Max Planck Institute for Mathematics
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New geometric correspondence between 4-manifolds and 2d gauge theories

Sergei Gukov

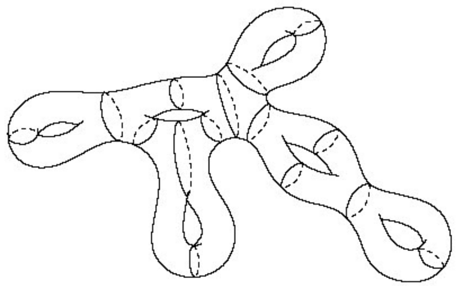
based on:

arXiv:1302.0015 (walls in 3d; elliptic genus of 2d $\mathcal{N} = (0,2)$ theories)
arXiv:1305.0266 (… more on "flavored elliptic genus" …)
arXiv:1306.4320 (2d $\mathcal{N} = (0,2)$ theories labeled by 4-manifolds)

with A.Gadde and P.Putrov

- <u>Class S:</u>

2-manifold C ⇒ 4d $\mathcal{N}=2$ theory
$T[C]$



[L.F.Alday, D.Gaiotto, Y.Tachikawa]
[A.Gadde, L.Rastelli, S.S.Razamat, W.Yan]
:

(geometric / topological) invariant of C = $Z_{T[C]}$

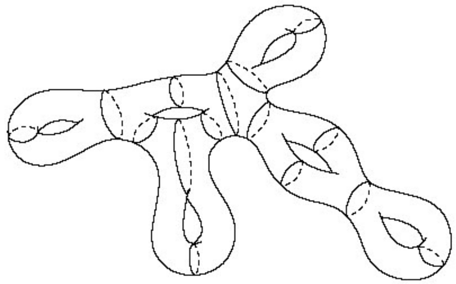
Liouville theory ⇐ S^4 partition function

deformed 2d Yang-Mills ⇐ superconformal index

- **Class S:**

2-manifold C ⇒ 4d $\mathcal{N}=2$ theory

T[C]



[L.F.Alday, D.Gaiotto, Y.Tachikawa]
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⋮

- **Class R:**

3-manifold M_3 ⇒ 3d $\mathcal{N}=2$ theory

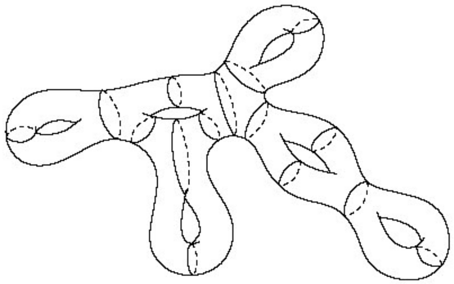
T[M_3]

$$\mathcal{M}_{\text{flat}}(M_3; G_{\mathbb{C}}) = \mathcal{M}_{\text{SUSY}}(T[M_3])$$

$$Z_{CS}(M_3; G_{\mathbb{C}}) = Z_{\text{vortex}}(T[M_3])$$

- **Class S:**

2-manifold C ⇒ 4d $\mathcal{N}=2$ theory $T[C]$



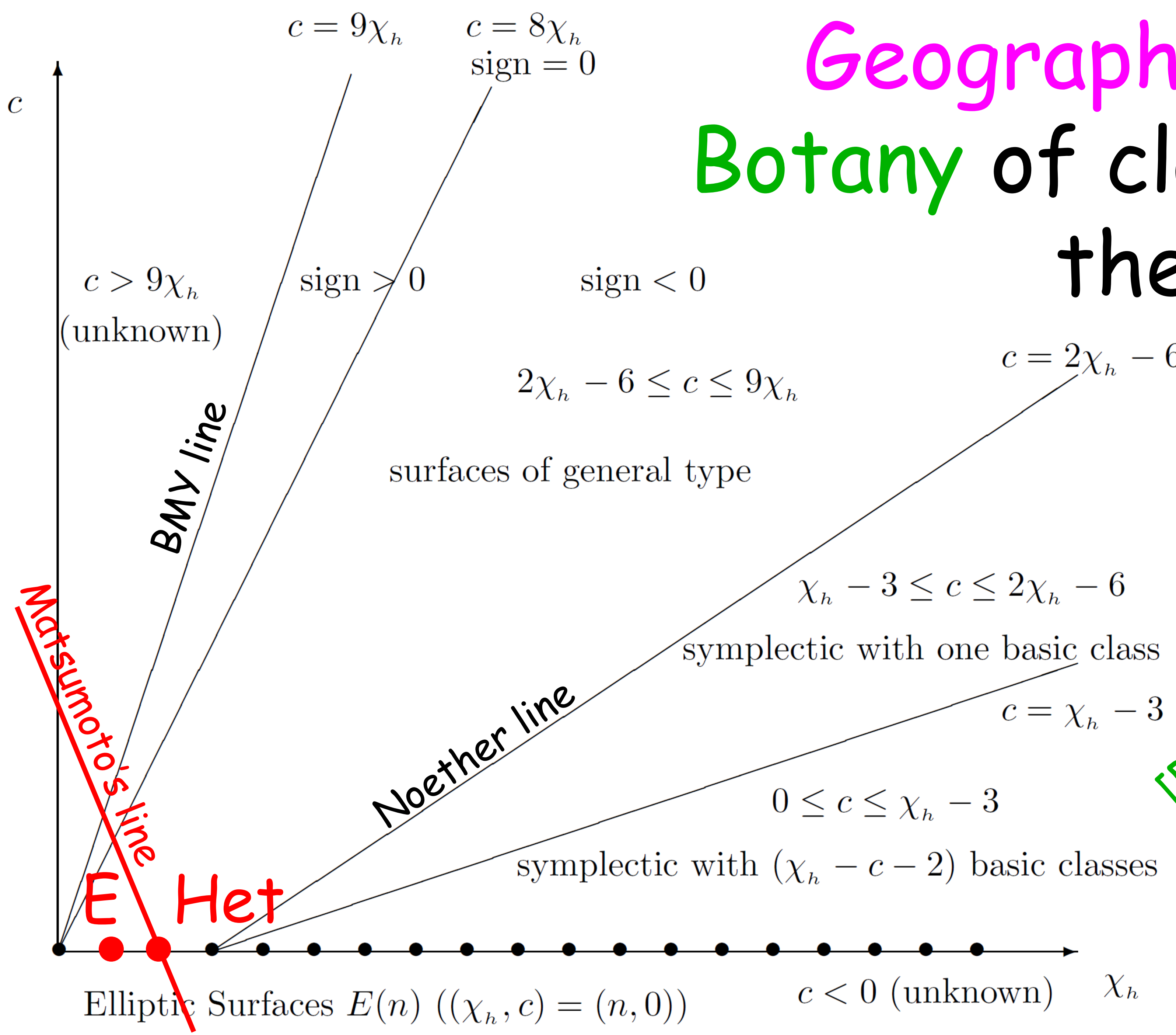
- **Class R:**

3-manifold M_3 ⇒ 3d $\mathcal{N}=2$ theory $T[M_3]$

- **Class H:**

4-manifold M_4 ⇒ 2d $\mathcal{N}=(0,2)$ theory $T[M_4]$

Geography and Botany of class H theories



Motivation

- Much richer structure than (2,2) models (new branches of vacua, gauge dynamics…)

