Computing complexity

Giorgio Parisi

- The mathematical definition of Complexity.
- An example taken from high energy physics.
- Two examples taken from statistical mechanics.
- Mean field approximation (no loops).
- A *supersymmetric* formulation of the problem
- The physical meaning of the supersymmetry.
- Spontaneous supersymmetry breaking.
- The typical scenario (in mean field theory).
- Open problems

Problem: given a function $f(\{x\})$ of N variables x, find the properties of the set of of critical or stationary points

$$\frac{\partial f}{\partial x_i} = 0 \ .$$

We want to compute their number and there relative location in the limit $N \to \infty$.

$$\text{Total Number} \ \propto \exp(N\Sigma_T)$$

Σ_T is the total complexity.

$$(\text{Number with} f = NF) \ \propto \exp(N\Sigma(F))$$

$\Sigma(F)$ is the complexity as function of F (an interesting quantity).

Gauge fixing

Given $A_\mu(x)$, find a gauge transform g such that

$$\partial_\mu A^g_\mu(x) = 0 \ ,$$

Here

$$f[g] = \int d^d x \left(A^g_\mu(x) \right)^2$$

Lattice theory gauge fixing

Given $U(i, \mu)$ defined on the links

$$U^g = g^*(i) U(i, \mu) g(i + \hat{\mu})$$

$$f[g] = -\sum_{i,\mu} \mathrm{Tr}(U^g(i, \mu))$$

Spin Glasses

$$H = -\frac{1}{2}\sum_{i,k} J_{i,k}\sigma(i)\sigma(k)$$

Approximated equations: Magnetization $m(i) = \langle\sigma(i)\rangle$.

$$\mathrm{arth}(m(i)) = \beta\sum_{k} J_{i,k}m(k)$$

Approximated field free energy:

$$f(\{m\}) = -\frac{1}{2}\sum_{i,k} J_{i,k}m(i)m(k) - \beta^{-1}\sum_{i} S(m(i));$$

$$-S(m) = \frac{1+m}{2}\log\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\log\left(\frac{1-m}{2}\right)$$

If $J_{i,k} = \pm 1$ randomly $\Longrightarrow$ gauge fixing for a Z_2 theory at $\beta = \infty$

Long range case;

Sherrington-Kirkpatrick model

$$\forall i, k \quad J_{i,k} = \pm 1$$

Glasses

x_i are three dimensional vectors.

$$f[x] = \sum_{i,k} V(x_i - x_k)$$

$V(x)$ is an appropriate two body potential.

Explicit computation in soluble cases.

- Spin glasses (gauge fixing) in the infinite dimensions limit.
- Spin glasses on special lattices (random lattices) or equivalently partial resummation of the high temperature expansion.
- Mean field approximation (no loops).

General considerations

In the generic case the determinant $D(x)$ of the Hessian matrix H is non zero at the critical points, i.e.

$$D(x) \equiv \det|H| \neq 0, \qquad H_{i,k} = \frac{\partial^2 f}{\partial x_i \partial x_k} \ .$$

The index $I(x)$ of a critical point is the number of negative eigenvalues of the Hessian; a minimum has index 0 while a maximum has an index N.

$$(-1)^{I(x)} = \mathrm{sign}(D(x)) = D(x)/|D(x)|$$

Morse theorem states that

$$\sum_{\alpha \in C} (-1)^{I(x_\alpha)} = \text{topological constant} \ ,$$

where C denotes the set of critical points.

