Dlog Form of Super-Yang-Mills Loop Integrands

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Based on 1212.6228 and work in progress, by AL and Lionel Mason

Introduction

- I will consider on-shell, color-ordered scattering amplitudes in the planar limit of N=4 super-Yang-Mills theory

- The amplitudes have a lot of hidden symmetry and many remarkable properties.

- In this talk, I will describe a new form of hidden simplicity:

Loop integrands can be written in dlog form!

Example: 1-loop MHV amplitude

$$\int \frac{ds_0}{s_0}\frac{dt_0}{t_0}\frac{ds}{s}\frac{dt}{t}$$

where

$$s_0 = \bar{s}_0$$

$$t_0 = \bar{t}_0$$

$$s = -\frac{\bar{t}\,(a_{i-1j} + iv) + a_{i-1j-1} + iv}{\bar{t}\,(a_{ij} + iv) + a_{ij-1} + iv} \qquad v = s_0 - t_0$$

Review of N=4 sYM

- Maximal supersymmetry
- Conformal
- Dual to type IIB string theory on AdS_5xS^5
- Believed to be exactly solvable in planar limit
- Toy model for QCD

Spinor-Helicity

- 4d null momentum:

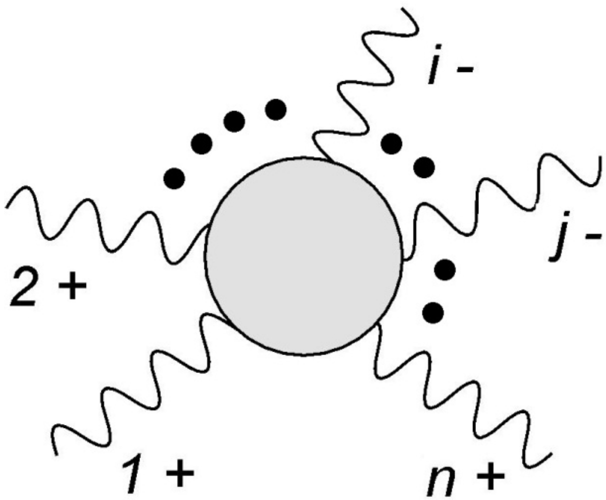
$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

where $\alpha = 0, 1$ and $\dot{\alpha} = \dot{0}, \dot{1}$

- Gluon amplitudes can be written in terms of these spinors, leading to very simple expressions.

MHV Amplitudes

At tree-level:



$$\mathcal{A}_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

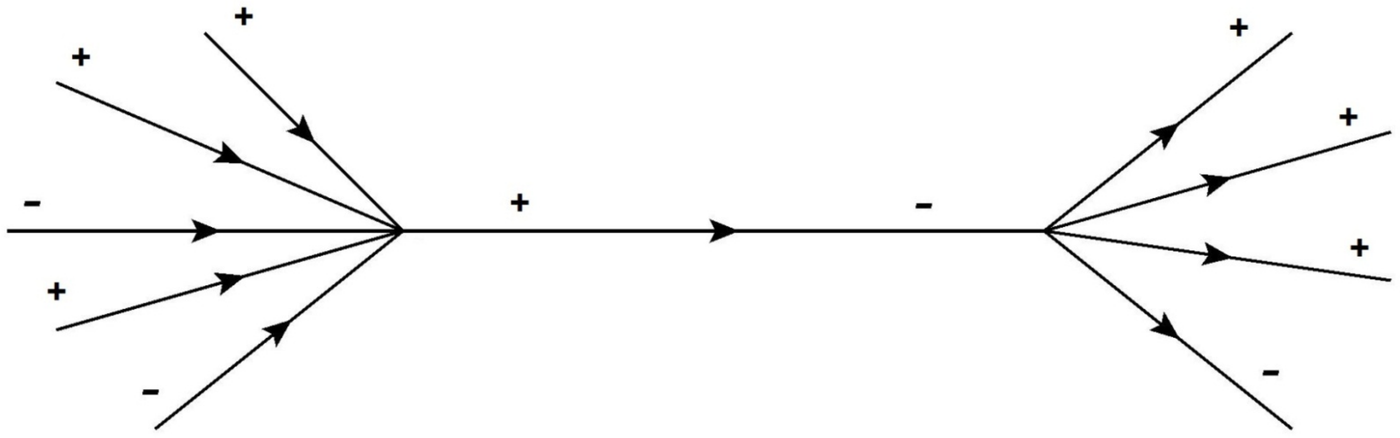
(Parke,Taylor)

where $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$

CSW Formalism

- Use tree-level MHV amplitudes as Feynman vertices for constructing tree-level non-MHV amplitudes.