[I.Melnikov, C.Quigle, S.Sethi, M.Stern, 2012]

- (0,2) mirror symmetry

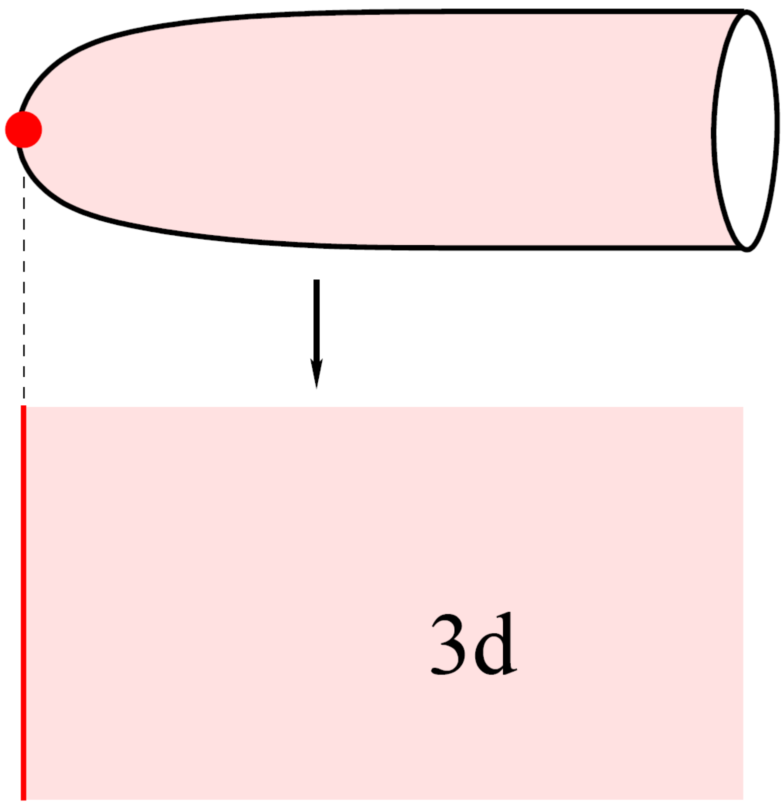
see e.g. [I.Melnikov, S.Sethi, E.Sharpe, 2012]

- Membranes (ABJM) with boundary and defect walls

- Fusion of defect ~~lines~~ walls in ~~2d~~ 3d

Surface Operators in 4d $\mathcal{N}=1$ gauge theories

w/ D.Gaiotto and N.Seiberg



A half-BPS surface operator in 4d $\mathcal{N}=1$ gauge theory defines a half-BPS boundary condition in 3d $\mathcal{N}=2$ theory

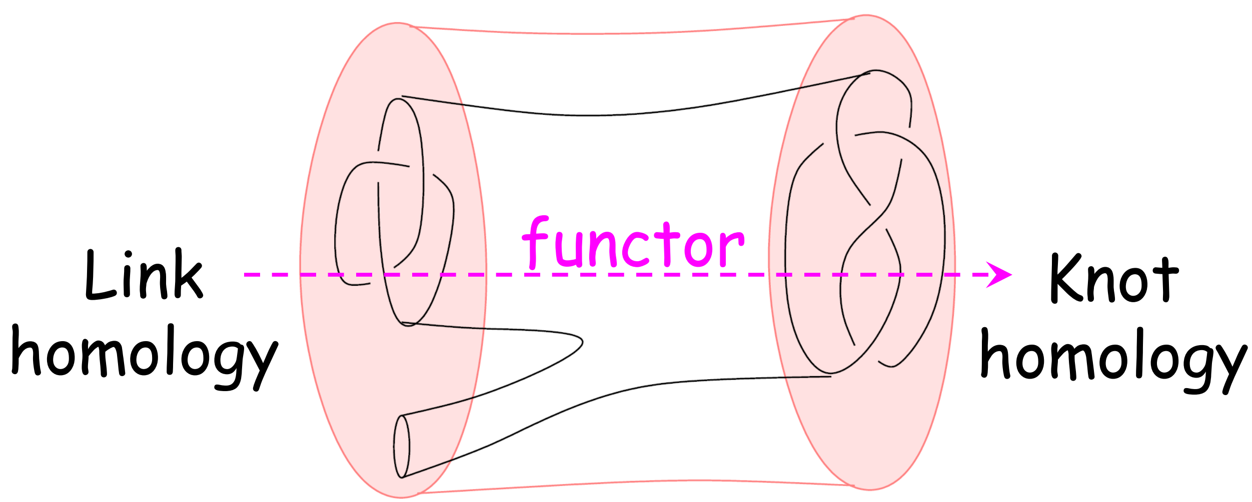
Representations of BPS algebras

$$\mathcal{H}^{(\text{closed})}_{\text{refined BPS}} = \text{algebra}$$

[J.Harvey, G.Moore]
[M.Kontsevich, Y.Soibelman]

[E.Gorsky, S.G., M.Stosic]

$$\mathcal{H}^{(\text{open})}_{\text{refined BPS}} = \text{module over } \mathcal{H}^{(\text{closed})}_{\text{refined BPS}}$$



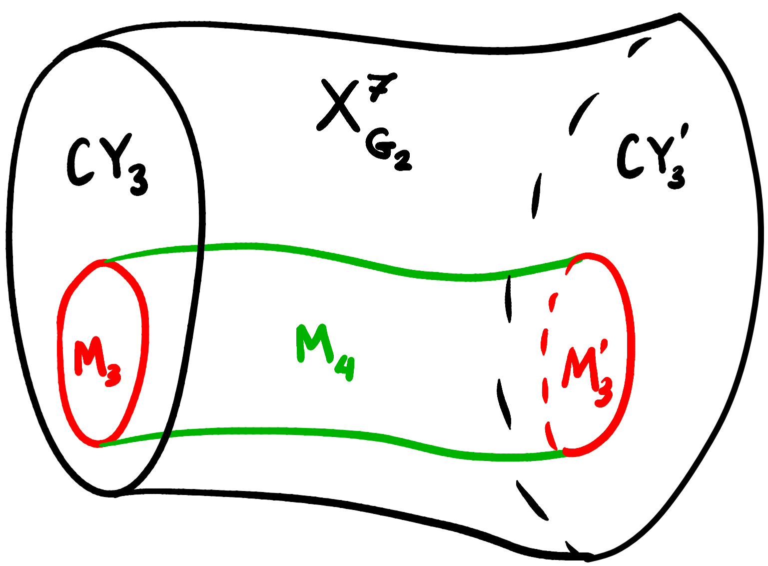
q-grading

homological grading

$$U(1)_P \times U(1)_F$$

$$\begin{array}{lccccc}
\text{space-time:} & \mathbb{R} & \times & \mathbb{R}^4_{q,t} & \times & T^*M_3 \\
 & \| & & \cup & & \cup \\
N \text{ M5-branes:} & \mathbb{R} & \times & \mathbb{R}^2_q & \times & M_3
\end{array}$$

$$M_4 = \mathbb{R} \times M_3$$



Vafa-Witten partition function

6d $(2,0)$ theory
on $T^2 \times M_4$

[C.Vafa, E.Witten]

$$\swarrow \qquad\qquad \searrow$$

$\mathcal{N} = 4$ super-Yang-Mills
on M_4

2d $(0,2)$ theory $T[M_4]$
on T^2

$$Z_{VW} = \sum_n \mathbf{x}^c q^n \chi(\mathcal{M}_{n,c})$$ = "flavored" elliptic genus of the (0,2) theory

