Notes of the lectures “Introduction to Nucleon-Nucleon Interaction” presented at the postgraduate School “Frontiers in Nuclear and Hadronic Physics”

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1 Introduction to quantum field theory

A change of inertial frame is defined by the transformation $x' = \Lambda x + s$ describing the relation between the space-time coordinates $x \equiv \{t, \boldsymbol{x}\}$ of an “event” in frame S with the corresponding coordinates x' as seen by the frame S'. Above, Λ is a 4×4 matrix describing the relative rotation and velocity of the two frames, and s a 4-vector describing the relative space-time translation. If $s = 0$, Λ has to verify the property $x'^\mu x'_\mu = x^\mu x_\mu$. In general, if $|A\rangle$ is a state of the Hilbert space describing some physical system as seen by the frame S, the state seen by the frame S' will be given by $U(\Lambda, s)|A\rangle$, where $U(\Lambda, s)$ is in general an unitary (or antiunitary) operator.

In general, the interaction Hamiltonian $H_I(t)$ in interaction picture (IP) can be written as an Hamiltonian density $\mathcal{H}(x) \equiv \mathcal{H}(t, \boldsymbol{x})$, so that $H_I(t) = \int d^3x\, \mathcal{H}_I(x)$. It can be proved that the Hamiltonian density must transform as [1]

$$U(\Lambda, s)\mathcal{H}_I(x)U(\Lambda, s)^\dagger = \mathcal{H}_I(\Lambda x + s) \ . \tag{1}$$

We shall see that this can be fulfilled using the operators known as quantum fields.

In the following we will work in a finite volume $\Omega = L^3$; the momentum values are discrete, *i.e.* $k_x = 2\pi n_x / L$, $n_x = 0, \pm 1, \pm 2, \ldots$. At the end one can substitute the sum over the discrete values with an integration as following

$$\sum_{\boldsymbol{k}} \to \Omega \int \frac{d\boldsymbol{k}}{(2\pi)^3} \tag{2}$$

The creation/annihilation operators of a scalar particle of type "i" verify the commutation rules

$$[a_{\boldsymbol{k},i}, a^\dagger_{\boldsymbol{k}',j}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{i,j} \ , \qquad [c_{\boldsymbol{k},i}, c^\dagger_{\boldsymbol{k}',j}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{i,j} \ , \tag{3}$$

where a is the annihilation operator of the particle and c the corresponding operator for the antiparticle. Moreover, $[a_{\boldsymbol{k},i}, a_{\boldsymbol{k}',j}] = 0$, etc. The creation/annihilation operators of spin 1/2 particles of type "t" verify the anticommutator rules

$$\{b_{\boldsymbol{p},s,t}, b^\dagger_{\boldsymbol{p}',s',t'}\} = \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{s,s'}\delta_{t,t'} \ , \qquad \{d_{\boldsymbol{p},s,t}, d^\dagger_{\boldsymbol{p}',s',t'}\} = \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{s,s'}\delta_{t,t'} \ , \tag{4}$$

where b is the annihilation operator of the particle and d the corresponding operator for the antiparticle. Again, $\{b_{\boldsymbol{p},s,t}, b_{\boldsymbol{p}',s',t'}\} = 0$, etc. Under Lorentz transformations, these operators transform in a very complicate way [1]. Therefore, it is necessary to take special combinations of the creation/annihilation operators which transform in a simpler way under Lorentz. These combinations are the so-called "quantum fields".

The field in IP for a scalar particle of type i and mass m is defined as

$$\phi_i(x) \ = \ \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\omega_k\Omega}} \left(a_{\boldsymbol{k},i} e^{-ik\cdot x} + c^\dagger_{\boldsymbol{k},i} e^{ik\cdot x} \right) \ , \tag{5}$$

where $\omega_k = \sqrt{m^2 + k^2}$. For a spin 1/2 particle of type "t" and mass M, the fields are "4-spinors" defined as

$$\Psi_t(x) \ = \ \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{2E_p\Omega}} \left(b_{\boldsymbol{p},s,t} u(\boldsymbol{p},s) e^{-ip\cdot x} + d^\dagger_{\boldsymbol{p},s,t} v(\boldsymbol{p},s) e^{ip\cdot x} \right) \ , \tag{6}$$

where $E = \sqrt{p^2 + M^2}$ and $u(\boldsymbol{p},s)$, $v(\boldsymbol{p},s)$ are the so-called Dirac 4-spinors [2]:

