{p}+\mathfrak{q})$ has the following 5-graded decomposition with respect to its subalgebra $\mathfrak{su}\,(1,1\,|\,\mathfrak{p}+\mathfrak{q}) \oplus \mathfrak{u}\,(1) \oplus \mathfrak{so}\,(1,1)$:

$$\mathfrak{su}\,(2,2\,|\,\mathfrak{p}+\mathfrak{q}) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)}$$

$$1^{(-2)} \oplus 2\,(2,\mathfrak{p}+\mathfrak{q})^{(-1)} \oplus [\mathfrak{su}\,(1,1\,|\,\mathfrak{p}+\mathfrak{q}) \oplus \mathfrak{u}\,(1) \oplus \mathfrak{so}\,(1,1)] \quad \oplus 2\,(2,\mathfrak{p}+\mathfrak{q})^{(+1)} \oplus 1^{(+2)}$$

$$F = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left[\left(N_d - N_g + N_\alpha - N_\beta + \zeta\right)^2 - \frac{1}{4}\right].$$

$\zeta$ is the deformation parameter. For $\zeta = 0$ one obtains the minimal unitary supermultiplet.

For $PSU(2,2|4)$ the minimal unitary supermultiplet is simply the $N = 4$ Yang-Mills supermultiplet. The deformations of the Yang-Mills supermultiplet are the non-CPT self-conjugate higher spin doubleton supermultiplets ( of maximal spin range 2).

For $SU(2,2|8)$ the minimal unitary supermultiplet corresponds to the CPT self-conjugate massless $N = 8$ Poincare supermultiplet of maximal supergravity in four dimensions. The deformations of the minimal unitary supermultiplet $SU(2,2|8)$ are higher spin supermultiplets of maximal spin range 4.

# Minimal Unitary Representations of Supergroups $OSp(8^*|2N)$ with Even Subgroups $SO^*(8) \times USp(2N)$

MG, Fernando (2010)

The superalgebra $\mathfrak{osp}(8^*|2N)$ has a 5-grading

$$\mathfrak{osp}(8^*|2N) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)}$$

with respect to the subsuperalgebra

$$\mathfrak{g}^{(0)} = \mathfrak{osp}(4^*|2N) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(1,1)_\Delta \, .$$

The grade $+2$ generator is given by

$$K_+ = \frac{1}{2} p^2 + \frac{1}{4\,x^2} \left( 8\,\mathcal{T}^2 + \frac{3}{2} \right)$$

$$\mathcal{C}_2 \left[ \mathfrak{osp}(4^*|2N) \right] = \mathcal{C}_2 \left[ \mathfrak{su}(2)_{\mathcal{T}} \right] - \frac{N\,(N-4)}{16} \equiv \mathcal{T}^2 - \frac{N\,(N-4)}{16}$$

The minimal unitary supermultiplet of $OSp(8^*|4)$ is the $(2,0)$ supermultiplet whose interacting theory is believed to be dual to M-theory over $AdS_7 \times S^4$.

The deformations of the $6d$ conformal group $SO^*(8) = SO(6,2)$ and $OSp(8^*|2N)$ are labelled by the eigenvalues of the quadratic Casimir of an $SU(2)$ subgroup of the little group $SO(4)$ of massless particles in $6d$.

# Minimal unitary representations and Joseph ideals

Among all the unitary representations of a noncompact Lie group the minimal one is distinguished by the fact that it is annihilated by the Joseph ideal inside its universal enveloping algebra. Denoting the generators of the Lie algebra of $SO(n-2,2)$ as $G_{ab}$ the Joseph ideal is generated by the following elements of the enveloping algebra:

Eastwood et.al.(2005)

$$J_{abcd} = G_{ab}G_{cd} - G_{ab} \circledcirc G_{cd} - \frac{1}{2}[G_{ab}, G_{cd}] + \frac{n-4}{4(n-1)(n-2)}.\langle G_{ab}, G_{cd}\rangle$$

where $\langle G_{ab}, G_{cd}\rangle$ is the Killing form,
$G_{ab} \circledcirc G_{cd}$ is the Cartan product which for orthogonal groups ($SO(n-2,2)$) can be written as:

$$\begin{aligned} G_{ab} \circledcirc G_{cd} \quad \equiv \quad & \frac{1}{3}G_{ab}G_{cd} + \frac{1}{3}G_{dc}G_{ba} + \frac{1}{6}G_{ac}G_{bd} - \frac{1}{6}G_{ad}G_{bc} + \frac{1}{6}G_{db}G_{ca} - \frac{1}{6}G_{cb}G_{da} \\ & - \frac{1}{2(n-2)}\left(G_{ae}G^{e}_{c}\eta_{bd} - G_{be}G^{e}_{c}\eta_{ad} + G_{be}G^{e}_{d}\eta_{ac} - G_{ae}G^{e}_{d}\eta_{bc}\right) \\ & - \frac{1}{2(n-2)}\left(G_{ce}G^{e}_{a}\eta_{bd} - G_{ce}G^{e}_{b}\eta_{ad} + G_{de}G^{e}_{b}\eta_{ac} - G_{de}G^{e}_{a}\delta_{bc}\right) \\ & + \frac{1}{(n-1)(n-2)}G_{ef}G^{ef}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) \end{aligned}$$

where $\eta_{ab}$ is the $SO(n-2,2)$ invariant metric.

# Higher spin algebras and superalgebras

Early work on the connection between high spin (super)algebras and the universal enveloping algebras of singleton representations of $AdS$ Lie(super)algebras

MG (1989); Konstein and Vasiliev (1989/1990)

A more modern definition by Vasiliev (2003) is as follows: Bekaert (2012)
The quotient of the universal enveloping algebra $\mathcal{U}(o(n-2,2))$ of $o(n-2,2)$ by its annihilator on the scalar singleton module is the $AdS_{n-1}/CFT_{n-2}$ higher-spin algebra.

Since the scalar singleton module corresponds to the minimal unitary representation whose annihilator is the Joseph ideal I will adopt Vasiliev's definition and define the higher spin algebras of generalized space-times defined by Jordan algebras $J$ as the universal enveloping algebras $\mathcal{U}(J)$ of their conformal groups quotiented by their Joseph ideals. These space-times include the standart Minkowskian space-times with conformal groups $SO(n-2,2)$.
Now consider the minimal unitary representations obtained by quantization of the quasiconformal realization of $SO(4,2), SO(6,2)$ as well as $SO(3,2)$. One finds that the Josephian $J_{abcd}$ that generates the Joseph ideal vanishes identically as operators for these quasiconformal realizations of their minreps:

Govil & MG (2013)

$$J_{abcd} \equiv 0$$

Hence their universal enveloping algebras yield directly the higher spin algebras in the respective dimensions!

- Quasiconformal realizations of the minimal unitary representations of non-compact groups and supergroups are *interacting* realizations since they contain generators which are cubic and quartic in the oscillators. Most of the work on higher spin algebras until now have utilized the realizations of underlying Lie (super)algebras as bilinears of oscillators which correspond to free field realizations.
- The quasiconformal approach allows one to give a natural definition of super Joseph ideal and leads directly to the interacting realizations of the superextensions of higher spin algebras.
- For $SU(2,2|N)$ one obtains a continuous parameter family of inequivalent "interacting" $AdS_5/CFT_4$ higher spin superalgebras labelled by the deformation parameter $\zeta$ ( helicity).
- For $OSp(8^*|2N)$ one obtains a discrete infinity of inequivalent "interacting" $AdS_7/CFT_6$ higher spin superalgebra labelled by the eigenvalues $t(t+1)$ of an $SU(2)$ subalgebra of the little group $SU(2) \times SU(2) = SO(4)$ ( $6d$ analog of helicity).

# Comments and Open Problems

- Interacting realizations of $PSU(2,2|4)$ and quantum integrability of $N=4$ super Yang-Mills ?
- Interacting realization of the minrep of $OSp(8^*|4)$ and the interacting 6$d$ (2,0) theory ?
- The quasiconformal construction of the minrep of $D(2,1:\alpha)$ and its deformations describe the spectra of $N=8$ supersymmetric interacting quantum mechanical models constructed using Harmonic superspace thecniques Govil & MG
  Applications of the interacting quasiconformal realizations of the $AdS_3/CFT_2$ higher spin algebras defined by $D(2,1:\alpha)\times D(2,1;\beta)$ ? c.f the work of Gaberdiel & Gopakumar and others...
- Physical meaning of the deformed $AdS_7/CFT_6$ and $AdS_5/CFT_4$ higher spin superalgebras and their relevance to M-theory and IIB superstring?
- Reformulation of interacting quasiconformal realizations of higher spin algebras in terms of covariant fields.