  (Cachazo,Svrcek,Witten)
- Example: NMHV amplitude



Superamplitudes

- Supermomentum: $$q^{a\alpha} = \lambda^{\alpha}\eta^{a} \qquad a = 1, 2, 3, 4$$

- Tree-level MHV superamplitude:

$$A_n^{MHV} = \frac{\delta^4(p)\,\delta^8(q)}{\langle 12\rangle\,\langle 23\rangle \ldots \langle n1\rangle}$$

- N^{k}MHV superamplitude:

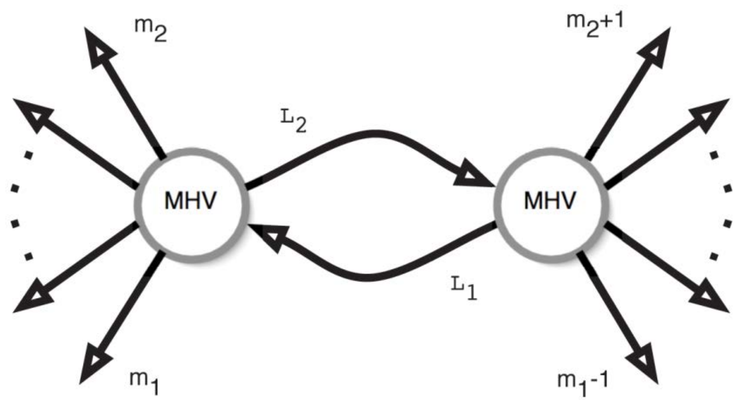
$$A_n^{N^k MHV} = \frac{\delta^4(p)\,\delta^8(q)}{\langle 12\rangle\,\langle 23\rangle \ldots \langle n1\rangle} M_n^k$$

where M_n^k has fermionic degree 4k

(super) CSW Formalism

- Use tree-level MHV superamplitudes as Feynman vertices to construct tree-level non-MHV amplitudes

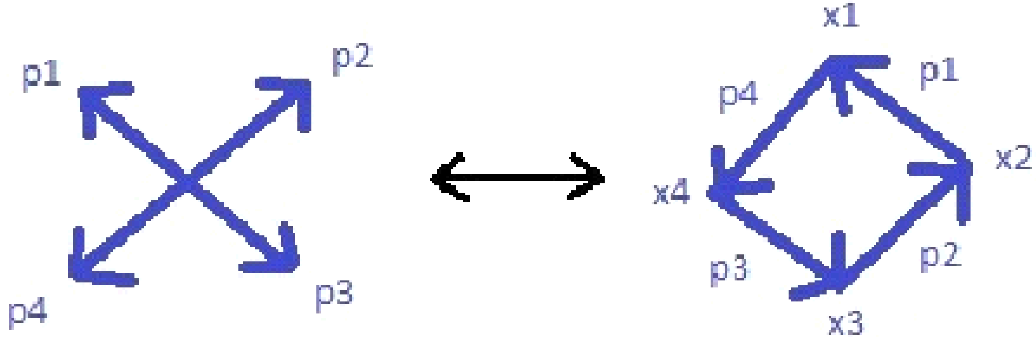
- Can also use these vertices to construct loop amplitudes. (Brandhuber,Heslop,Travaglini)



Dual Conformal Symmetry

- Dual variables:

$$x_i - x_{i+1} = p_i$$



- Amplitudes transform covariantly when

$$x_i \to x_i^{-1}$$

(Drummond, Henn, Korchemsky, Smirnov, Sokatchev)

Amplitude/Wilson Loop Duality

- Dual conformal symmetry can be extended to dual superconformal symmetry by defining fermionic dual variables:

$$\theta_i - \theta_{i+1} = q_i$$

- Dual superconformal symmetry corresponds to ordinary superconformal symmetry of a null-polygonal Wilson-loop

- This Wilson loop is dual to the planar S-matrix!