$$u(\boldsymbol{p},s) = \sqrt{E+M} \left(\begin{array}{c} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+M}\chi_s \end{array} \right) \ , \qquad v(\boldsymbol{p},s) = \sqrt{E+M} \left(\begin{array}{c} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+M}\eta_s \\ \eta_s \end{array} \right) \ , \tag{7}$$

where

$$\chi_{+\frac{1}{2}} = \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \ , \quad \chi_{-\frac{1}{2}} = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \ , \tag{8}$$

and $\eta_s = i\sigma_2\eta_{-s}$ (i.e., $s = \pm 1/2$ denote the spin state the particle). Above, $x \equiv x^\mu$ and $p \equiv p^\mu$ are 4-vectors.

These operators can be shown to have the following properties

1. They verify the equal-time commutator/anticommutator rules

$$[\phi_i(t,\boldsymbol{x}), -i\frac{\partial\phi_j(t,\boldsymbol{y})}{\partial t}] = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{i,j} \ , \qquad (9)$$

$$\left\{\Big(\Psi_t(t,\boldsymbol{x})\Big)_\lambda, \Big(\Psi^\dagger_{t'}(t,\boldsymbol{y})\Big)_{\lambda'}\right\} = \delta(\boldsymbol{x}-\boldsymbol{y})\delta t, t'\delta_{\lambda,\lambda'} \ , \qquad (10)$$

   where λ, $\lambda' = 1,\ldots,4$ are 4-spinor components.

   Exercise 1: verify these relations.

2. Under "proper" Lorentz transformations (namely frame transformations which do not involve time and space reflections), the fields transform in the following way

$$U(\Lambda,s)\phi_i(x)U(\Lambda,s)^\dagger = \phi_i(\Lambda x+s) \ , \qquad U(\Lambda,s)\Psi_t(x)U(\Lambda,s)^\dagger = S(\Lambda)\Psi_t(\Lambda x+s) \ , \qquad (11)$$

   where $S(\Lambda)$ is a (known) 4×4 matrix [2].

3. The presence of the creation operators of the antiparticle together the annihilation operators of the particle in the definition of the fields, it is necessary in order that the fields verify the conditions

$$[\phi_i(x), \phi_i^\dagger(x')] = 0 \ , \qquad \{\Psi_t(x), \Psi_t^\dagger(x')\} = 0 \ , \qquad \text{if } (x-x')^2 < 0 \ , \qquad (12)$$

   namely the boson (fermion) fields must commute (anticommute) for space-like events [1]. Constructed the Hamiltonian density $\mathcal{H}_I(x)$ as products of fields, in which the fermion fields must appear always in pairs, it is then easy to prove that $[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0$ if $(x-x')^2 < 0$, known as micro-causality condition. Namely, interactions happening in two "space-like" events cannot interfere.

It is also useful to introduce the so called Dirac matrices, a set of 4×4 matrices which can be used to construct Lorentz invariant quantities. A particular choice of the Dirac matrices is the following:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ , \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad (13)$$

where above "1" , "0", and "σ^i" denote the 2×2 identity matrix, the 2×2 matrix with all zero elements, and the Pauli matrices, respectively. These matrices verify the following relations

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \ , \qquad \{\gamma^\mu, \gamma^5\} = 0 \ , \qquad (\gamma^5)^2 = I \ , \qquad (14)$$

for $\mu, \nu = 0, \ldots, 3$. Above $\{, \cdots\}$ is the anticommutator, and $g^{\mu\nu}$ is the "metrix tensor" (in our notation, $g^{00} = 1$, $g^{ii} = -1$, and the off diagonal elements are zero).

The $S(\Lambda)$ matrix entering Eq. (11) verifies the relations

$$S(\Lambda)^\dagger \gamma^0 S(\Lambda) = I \ , \qquad S(\Lambda)^\dagger \gamma^0 \gamma^\mu S(\Lambda) = (\Lambda)^\mu{}_\nu \gamma^\nu \ . \tag{15}$$