A more detailed problem

We are interested in the distribution of f at the critical points:

$$\tilde{Z}(w) = \sum_{\alpha \in C} (-1)^{I(x_\alpha)} \exp(-w f(x_\alpha)) \quad \text{and} \quad Z(w) = \sum_{\alpha \in C} \exp(-w f(x_\alpha)) \ .$$

$$\sum_{\alpha \in C} = \int dx |D(x)| \delta \left(\frac{\partial f}{\partial x} \right)$$

$$\tilde{Z}(w) = \int dx D(x) \prod_{i=1,N} \delta(f_i(x)) \exp(-w f(x)) \ .$$

$$Z(w) = \int dx |D(x)| \prod_{i=1,N} \delta(f_i(x)) \exp(-w f(x)) \ .$$

The complexity as function of F

$$\tilde{\mathcal{N}}(F) \equiv \sum_{\alpha \in C} \delta(f_\alpha - N\ F)(-1)^{I_\alpha}$$

For large N

$$\tilde{\mathcal{N}}(F) \approx \exp(N\tilde{\Sigma}(F))$$

$$\tilde{Z}(w) \approx (N\tilde{\Phi}(w))$$

$\tilde{\Phi}(w)$ and $\tilde{\Sigma}(F)$ are related by an Legendre transform.

After the introduction of auxiliary variables

$$\tilde{Z}(w) = \int d\mu(X) \exp(-S(X)) \,,$$

$$S(X) = \sum_i \lambda_i f_i(x) + \sum_{i,k} \bar{\psi}_i \psi_k f_{i,k}(x) + w f(x) \,,$$

$$d\mu \equiv \prod_i dX_i = \prod_i dx_i d\psi_i d\bar{\psi}_i d\lambda_i \ .$$

X_i denote a point of the superspace with coordinates

$$X_i = \{x_i, \psi_i, \bar{\psi}_i, \lambda_i\} \,,$$

The integral over the variables λ goes from $-i\infty$ to $+i\infty$.

The irrelevance of the determinant

In order to have a similar representation for $\mathcal{N}(w)$ we should introduce $2n$ Fermionic variables, evaluate the result for integer n and making the analytic continuation to $n = 1/2$ as suggested firstly by Nicola d'Oresme.

We use d'Oresme identity:

$$|A| = (A^2)^{1/2} \ .$$

We expect that for $w > 0$ the sum is dominated by minima (or quasi-minima).

There are no strong cancellations and

$$\tilde{\Sigma}(F) = \Sigma(F)$$

Fermionic symmetries

The action S ad the measure $d\mu$ are invariant under the following supersymmetry (Cavagna, Garrahan and Giardina 1998):

$$\delta x_i = \epsilon_i \psi_i \quad \delta\lambda_i = -\epsilon_i \, w\psi_i \quad \delta\bar{\psi}_i = \epsilon_i \lambda_i \quad \delta\psi_i = 0.$$

The Fermionic symmetry is the BRST type. The Morse theorem, that states that

$$\tilde{Z}(w)|_{w=0} = \text{topological constant}$$

can be proved in a neat way using the supersymmetric formalism.

As usual supersymmetry implies identities among correlations of different quantities:

$$\langle \bar{\psi}_i \psi_k \rangle = \langle x_i \lambda_k \rangle, \qquad w \langle \bar{\psi}_i \psi_k \rangle = \langle \lambda_i \lambda_k \rangle \ .$$

A physical argument

Let us assume that for infinitesimal h and there is an one to one correspondence among the solutions of the equations

$$f_i(x) = h_i$$

at zero h and at non zero h.

A detailed computation shows that

$$\langle x_i \lambda_k \rangle = \langle \frac{\partial x_i}{\partial h_k} \rangle = \langle (H^{-1})_{i,k} \rangle = \langle \bar{\psi}_i \psi_k \rangle \ .$$

Supersymmetry takes care of purely geometrical relations among the correlations of the Bosonic variables, that play the role of Lagrangian multipliers and the Fermionic variables, that have been used to evaluate the inverse of the Hessian matrix and its determinant.

Spontaneously breaking of the supersymmetry

The proof of the Ward identities is based on the hypothesis that the Hessian matrix at the critical point has no zero-mode.

If we average the function f inside a given class, we may integrate over regions where zero-modes are always present and our assumptions fail.

If a zero mode is present H^{-1} is infinite and all kind of problems arise in defining the expectation value of $(H^{-1})_{i,k}$.