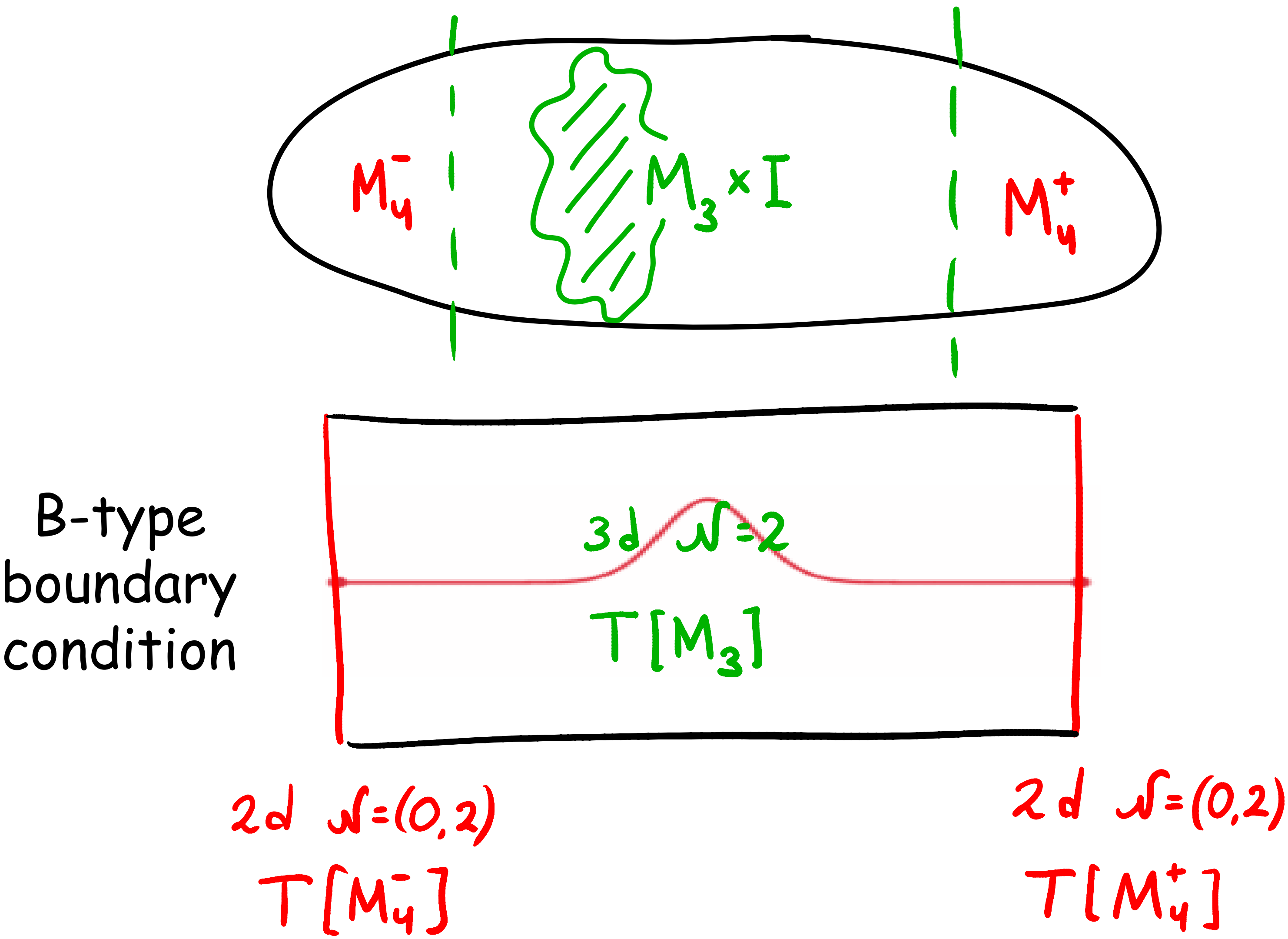
<u>Gluing Rule #1:</u>

$$G_{\text{flavor}}(T[M_4]) = U(1)^{b_2}$$

- Discrete vs continuous basis
- Integration measure = (0,2) vector multiplet superconfromal index

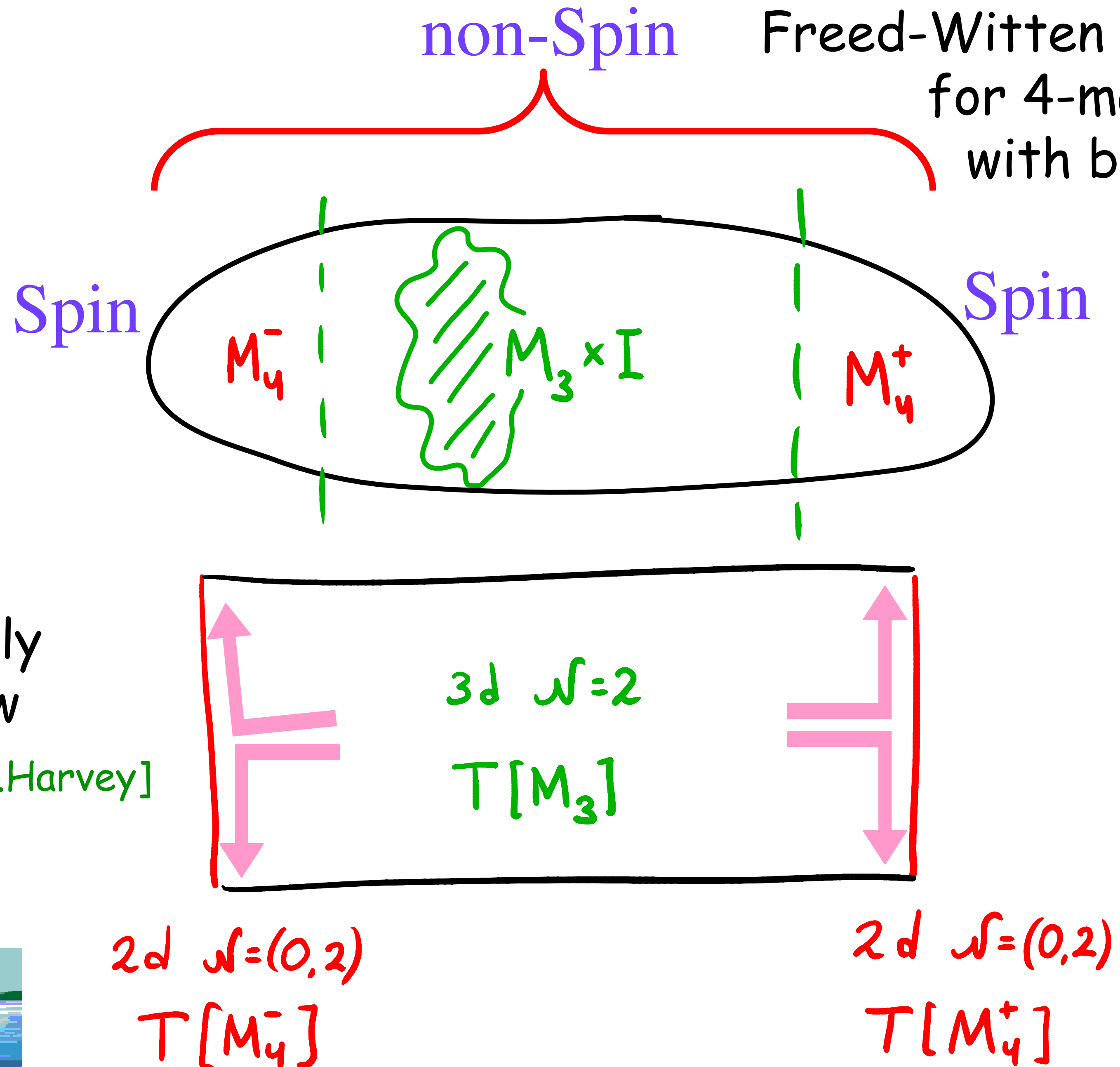
arXiv:1302.0015
with A.Gadde and P.Putrov