(Alday,Maldacena;Drummond,Henn,Korchemsky,Sokatchev; Brandhuber,Heslop,Travaglini;Mason,Skinner;Caron-Huot)

Back to Dlog Form

Our proof of the dlog form of loop integrands will make use of several ideas I just described:

- CSW formalism
- Dual superconformal symmetry
- Amplitude/Wilson loop duality

We just need one more ingredient…

Momentum Twistors

- Make dual superconformal symmetry manifest:

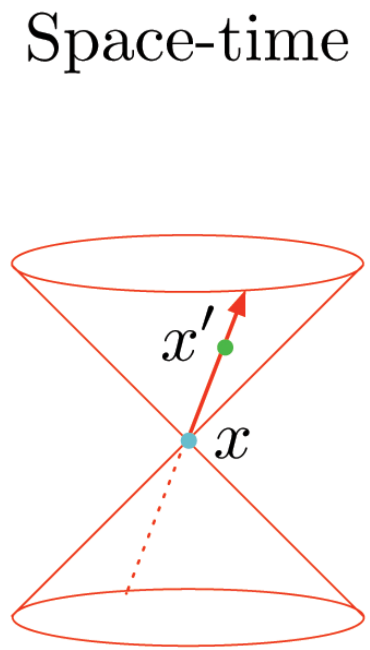
$$\begin{pmatrix} Z^A \\ \chi^a \end{pmatrix}, \quad Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix}$$

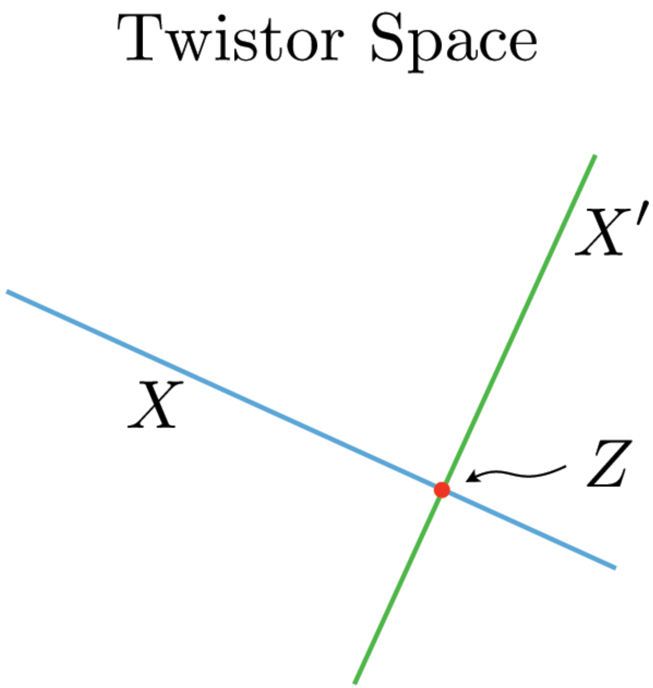
- Incidence relations: (Penrose)

$$\mu^{\dot\alpha} = -i x^{\dot\alpha\alpha}\lambda_\alpha, \quad \chi^a = -i\theta^{a\alpha}\lambda_\alpha$$

- Momentum conservation automatic (Hodges)