Moreover $S(\Lambda)$ commutes with γ^5. Using the notation $\overline{\Psi}_t = \Psi_t^\dagger \gamma^0$, we can easily construct bilinears of fermionic fields which have the following transformation rules under Lorentz transformations

$$\overline{\Psi}_t(x)\Psi_t(x) \to \overline{\Psi}_t(\Lambda x+s)\Psi_t(\Lambda x+s) \ , \qquad \overline{\Psi}_t(x)\gamma^5\Psi_t(x) \to \overline{\Psi}_t(\Lambda x+s)\gamma^5\Psi_t(\Lambda x+s) \ , \tag{16}$$

namely they transform as scalar quantities (actually the combination with γ^5 is a pseudoscalar, see below). Moreover,

$$\overline{\Psi}_t(x)\gamma^\mu\Psi_t(x) \ \to \ (\Lambda)^\mu{}_\nu \overline{\Psi}_t(\Lambda x + s)\gamma^\nu\Psi_t(\Lambda x + s) \ , \tag{17}$$

$$\overline{\Psi}_t(x)\gamma^\mu\gamma^5\Psi_t(x) \ \to \ (\Lambda)^\mu{}_\nu \overline{\Psi}_t(\Lambda x + s)\gamma^\nu\gamma^5\Psi_t(\Lambda x + s) \ , \tag{18}$$

namely they transform as 4-vectors, etc.

In the following, we need also the transformation of the field under parity, a particular case of Lorentz transformation, corresponding to

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{19}$$

By definition, this corresponds to an operator U_P which acts on a particle state with momentum $\boldsymbol{p}$ and spin projection s as $U_p|\boldsymbol{p}, s\rangle = \eta|-\boldsymbol{p}, s\rangle$, where η is a phase related to the intrinsic parity of the particle (it can be shown that $\eta = \pm 1$). The intrinsic parity of the boson (fermion) antiparticle is the same (opposite) of the one of the corresponding particle. Therefore, we can assume that

$$U_P a^\dagger_{\boldsymbol{k},i} U_P^\dagger = \eta_i a^\dagger_{-\boldsymbol{k},i} \ , \quad U_P c^\dagger_{\boldsymbol{k},i} U_P^\dagger = \eta_i c^\dagger_{-\boldsymbol{k},i} \ , \tag{20}$$

$$U_P b^\dagger_{\boldsymbol{k},s,t} U_P^\dagger = \eta_t b^\dagger_{-\boldsymbol{p},s,t} \ , \quad U_P d^\dagger_{\boldsymbol{k},s,t} U_P^\dagger = -\eta_t d^\dagger_{-\boldsymbol{p},s,t} \ . \tag{21}$$

Then, it is not difficult to prove that

$$\phi(t, \boldsymbol{x}) \ \xrightarrow{P} \ \eta_i \phi(t, -\boldsymbol{x}) \ , \tag{22}$$

$$\Psi_t(t, \boldsymbol{x}) \ \xrightarrow{P} \ \eta_t \gamma^0 \psi(t, -\boldsymbol{x}) \ , \tag{23}$$

and therefore

$$\overline{\Psi}_t(x)\gamma^5\Psi_t(x) \xrightarrow{P} -\overline{\Psi}_t(t,-\boldsymbol{x})\gamma^5\Psi_t(t,-\boldsymbol{x}) \ , \tag{24}$$

namely this quantity is odd under parity.

Exercise 2: prove these relations.

Another useful symmetry is the "charge conjugation", defined as the transformation of the particles in the corresponding antiparticles and viceversa. The representation of this operation in the Hilbert space is an unitary operator U_C, defined as

$$U_C a_{\boldsymbol{k},i} U_C^\dagger = \eta_i^c c_{\boldsymbol{k},i} \ , \qquad U_C b_{\boldsymbol{p},s,t} U_C^\dagger = \eta_t^c d_{\boldsymbol{p},s,t} \ , \tag{25}$$

where again η^c can assume the values ± 1. Also in this case it is possible to find as the fields transforms

$$U_C \phi(x) U_C^\dagger = \phi^\dagger(i) \ , \qquad U_C \Psi_t(x) U_C^\dagger = -i\,\gamma^0\gamma^2(\overline{\Psi}_t(x))^T \ , \tag{26}$$

where $(\cdots)^T$ denote the transpose.

Defined briefly the properties of the fields, let us now recall the quantum chromodynamics (Q.C.D) theory, describing the interaction of quarks, and introduce the chiral symmetry.