Gluing



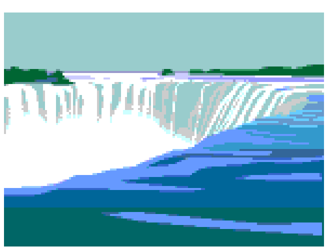
M_4^-
$M_3 \times I$
M_4^+
B-type boundary condition
3d $\mathcal{N}=2$
$T[M_3]$
2d $\mathcal{N}=(0,2)$
$T[M_4^-]$
2d $\mathcal{N}=(0,2)$
$T[M_4^+]$

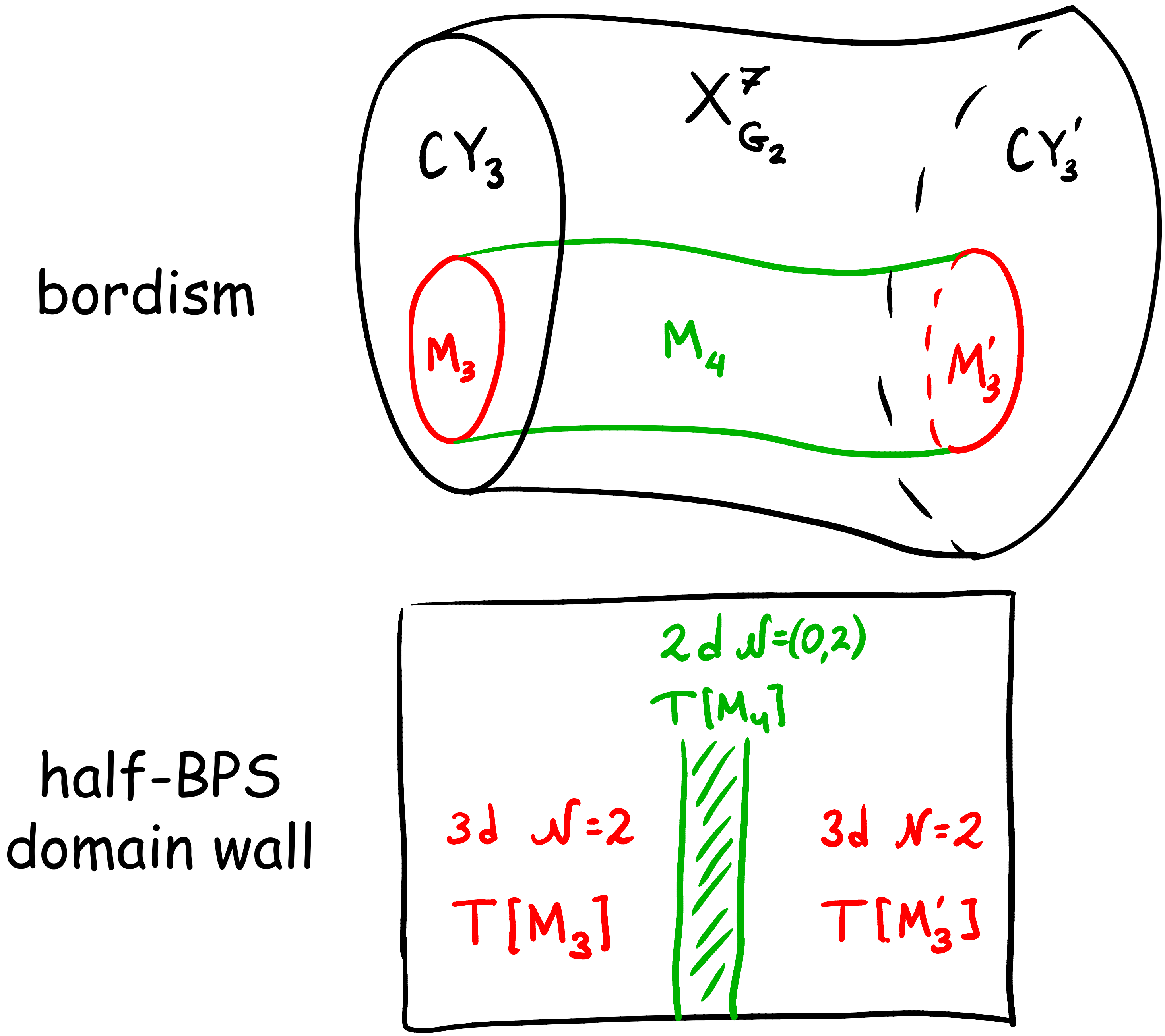
Freed-Witten anomaly
for 4-manifolds
with boundary