Spactime vs Twistor Space



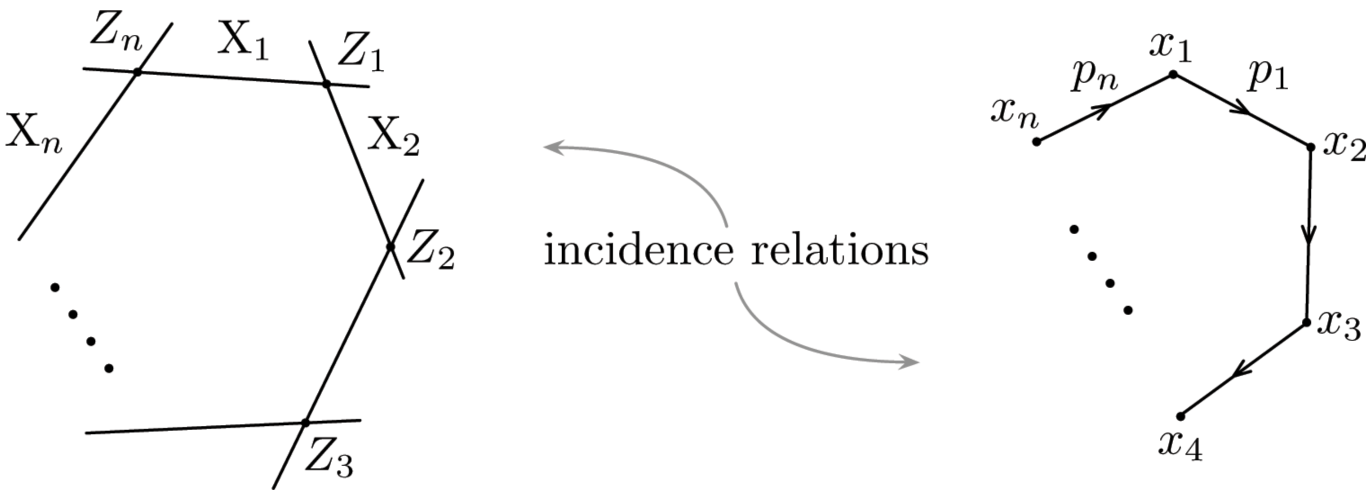


Point in spacetime ⟷ CP^1 in twistor space

Point in twistor space ⟷ null ray in spacetime

Twistor Wilson Loop

- Null polygon in spacetime corresponds to polygon in twistor space:



- Expectation value of the twistor Wilson loop computes planar S-matrix! (Mason,Skinner)

N=4 sYM in Twistor Space

- Superfield:

$$\mathcal{A} = g^+ + \chi^a \tilde{\psi}_a + \frac{1}{2}\chi^a\chi^b\phi_{ab} + \epsilon_{abcd}\chi^a\chi^b\chi^c\left(\frac{1}{3!}\psi^d + \frac{1}{4!}\chi^d g^-\right)$$

- Twistor action: (Boels,Mason,Skinner)

$$S[\mathcal{A}] = \frac{i}{2\pi}\int D^{3|4}Z\,\mathrm{Tr}\left(\mathcal{A}\wedge\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \longleftarrow \text{self-dual sector}$$

$$+ g^2\int d^{4|8}x \log\det\left(\left(\bar{\partial}+\mathcal{A}\right)\big|_X\right) \longleftarrow \text{MHV expansion}$$

- Axial Gauge: $\bar{Z}_* \cdot \mathcal{A} = 0$
- Feynman rules correspond to CSW formalism!

Feynman Rules

- Propagator:

$$\Delta(Z,Z') = \int \frac{\mathrm{d}s\,\mathrm{d}t}{st}\bar{\delta}^{4|4}(Z_* + sZ + tZ')$$

where $\delta^{4|4}(Z) = \Pi_{A=1}^{4}\bar{\delta}\left(Z^A\right)\Pi_{a=1}^{4}\chi^a$

- MHV Vertices:

$$\int_{\mathbb{M}\times(\mathbb{CP}^1)^n} \frac{\mathrm{d}^{4|4}Z_A\,\mathrm{d}^{4|4}Z_B}{\mathrm{Vol}\ \mathrm{GL}(2)} \prod_{i=1}^{n} \frac{D\sigma_i}{(\sigma_i\,\sigma_{i+1})}$$

where $\sigma = (\sigma^0, \sigma^1)$ are homogeneous coordinates for the line (Z_A, Z_B) and

$$D\sigma = (\sigma d\sigma)\,, \quad (\sigma_i\,\sigma_j) = \sigma_i^0\sigma_j^1 - \sigma_i^1\sigma_j^0$$