2 Chiral symmetry and Q.C.D.

Consider the Q.C.D. Lagrangian in case of the two lightest quarks of flavor u and d [3]:

$$\mathcal{L} = \overline{q}\left(i\gamma_\mu D^\mu - \mathcal{M}\right) q - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}, \tag{27}$$

where $D_\mu = \partial_\mu - igG^a_\mu T^a$, T^a $(a = 1\cdots 8)$ being the Gell-Mann matrices (generators of $SU(3)$ in the representation of dimension 3), while :

$$q = \begin{pmatrix} \Psi_u \\ \Psi_d \end{pmatrix} \ , \tag{28}$$

are the quark fields (spinor and color indices are omitted). The G^a_μ are the gluon fields while

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + igf^{abc} G^b_\mu G^c_\nu \ , \tag{29}$$

indicates the field strength of gluon fields, f^{abc} being the structure constants of the "color" group $SU(3)$ defining the commutator between the generators

T_a of the $SU(3)$ algebra $[T_a, T_b] = i f_{abc} T_c$. Finally, the mass matrix of the quarks in the case of 2 lightest flavors is:

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} . \tag{30}$$

Let us define the left q_L and right q_R spinors as

$$q_R = \frac{1+\gamma^5}{2} q = \begin{pmatrix} \Psi_{u,R} \\ \Psi_{d,R} \end{pmatrix} , \tag{31}$$

$$q_L = \frac{1-\gamma^5}{2} q = \begin{pmatrix} \Psi_{u,L} \\ \Psi_{d,L} \end{pmatrix} . \tag{32}$$

The Lagrangian (27) can be rewritten in the following way

$$\mathcal{L}_{QCD} = \overline{q}_L i \gamma_\mu D^\mu q_L + \overline{q}_R i \gamma_\mu D^\mu q_R - \overline{q}_L \mathcal{M} q_R - \overline{q}_R \mathcal{M} q_L - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}. \tag{33}$$

The left and right components of the quark fields are only connected through the mass term $\mathcal{M}$. Experimentally, the masses of the u and d quarks are:

$$m_u \simeq 1.5 \cdots 3.3 \text{ MeV}, m_d \simeq 3.5 \cdots 6.0 \text{ MeV}, \tag{34}$$

therefore, they are very small compared to the typical hadron masses ($\sim$ 1GeV). The quark mass term in first approximation can be neglected. In this way the Lagrangian (33) turns out to be invariant under independent "flavor" rotations for the components left and right, namely [4]:

$$q_R \rightarrow q'_R = R q_R = \exp\left(-i\vec{\theta}_R \cdot \vec{\tau}/2\right) q_R , \tag{35}$$

$$q_L \rightarrow q'_L = L q_L = \exp\left(-i\vec{\theta}_L \cdot \vec{\tau}/2\right) q_L , \tag{36}$$

where $\vec{\tau}$ denotes the Pauli matrices in the space of flavors and $\vec{\theta}_{L,R}$ the corresponding angles of rotation. The symmetry group corresponding to these transformations is $G = SU(2)_L \otimes SU(2)_R$. In accordance with Noether's theorem [2], neglecting the mass term, the currents associated with the transformations (35) and (36),

$$L^i_\mu = \overline{q}_L \gamma_\mu \frac{\tau^i}{2} q_L, \tag{37}$$

$$R^i_\mu = \overline{q}_R \gamma_\mu \frac{\tau^i}{2} q_R, \tag{38}$$

result to be conserved: $\partial^\mu L^i_\mu = 0$ e $\partial^\mu R^i_\mu = 0$.

Alternatively, we can define "vector" and "axial" rotations corresponding to the cases $\vec{\theta}_R = \vec{\theta}_L \equiv \vec{\theta}_V$ and $\vec{\theta}_R = -\vec{\theta}_L \equiv \vec{\theta}_A$. In this case, noting that $\gamma^5(1+\gamma^5)/2 = (1+\gamma^5)/2$ and $\gamma^5(1-\gamma^5)/2 = -(1-\gamma^5)/2$, we can easily prove that the transformations (35) and (36) can be rewritten in the following form

$$
\begin{aligned}
q \quad &\rightarrow \quad q' = Vq = \exp\left(-i\vec{\theta}_V \cdot \vec{\tau}/2\right) q \quad \text{(vector transformation)}, \qquad (39)\\
q \quad &\rightarrow \quad q' = Aq = \exp\left(-i\vec{\theta}_A \cdot \vec{\tau}/2\gamma^5\right) q \quad \text{(axial transformation)}. \qquad (40)
\end{aligned}
$$