anomaly
inflow

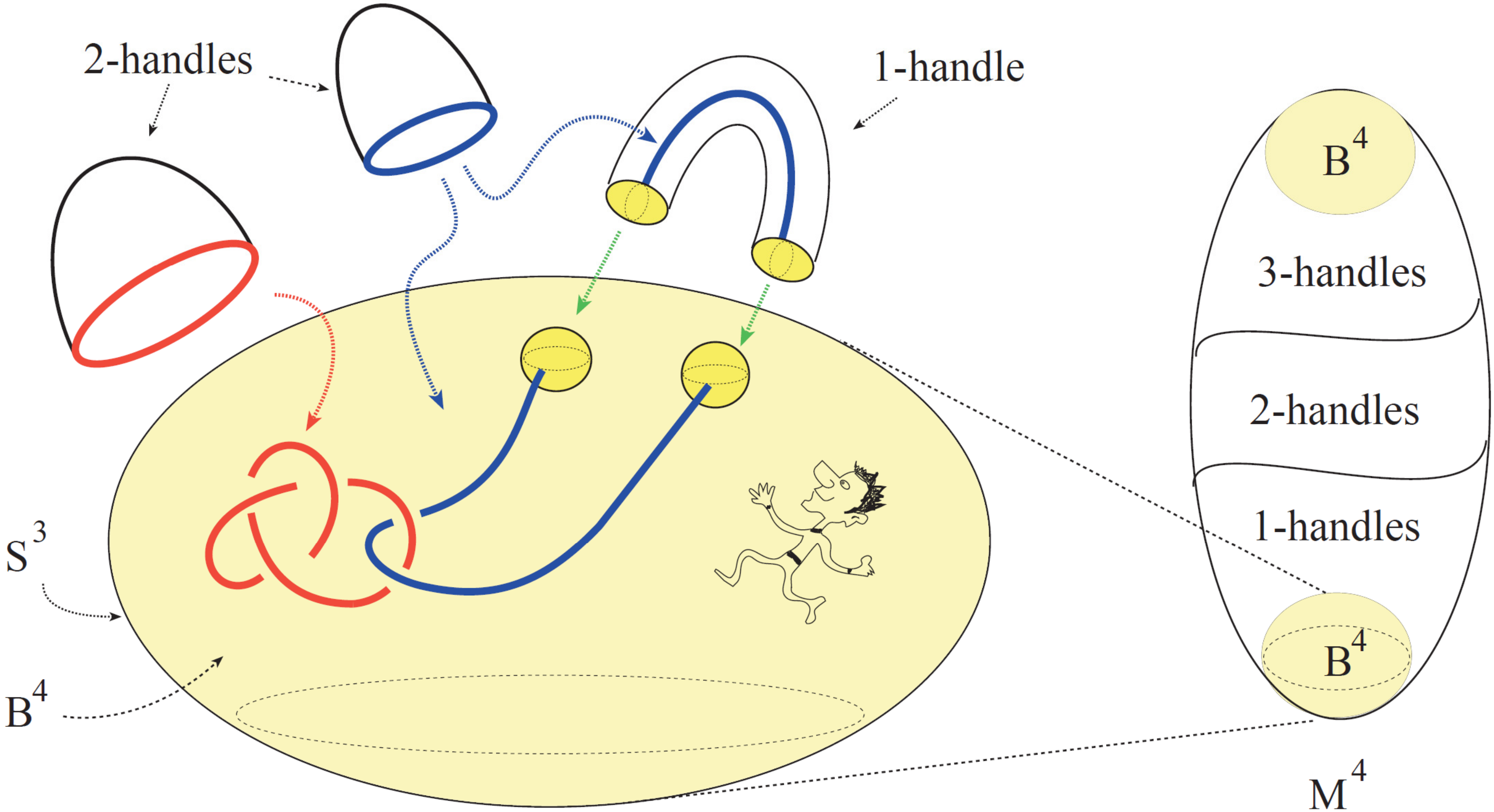
[C.Callan, J.Harvey]





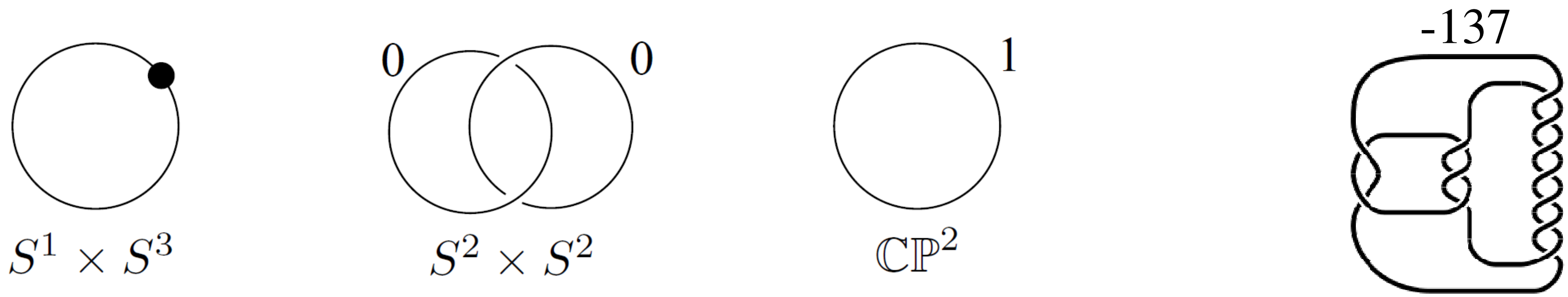
bordism
CY_3
$X^7_{G_2}$
CY'_3
M_3
M_4
M'_3
half-BPS domain wall
2d $\mathcal{N}=(0,2)$
$T[M_4]$
3d $\mathcal{N}=2$
$T[M_3]$
3d $\mathcal{N}=2$
$T[M'_3]$

Building blocks



S. Akbulut, 2012

Kirby diagrams

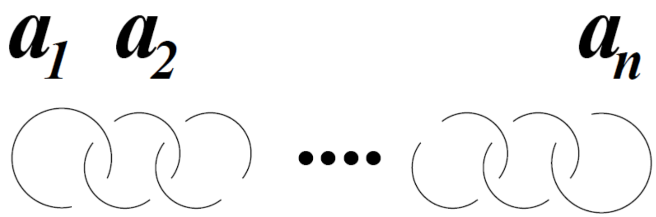


$$\boxed{M_4 \quad : \qquad K_1^{a_1} \, K_2^{a_2} \, \dots \, K_n^{a_n}}$$

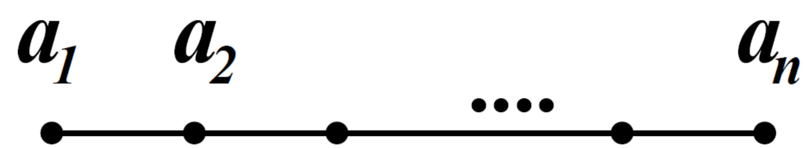
Intersection form on $H_2(M_4;\mathbb{Z})$:

$$Q_{ij} = \begin{cases} \mathrm{lk}(K_i, K_j), & \text{if } i \neq j \\ a_i, & \text{if } i = j \end{cases}$$