Back to the Twistor WL

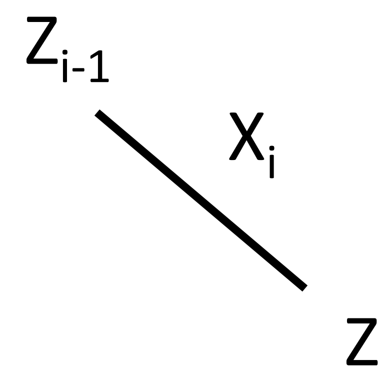
- Wilson loop expectation value:

$$\langle W[C] \rangle \propto \int \mathcal{D}\mathcal{A} e^{-S[\mathcal{A}]} W[C]$$

where W[C] follows from parallel transport:

- For each side (Z_{i-1}, Z_i) of the twistor polygon, find $H_i(Z)$ such that

$$\left(\bar{\partial} + \mathcal{A}\right) H_i|_{X_i} = 0, \;\; H_i\left(Z_{i-1}\right) = 1$$



- Then

$$W[C] := \mathrm{Tr} \prod_{i=1}^{n} H_i(Z_i)$$

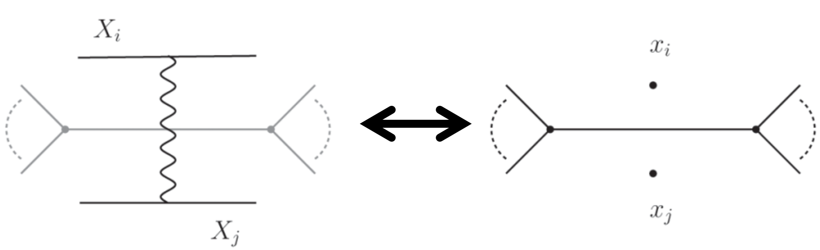
Planar Duality

- Amplitude Diagrams vs Wilson loop diagrams:

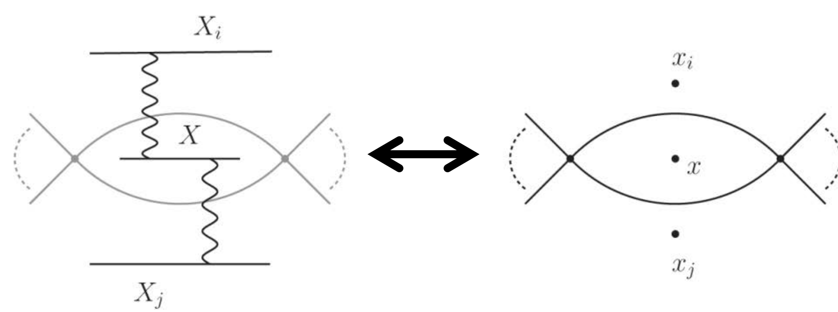
Amplitude	**Wilson Loop**
# of legs	# of sides
# of loops	# of MHV vertices
MHV degree	# of propagators – 2 x (# of MHV vertices)

- Examples:

Tree-level NMHV:

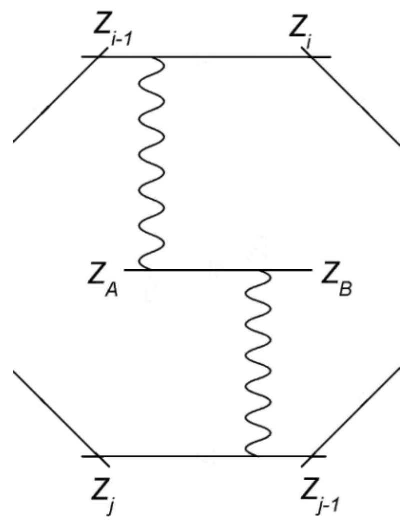


1-loop MHV:



Example: 1-loop MHV

- Twistor Wilson loop diagram:



- Feynman rules give

$$\int \frac{ds_1}{s_1}\frac{dt_1}{t_1}\frac{\mathrm{d}s_2}{s_2}\frac{\mathrm{d}t_2}{t_2}\int \mathrm{d}^{4|4}Z_A\,\mathrm{d}^{4|4}Z_B\,\bar{\delta}^{4|4}\left(Z_A - Z_* - s_1 Z_{i-1} - t_1 Z_i\right)$$
$$\bar{\delta}^{4|4}\left(Z_B - Z_* - s_2 Z_{j-1} - t_2 Z_j\right)$$