The conserved currents in the limit of negligible quark masses compared to the hadron energies, will be:

$$
\begin{aligned}
V^i_\mu &= R^i_\mu + L^i_\mu = \overline{q}\gamma_\mu \frac{\tau^i}{2} q \qquad (41)\\
A^i_\mu &= R^i_\mu - L^i_\mu = \overline{q}\gamma_\mu \gamma^5 \frac{\tau^i}{2} q \qquad (42)
\end{aligned}
$$

In general, the symmetry group of the Lagrangian (33) in case of massless u and d quarks would be $G' = SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A$, where $U(1)_V$ corresponds to the transformations $q \rightarrow \exp(-i\alpha)q$, related simply to the conservation of the number of quarks, while $U(1)_A$ corresponds to the transformations $q \rightarrow \exp(-i\alpha\gamma^5)q$. The $U(1)_A$ symmetry is at the end broken by quantum effects, and although it would represent a symmetry of the classical theory, it is not a symmetry of the quantum theory.

Returning to the group $G = SU(2)_L \otimes SU(2)_R$, the invariance of (33) is not completely reflected in the physical ground state; the phenomenon is called spontaneous symmetry breaking. In particular, the vacuum state is not invariant under the $SU(2)_A$ axial transformation contained in G. There are several experimental and theoretical evidences that motivate this phenomenon. Hadrons in nature occur in multiplets of $SU(2)_V$ almost degenerate in mass, therefore $SU(2)_V$ (corresponding to the isospin symmetry) is a good symmetry of the physical states. If also $SU(2)_A$ would be a good symmetry, particles would occurs in multiplets containing particles with opposite parities. However, no parity doublets are observed in the spectrum of low-energy hadrons.

In addition, in presence of a spontaneously broken symmetry, one of the consequences is the existence of massless particles called Nambu-Goldstone bosons. So the existence of very light mesonic particles (pions are the natural candidates for this role) is another strong argument in support of presence of spontaneous symmetry breaking. Within this scheme the non-zero mass of the pion is justified by the fact that the symmetry is approximate due to the (very small but definitely non zero) quarks masses. These and other arguments suggest that the chiral group $G = SU(2)_L \otimes SU(2)_R$ is spontaneously broken to the subgroup $SU(2)_V$.

3 Experimental evidences of the chiral symmetry in hadron physics

We saw in the previous section that the Lagrangian that governs the strong interactions between quarks u and d, in the limit of vanishing quark masses, is invariant under the transformations of the chiral group. Therefore, this symmetry should play a important role in the dynamics of nucleons and pions. The study of these systems starting from the interaction between quarks is still very complicated. On the other hand in many processes of nuclear physics (all reactions of astrophysical interest and many of the reactions that can be studied in laboratory), one is interested in studying processes of low energy. It is therefore convenient from an operational standpoint to make a "effective" theory and assume as "active" degrees of freedom only nucleons and pions. The validity of this theory will be limited only to processes where the involved energies do not permit the excitation of our elementary particles, such as the excitation of the pion in the meson ρ, or of a nucleon in a particle Δ. One can then assume that our effective theory will be valid up to energies of a few hundred MeV, well above the typical energies of nuclear processes such as the beta decays of nuclei, reactions of nucleosynthesis of stars, and so on. At this point the problem is the construction of the effective Lagrangian in terms of the nucleon and pion fields. For this task we will be guided by symmetry principles, in particular the conditions imposed by the chiral symmetry. Let us discuss first the experimental evidences that show how this symmetry plays an important role in the physics of low-energy nucleons and pions.

From now on, $\Psi_t(x)$ will denote the nucleon field of type t ($t = +1/2$ for the proton and $t = -1/2$ for a neutron). Often we will use the notation $\Psi_{+\frac{1}{2}}(x) \equiv \Psi_p(x)$ for the proton field and $\Psi_{-\frac{1}{2}}(x) \equiv \Psi_n(x)$ for the neutron field. Moreover,

$$\Psi(x) = \begin{pmatrix} \Psi_p(x) \\ \Psi_n(x) \end{pmatrix} . \tag{43}$$

The fields $\phi^{(+)}(x)$, $\phi^{(-)}(x)$ and $\phi^{(0)}(x)$ are the fields of π^+, π^- e π^0 mesons, respectively. It is convenient to use the "cartesian" components of such fields, i.e.,

$$\phi_1(x) = \frac{\left(\phi^{(+)}(x) + \phi^{(-)}(x)\right)}{\sqrt{2}} \ , \quad \phi_2(x) = \frac{i\left(\phi^{(+)}(x) - \phi^{(-)}(x)\right)}{\sqrt{2}} \ , \quad \phi_3(x) = \phi^{(0)}(x) \ . \tag{44}$$