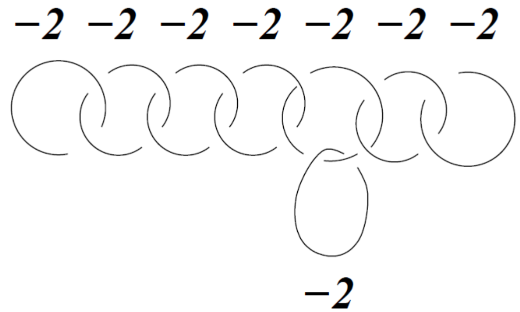
Plumbing graphs



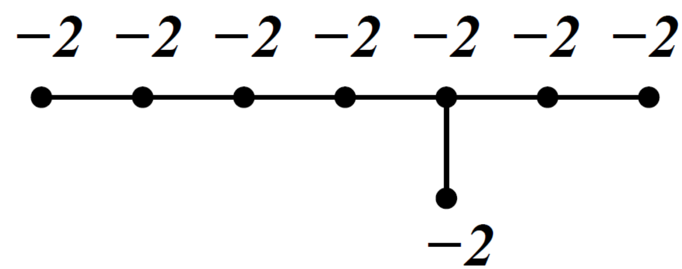
$=$



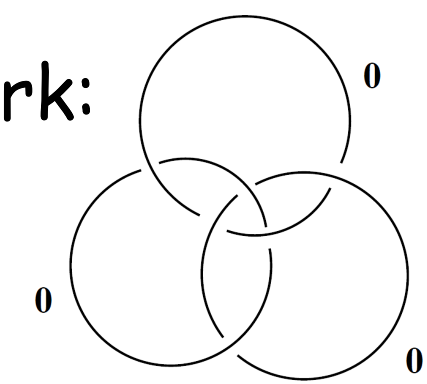
E_8 $=$



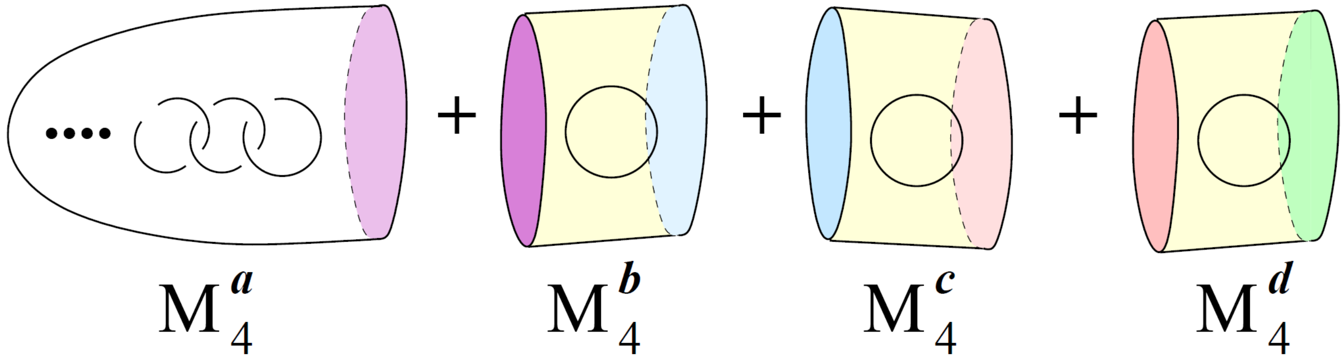
$=$



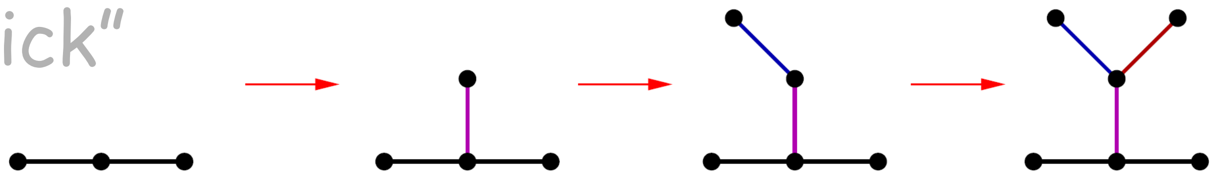
does not always work:



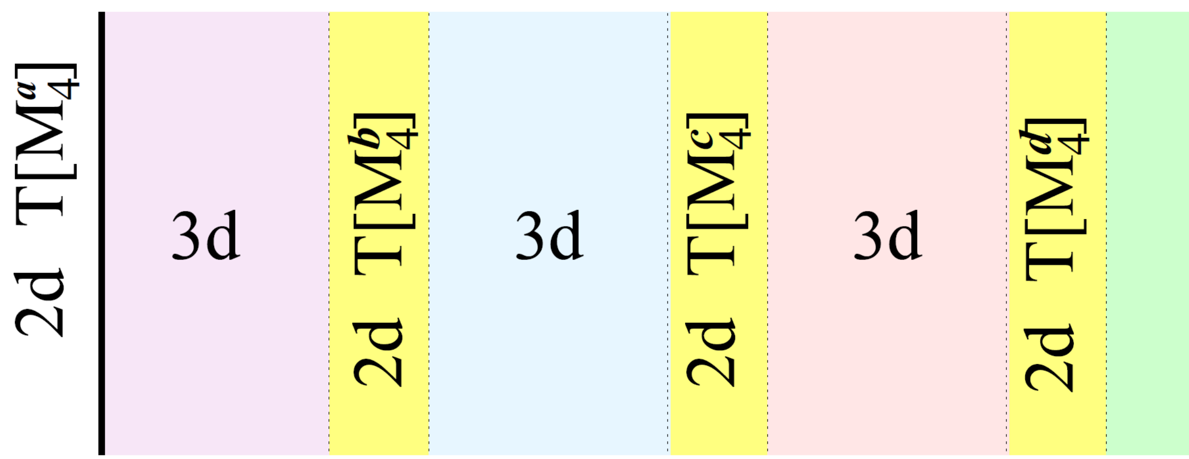
4-manifold bounded by a 3-torus



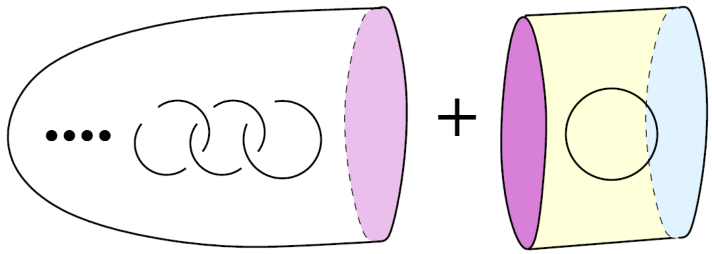
"Norman trick"



$$\chi_\rho^G = \sum_{\rho'} C_\rho^{\rho'} \chi_\rho^H$$



Gluing rule #2: Z_{vw}=coset branching function

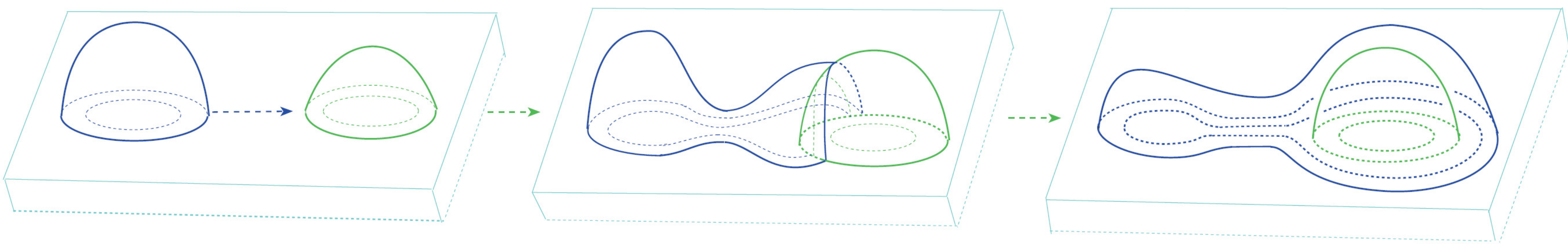


According to Nakajima, the Vafa-Witten partition function Z_{vw} (U(N), A_k) is a level N **character** of SU(k+1).