- Integrating Z_A and Z_B against delta functions leaves us with

$$\int \frac{ds_1}{s_1}\frac{dt_1}{t_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2}$$

and the constraints

$$Z_A = Z_* + s_1 Z_{i-1} + t_1 Z_i \,, \qquad Z_B = Z_* + s_2 Z_{j-1} + t_2 Z_j$$

- Let

$$(s_1, t_1) = -\frac{i}{s_0\,(1+s)}\,(1, s)\,, \quad (s_2, t_2) = -\frac{i}{t_0\,(1+t)}\,(1, t)$$

- Then the integrand still has the dlog form but

$$Z_A = i s_0 Z_* + \frac{1}{1+s}\,(Z_{i-1} + s Z_i)\;,\; Z_B = i t_0 Z_* + \frac{1}{1+t}\,(Z_{j-1} + t Z_j)$$

Reality constraints:

$$Z_A \cdot \bar{Z}_A = 0 \to s_0 = \bar{s}_0$$

$$Z_B \cdot \bar{Z}_B = 0 \to t_0 = \bar{t}_0$$

$$Z_A \cdot \bar{Z}_B = 0 \to$$

$$s = -\frac{\bar{t}\,(a_{i-1j} + iv) + a_{i-1j-1} + iv}{\bar{t}\,(a_{ij} + iv) + a_{ij-1} + iv}, \quad v = s_0 - t_0$$

where

$$a_{ij} = Z_i \cdot \bar{Z}_j$$

and we used

$$\bar{Z}_* \cdot Z_* = \bar{Z}_i \cdot Z_i = \bar{Z}_{i-1} \cdot Z_i = 0 \qquad \bar{Z}_* \cdot Z_i = 1$$

Generalities

- Feynman rules for the twistor Wilson loop can be expressed as integrals of dlogs and δ-functions

- Integration variables correspond to insertion points of propagators on MHV vertices or edges of the twistor Wilson loop

- External data is encoded in the integration contour, which is determined by reality constraints

- The integrand of an L-loop N^k MHV amplitude can be expressed in terms of 4(L+k) dlogs and k $\delta^{4|4\prime}s$

Integration

- For 1-loop MHV, we have

$$\int \frac{ds_0}{s_0}\frac{dt_0}{t_0}\frac{ds}{s}\frac{dt}{t}$$

- s,t integrals are easy to perform:

$$\int d\ln t d\ln s\,(v,\bar{t}) = 2\pi i \ln\left|\frac{(a_{i-1j}+iv)\,(a_{ij-1}+iv)}{(a_{i-1j-1}+iv)\,(a_{ij}+iv)}\right|^2$$

where $v = s_0 - t_0$

- Poles in s_0 and t_0 are real, so require regularization.

- This can be achieved using the Feynman iε prescription.

Integration

- In the end we obtain

$$\boxed{8\pi^2\left[Li_2\left(\frac{ia_{i-1j}}{v_*}\right)+Li_2\left(\frac{ia_{ij-1}}{v_*}\right)-Li_2\left(\frac{ia_{ij}}{v_*}\right)-Li_2\left(\frac{ia_{i-1j-1}}{v_*}\right)+c.c.\right]}$$

where

$$v_* = i\left(\frac{a_{ij-1}a_{i-1j}-a_{ij}a_{i-1j-1}}{a_{i-1j}-a_{ij}-a_{i-1j-1}+a_{ij-1}}\right)$$

- 1-loop MHV amplitude obtained by summing over all i,j
- Matches previous results!

Conclusions

- The loop integrands of planar amplitudes in N=4 sYM can be expressed in dlog form.
- The twistor Wilson loop naturally gives Feynman rules in dlog form and provides a simple geometric interpretation.
- The 1-loop MHV amplitude can be easily computed from its dlog loop integrand using the Feynman iε prescription .
- This method is dramatically simpler than other methods and yields compact, elegant expressions.

Future Directions

- Use the techniques described in this talk to compute planar amplitudes with more loops and higher MHV degree.

- A dlog form can also be obtained using “on-shell diagrams” and the “Grassmannian integral formula.” (Arkani-Hamed, Bourjaily, Cachazo,Goncharov, Postnikov, Trnka) How is this related to the twistor Wilson-loop?

- Can the techniques described in this talk be extended to non-planar amplitudes in N=4 sYM?