Since $[\phi^{(+)}(x)]^\dagger = \phi^{(-)}(x)$, $\phi_i(x)$, $i = 1, 2, 3$, are hermitean fields, therefore for these fields we can assume $c_{\boldsymbol{k},i} = a_{\boldsymbol{k},i}$. The advantage of using these fields is due to the fact that under isospin transformations $\phi_i(x)$ transform as an isovector. Often then we will use also the notation $\vec{\phi}(x)$ to indicate the three pion fields. Moreover, hereafter, M will denote the nucleon mass and m the pion mass, $E_p = \sqrt{p^2 + M^2}$ and $\omega_k = \sqrt{k^2 + m^2}$ are the single particle energies. Note that under parity $U_P \phi_i(t, \boldsymbol{x}) U_P^\dagger = -\phi_i(t, -\boldsymbol{x})$, since the pions have negative intrinsic parity.

In the following we denote $\alpha \equiv \boldsymbol{p}, s, t$ and we will use standard time-ordered perturbation theory (also called "old-fashioned perturbation theory") to compute transition probabilities, then it is convenient to work with the interaction in Schroedinger picture (SP), which can be constructed in practice using the interaction picture fields at $t = 0$. Clearly, we must take care first to compute the time derivatives and then put $t = 0$. For example, the fields in SP are

$$\psi_t(x)|_{t=0} = \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{2E_p\Omega}} \left(b_{\boldsymbol{p},s,t} u(\boldsymbol{p}, s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + d^\dagger_{\boldsymbol{p},s,t} v(\boldsymbol{p}, s) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \ , \tag{45}$$

$$\partial_\mu \Psi_t(x)|_{t=0} = \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{2E_p\Omega}} \left(b_{\boldsymbol{p},s,t} u(\boldsymbol{p}, s)(-ip_\mu) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + d^\dagger_{\boldsymbol{p},s,t} v(\boldsymbol{p}, s)(ip_\mu) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) , \tag{46}$$

$$\phi_i(x)|_{t=0} = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\omega_k\Omega}} \left(a_{\boldsymbol{k},i} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{k},i} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right) \ , \tag{47}$$

$$\partial_\mu \phi_i(x)|_{t=0} = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\omega_k\Omega}} \left(a_{\boldsymbol{k},i}(-ik_\mu) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{k},i}(ik_\mu) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right) \ . \tag{48}$$

In SP the probability T of a process can be evaluated in "old-fashioned perturbation theory", as

$$T = H_I + H_I \frac{1}{E_i - H_0 + i\epsilon} H_I + H_I \frac{1}{E_i - H_0 + i\epsilon} H_I \frac{1}{E_i - H_0 + i\epsilon} H_I + \cdots \ , \tag{49}$$

where H_0 is the "free" Hamiltonian and H_I the interaction Hamiltonian (in SP).

3.1 Nucleon-pion scattering lengths

The simplest term describing nucleon-pion interaction is the following [2] (pseudo-scalar coupling):

$$\mathcal{H}_{PS} = g_\pi \overline{\Psi}(x)\gamma^5 \vec{\tau} \cdot \vec{\phi}(x)\Psi(x) \; , \tag{50}$$

or more explicitly

$$\mathcal{H}_{PS} = g_\pi \sum_{i,t',t} \overline{\Psi}_{t'}(x)\gamma^5 (\tau_i)_{t',t}\phi_i(x)\Psi_t(x) \; . \tag{51}$$

Note that $\mathcal{H}_{PS}$ has the correct transformation properties under Lorentz, and it is invariant under parity, charge conjugation, and isospin transformations [2].

Exercise 3: verify that the expression (51) is invariant under (proper) Lorentz and parity transformations.

The constant g_π is a dimensionless constant. From this term, using a second order calculation in perturbation theory, it can be derives the one pion exchange potential (OPEP),

$$v_{OPEP}(\boldsymbol{r}) = \frac{g_\pi^2}{12\pi}\frac{m^2}{4M^2}\vec{\tau}_1 \cdot \vec{\tau}_2 \left[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + S_{12}\Big(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\Big)\right] \frac{e^{-mr}}{r} \; . \tag{52}$$