4d Kirby moves

- identity for Vafa-Witten partition function (= flavored elliptic genus):

$$(q;q)_\infty \int dz \frac{1}{\theta(z^{-1}w)} \prod_{i=1}^{N_f} \theta(q^{\frac{1}{2}} z x_i) = \prod_{i=1}^{N_f} \theta(q^{\frac{1}{2}} x_i w)$$



Class H moves

- identity for Vafa-Witten partition function (= flavored elliptic genus):

$$(q;q)_\infty \int dz \underbrace{\frac{1}{\theta(z^{-1}w)}}\ \underbrace{\prod_{i=1}^{N_f} \theta(q^{\frac{1}{2}} z x_i)} = \underbrace{\prod_{i=1}^{N_f} \theta(q^{\frac{1}{2}} x_i w)}$$

2d $\mathcal{N} = (0,2)$ SQED

	Φ	$\Psi_{i=1,\dots,N_f}$
$U(1)_{\text{gauge}}$	-1	$+1$
$U(1)_{\text{flavor}}$	$+1$	0

$\cong$

$$\left\{ \begin{array}{l} N_f \quad \text{Fermi multiplets } \Psi'_{i=1,\dots,N_f} \\ \text{with charge } +1 \text{ under } U(1)_{\text{flavor}} \end{array} \right\}$$

"mesons" $\boxed{\Psi'_i = \Phi\Psi_i}$

2d (0,2) "twisted superpotential"

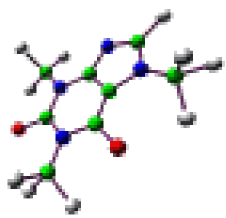
$$\widetilde{J} = -\frac{i}{8\pi}(N_f - 1)\log(\Phi)$$

$\mathcal{N}=2$ quiver Chern-Simons theory

vertex a $\longleftrightarrow$ U(1) Chern-Simons at level a

$$S = \frac{a}{4\pi}\int d^3x d^4\theta\ V\Sigma$$

$$= \frac{a}{4\pi}\int (A\wedge dA - \bar{\lambda}\lambda + 2D\sigma)$$



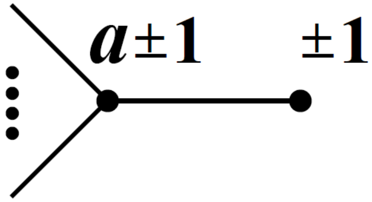
a_i — a_j $\longleftrightarrow$

$$S = \frac{1}{2\pi}\int d^3x d^4\theta\ V_i\Sigma_j$$

edge

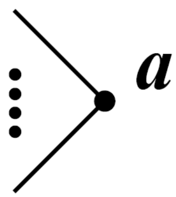
cf. [D.Belov, G.Moore]
[A.Kapustin, N.Saulina]
[J.Fuchs, C.Schweigert, A.Valentino]
:

$\mathcal{N} = 2$ quiver Chern-Simons theory

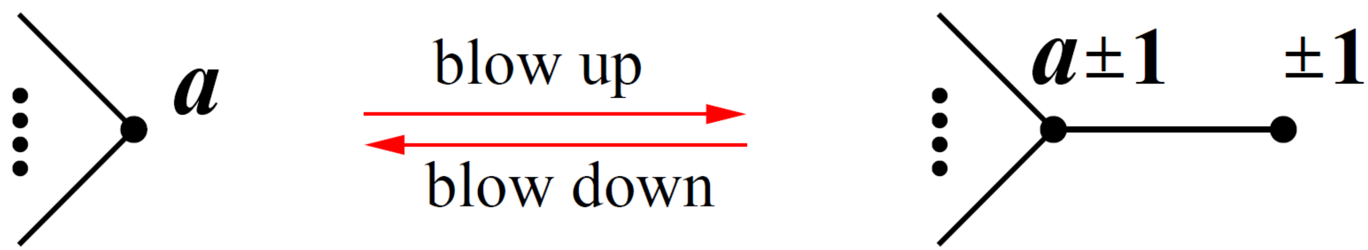


$$= \frac{1}{4\pi}\int d^4\theta \left(\pm V\Sigma + 2\widetilde{V}\Sigma + (a\pm 1)\widetilde{V}\widetilde{\Sigma} + \ldots\right)$$

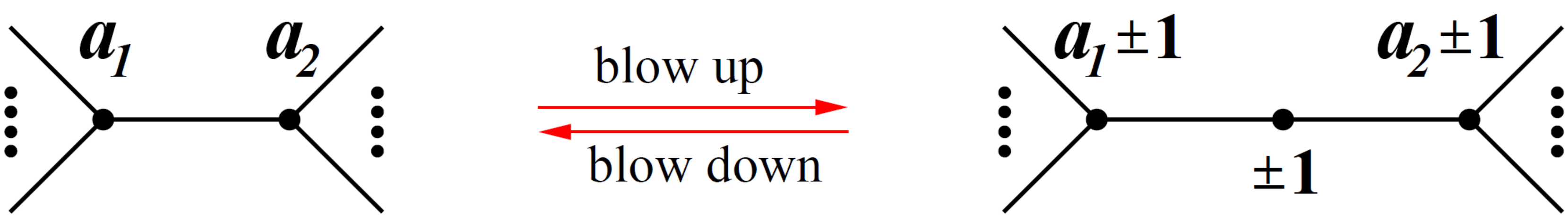
integrate out V $$= \frac{1}{4\pi}\int d^4\theta \left(\pm \widetilde{V}\widetilde{\Sigma} \mp 2\widetilde{V}\widetilde{\Sigma} + (a\pm 1)\widetilde{V}\widetilde{\Sigma} + \ldots\right)$$

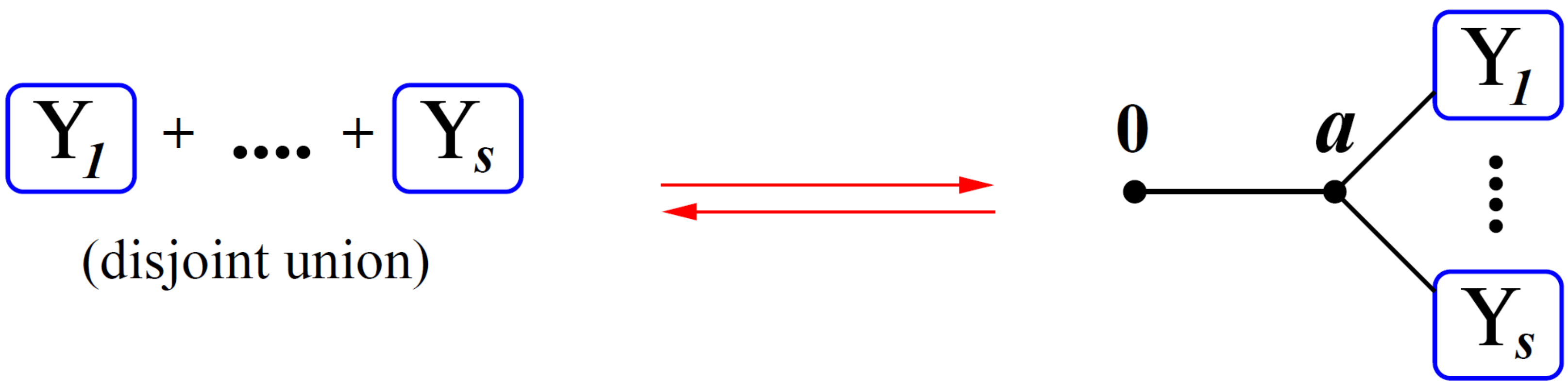


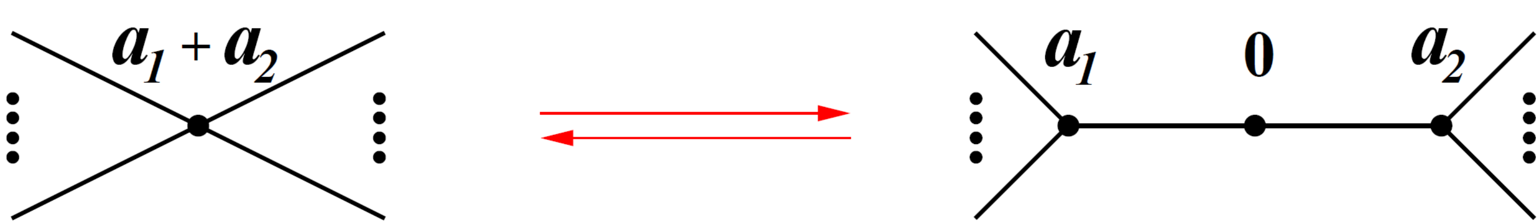
$$= \frac{1}{4\pi}\int d^4\theta \left(a\widetilde{V}\widetilde{\Sigma} + \ldots\right)$$