We could add a small term in the weight to suppress the critical points with zero modes. e.g. an extra factor

$$\exp(-\Delta r(x)) \, .$$

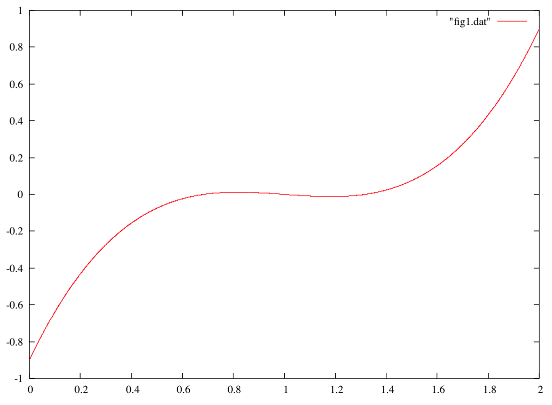
This breaks explicitly supersymmetry and in the limit $\Delta \to 0$ the breaking may survive.

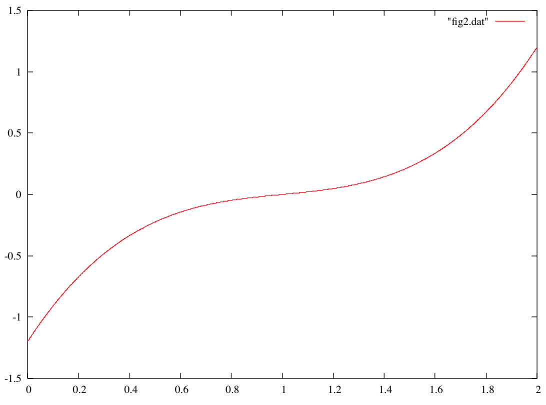
(Aspelmeier, Bray, and Moore) If supersymmetry is broken there is exactly one near zero mode for $N \to \infty$ and critical points come in pairs for N (they becomes saddles when $N \to \infty$).

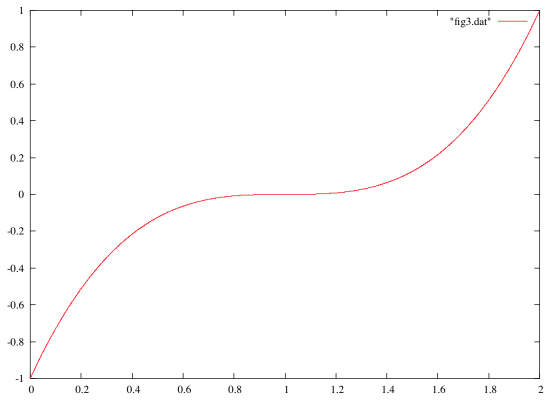
(Cavagna, Giardina, Parisi) Careful numerical verification.

(Parisi Rizzo) The zero mode is the Goldstone Fermion of supersymmetry.

Plots of f along the direction of the zero mode.







Three possibilities

Now for each value of F we can stay in one of the following cases

- I: The generic stationary point is a minimum (index 0).
- II_1: The generic stationary point is may be not a minimum but the number of negative modes remains bounded whin $N \to \infty$.
- II_∞: The generic stationary point has a number of negative modes diverging when $N \to \infty$.

For $w > 0$ only cases I or II_1 are realized.

What has been done

Most studied problem:

f= free energy as function of the magnetization.

Techniques used for supersymmetry broken case:

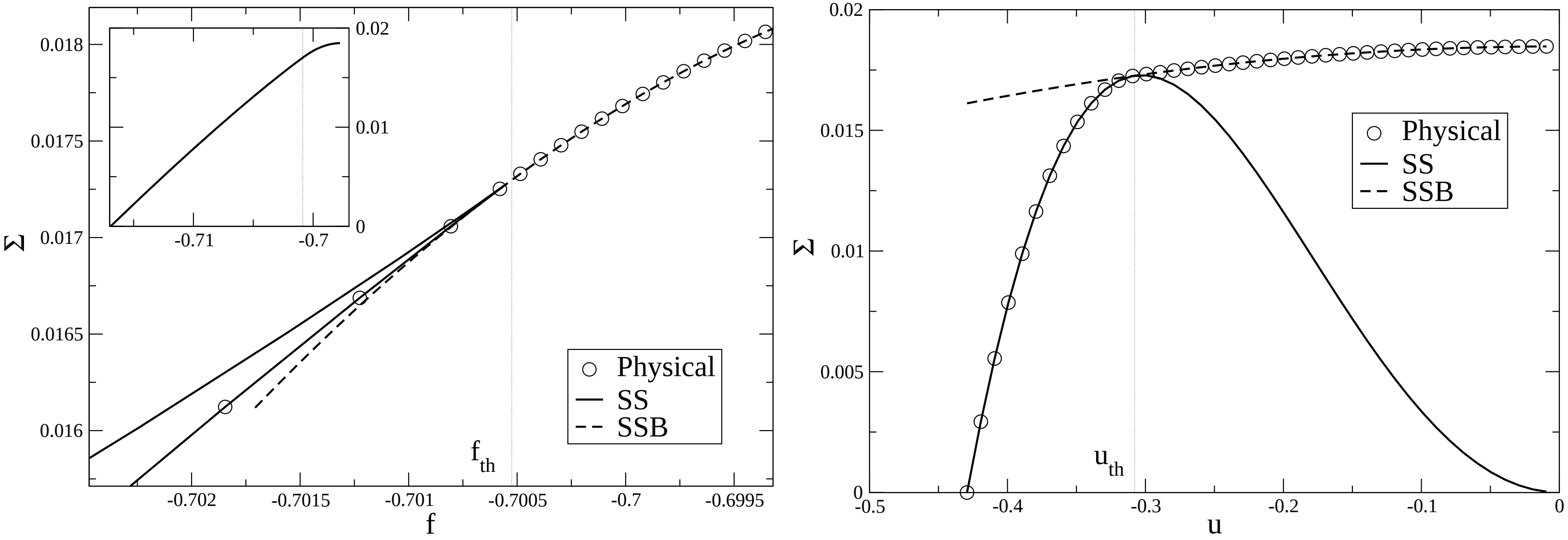
- Original fermionic representation.
- Cavity approach: $N \Longleftrightarrow N+1$ (Cavagna Giardina Parisi).
- Explicit supersymmetry breaking and Cavity (Rizzo).

Model considered

- SK model for spin glasses: $F^* = F_m$.
- Spherical model for spin glasses $F^* = F_M$.
- Spin glasses on Bethe Lattice $F^* = F_m$, but more difficult to study.

An example in a soluble model.

An example of the complexity as function of w (u), and an example of the complexity as function of the free energy.



Open problems

The general principles in the mean field approximation (no loops) are well understood.

There are some technical problems unsolved:

- More complex models.
- The critical point form clusters and there are many clusters (with broken supersymmetry).

The effects of loops has never been studied:

- Perturbation theory (easy).
- Non-perturbative effects in finite dimensions (difficult).

Some recent references

Cavagna, Giardina, Mezard, Parisi: *On the formal equivalence of the TAP and thermodynamic methods*; cond-mat 0210665.

Annibale, Cavagna, Giardina, Parisi, Trevigne: *The role of the BRST supersymmetry in the calculation of the complexity ...*; cond-mat 0307475.

Aspelmeier, Bray, Moore: *The Complexity of Ising Spin Glasses*; cond-mat 0390113.

Crisanti, Leuzzi, Parisi, Rizzo: *Quenched Computation of the Complexity of the Sherrington-Kirkpatrick Model* cond-mat 0309256.

Cavagna, Giardina, Parisi: *Numerical study of metastable states in Ising spin glasses*; cond-mat 0312534.

Parisi Rizzo: *On Supersymmetry Breaking in the Computation of the Complexity*; cond-mat 0401509.

Rizzo: *Tap Complexity, the Cavity Method and Supersymmetry*; cond-mat 0403261.

Cavagna, Giardina, Parisi: *Cavity Method for Supersymmetry Breaking Spin Glasses*; cond-mat 0407440.