Exercise 4: Compute the matrix element

$$T^{(2)} = \langle \alpha_1' \alpha_2' | H_I \frac{1}{(E_0 - H_I + i\epsilon} H_I | \alpha_1 \alpha_2 \rangle \tag{53}$$

giving the probability of the process $NN \to N'N'$. Above $\alpha_i \equiv \boldsymbol{p}_i, s_i, t_i$ ($\alpha_i' \equiv \boldsymbol{p}_i', s_i', t_i'$) are the momentum, spin projection and isospin projection of nucleon i in the initial (final) state. H_I is the Hamiltonian in Schrodinger picture, which in practice can be obtained as $H_I = \int d^3x \mathcal{H}(t=0,\boldsymbol{x})$. Consider as Hamiltonian density the one given in Eq. (51). Perform the calculation in the center-of-mass (CM). At the end, perform a non-relativistic approximation expanding the Dirac 4-spinors $u(\boldsymbol{p}, s)$ entering the nucleon fields in power of $1/M$, by retaining only the lowest order. The final result should be proportional to

$$T^{(2)} \sim \vec{\tau}_1 \cdot \vec{\tau}_2 \, \frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{k} \; \boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{\omega_k^2} \; , \tag{54}$$

where $(\vec{\tau}_1) \equiv (\vec{\tau})_{t_1',t_1}$, etc, and $\boldsymbol{k} = \boldsymbol{p}_1' - \boldsymbol{p}_1$.

From the large number experimental NN scattering data (in particular from the analysis of the "peripheral" phase shifts) we know that v_{OPEP} describes very well the long-range NN interaction. From an accurate fit of the experimental data it has been possible to extract the value of the constant g_π with the result $(g_\pi)^2/4\pi \approx 13.5$ [5], and then $g_\pi \approx 13.0$.

The term $\mathcal{H}_{PS}$ contributes also the the $\pi - N$ scattering. At very low energies, the cross section of the process

$$\text{nucleon of type } t + \text{pion of type } i \rightarrow \text{nucleon of type } t' + \text{pion of type } i' \ , \tag{55}$$

has been parametrizes as [6]

$$\sigma = 4\pi |a^+ \ \delta_{t,t'}\delta_{i,i'} + a^- \ \mathrm{i}\epsilon_{i,i',j}(\tau_j)_{t,t'}|^2 \ , \tag{56}$$

where a^+ and a^- are the so-called "even" and "odd" scattering lengths, respectively. Experimentally we know that [6]

$$a^- = (0.097 \pm .007)m^{-1} \qquad a^+ = (-0.015 \pm .015)m^{-1} \ , \tag{57}$$

where M is the nucleon mass and m the pion mass. Using the interaction (50), a second order calculation (of the perturbative series) gives

$$a^- = \frac{g_\pi^2}{4\pi}\frac{m^2}{2M^2}\frac{1}{m} \approx 0.14\frac{1}{m} \ , \qquad a^+ = -\frac{g_\pi^2}{4\pi}\frac{m}{M}\frac{1}{m} \approx -2\frac{1}{m} \ . \tag{58}$$

As is possible to see, a^- is of the correct order of magnitude, while a^+ is about 200 times larger than the experimental value. Calculations of subsequent orders considerably worsen this situation [6]. Two possible ways to solve this problem are:

1. the presence of additional terms in the Hamiltonian density that cancels the contribution of $\mathcal{H}_{PS}$ in a^+ without modifying substantially the result for a^-;

2. the replacement of the term $\mathcal{H}_{PS}$ in the Hamiltonian density with an interaction term containing a "4-gradient".

Le us discuss briefly this second possibility. In particular we propose to replace the pseudo-scalar coupling described by $\mathcal{H}_{PS}$ with the following "pseudo-vector" coupling

$$\mathcal{H}_{PV} = \frac{f}{m}\overline{\Psi}(x)\gamma^5\gamma^\mu\vec{\tau}\cdot\partial_\mu\vec{\phi}(x)\Psi(x) \ . \tag{59}$$

Exercise 5: prove that $\overline{\Psi}(x)\gamma^5\gamma^\mu\Psi(x)$ transform like an axial four-vector, and that $\mathcal{H}_{PV}$ is invariant under (proper) Lorentz transformation and parity.