3d Kirby moves



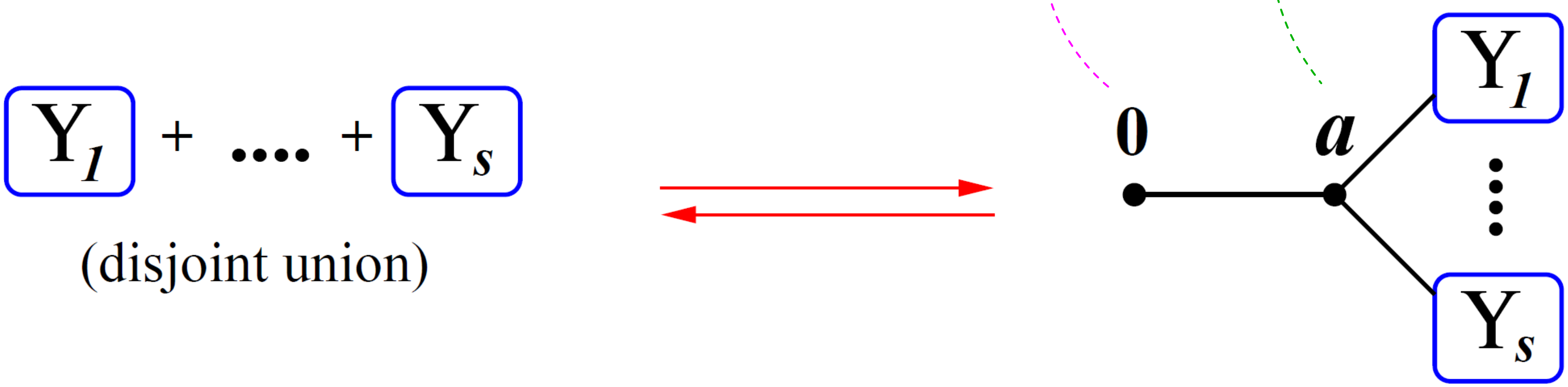




3d Kirby moves

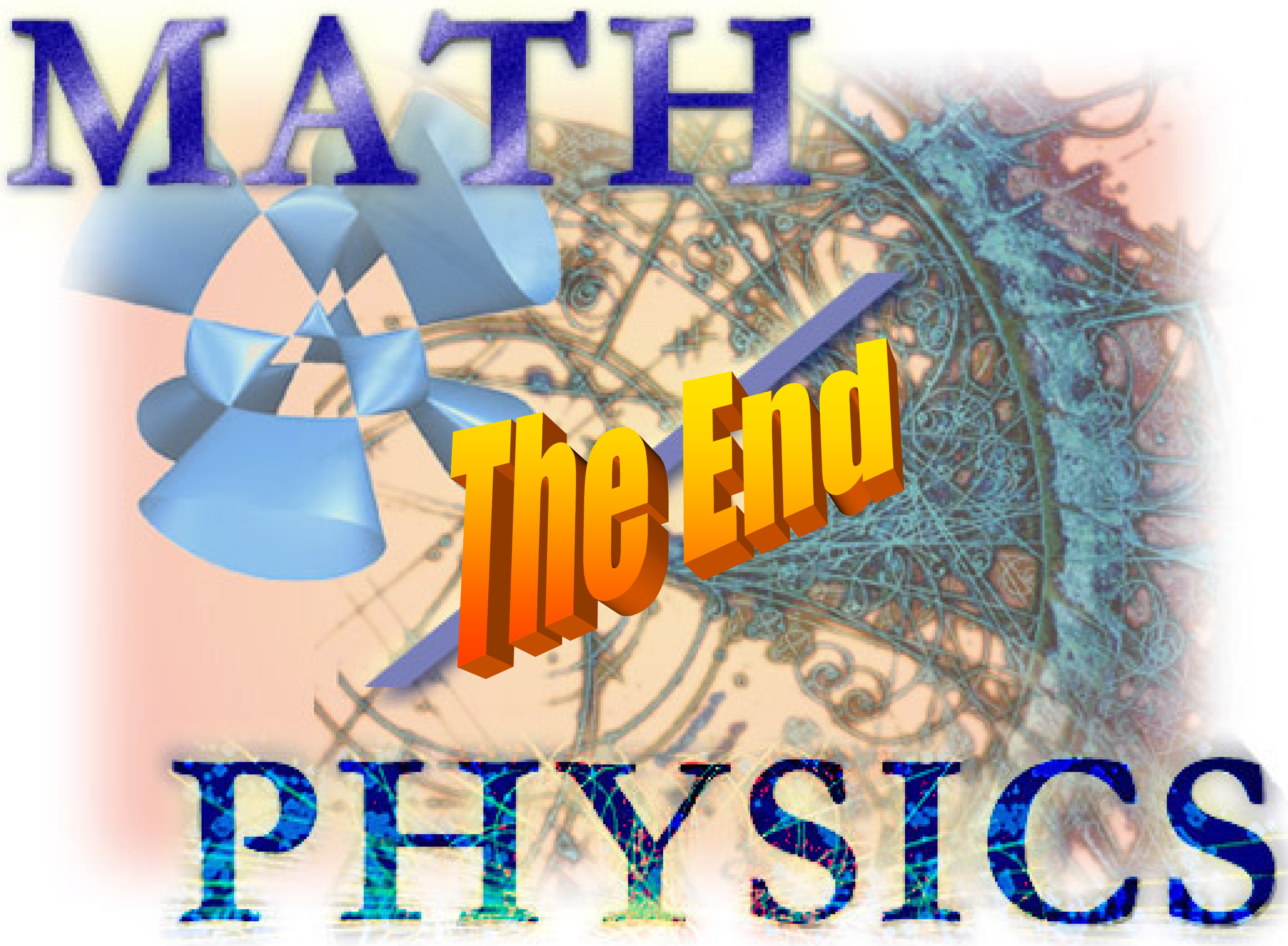
$$\mathcal{L} = \frac{1}{4\pi}\int d^4\theta \left(2V\widetilde{\Sigma} + a\widetilde{V}\widetilde{\Sigma} + \ldots\right)$$

V is Lagrange multiplier



Integrating out V makes $\widetilde{V}$ pure gauge
and removes all its Chern-Simons couplings

4-manifold M_4	**2d** $(0,2)$ **theory** $T[M_4]$
handle slides	dualities of $T[M_4]$
boundary conditions	vacua of $T[M_3]$
3d Kirby calculus	dualities of $T[M_3]$
cobordism from M_3^- to M_3^+	domain wall (interface) between $T[M_3^-]$ and $T[M_3^+]$
gluing	fusion
Vafa-Witten partition function	flavored (equivariant) elliptic genus
Z_{VW}(cobordism)	branching function
instanton number	L_0
embedded surfaces	chiral operators
Donaldson polynomials	chiral ring relations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