Also an interaction term of this type gives the OPEP NN potential given in Eq. (52), once $f = (m/2M)g_\pi$ and therefore $f^2/4\pi \approx 0.075$ (note that the coupling constant f is much smaller than g_π). Furthermore, with $\mathcal{H}_{PV}$, the $\pi - N$ scattering lengths at second order comes out to be

$$a^- = -\frac{g_\pi^2}{4\pi}\frac{m^4}{8M^4}\frac{1}{m} \approx -0.001\frac{1}{m} \ , \qquad a^+ = \frac{g_\pi^2}{4\pi}\frac{m^3}{4M^3}\frac{1}{m} \approx 0.01\frac{1}{m} \ , \tag{60}$$

both very small and in reasonable agreement with the experimental data. This result essentially follows from the fact that the presence of the term $\partial_\mu\phi_i(x)$ in $\mathcal{H}_{PV}$ is proportional to the 4-momentum of the pion. So for pion three-momentum very close to zero, the term $\partial_\mu\phi_i(x)$ contributes only with the time component (proportional to m). This fact, combined with the assumption that $f = (m/2M)g_\pi$ gives the additional factor $(m/M)^2$ in Eq. (60).

This possibility (as well as the one mentioned above, namely the presence of additional terms in the Hamiltonian density that cancels the contribution of $\mathcal{H}_{PS}$) can be justified assuming that the Hamiltonian density of pions and nucleons is invariant under chiral transformations. Thus, chiral symmetry plays a very important role in the strong interaction between nucleons and pions, and in general for all nuclear physics. The pion-nucleon interaction is suppressed for small momenta (as seen by the the fact that the $\pi - N$ scattering lengths are small compared to their "natural" scale $\sim 1/m$). This makes the NN interaction, mediated mainly by pions, weak and therefore the binding energies of nuclei (per nucleon) turn out to be of the order of a few MeV, instead of the order of the strong energy scale, namely ~ 1 GeV.

3.2 The neutron and pion weak decay and the Goldberger-Treiman relation

At hadronic level and low energies, the weak interaction (the part arising from the exchange of $W^\pm$) of nucleons and pions is included by adding to the Hamiltonian density an interaction of the type [6]

$$\mathcal{H}_{\rm weak}(x) = \frac{G_F}{\sqrt{2}}\left[J_\mu^{(W)}(x)\right]^\dagger J^{(W)\mu}(x) \ , \tag{61}$$

where

$$J_\mu^{(W)}(x) = J_\mu^{(l)}(x) + J_\mu^{(h)}(x) \ , \tag{62}$$

and $J_\mu^{(l)}(x)$ ($J_\mu^{(h)}(x)$) is the leptonic (hadronic) part. The coupling constant $G_F \approx 1.15\ 10^{-5}$ GeV^{-2} is the Fermi constant, whose value has been extracted from the study of super-allowed beta decays [7]. The leptonic current is given by

$$J_\mu^{(l)}(x) = \sum_\ell \overline{\Psi}_{\nu_\ell}(x)\gamma_\mu(1-\gamma^5)\Psi_\ell(x) \ , \tag{63}$$

where ℓ runs over the different leptonic families (electrons, muons, and τ's), while $\Psi_\ell(x)$ and $\Psi_{\nu_\ell}(x)$ are the lepton fields of type ℓ and of the corresponding neutrino, respectively. The hadronic current instead is very complicated by the fact that hadrons are systems formed by quarks. After many studies it has been seen that $J_\mu^{(h)}(x)$ can be written as [7, 6]

$$J_\mu^{(h)}(x) = V_\mu(x) - A_\mu(x) \ , \tag{64}$$

namely in a "vector" and "axial" part. More precisely, under parity $V_\mu(x)$ transforms like a four-vector while $A_\mu(x)$ as a pseudo-four-vector.

The vector part is closely related to the electromagnetic hadronic current, in fact in a first approximation

$$J^\mu_{e.m.}(x) \sim \overline{\Psi}(x)\gamma^\mu \frac{1+\tau_z}{2}\Psi(x) \ , \qquad V^\mu(x) \sim \overline{\Psi}(x)\gamma^\mu\tau_+\Psi(x) \ . \tag{65}$$

In fact, $J_\mu^{em}(x)$ represents the current of a particle of spin $\frac{1}{2}$ and the operator $(1+\tau_z)/2$ projects $\Psi(x)$ on $\Psi_p(x)$. In the weak current operator τ_+ gives $V_\mu(x) \sim \overline{\Psi}_p(x)\gamma_\mu\Psi_n(x)$, thus it describes the transformation of a neutron into a proton. With the further distinction of the electromagnetic current in a isoscalar (IS) part and (IV) isovectorial part

$$\begin{aligned} J_\mu^{e.m.}(x) &= J_\mu^{IS}(x) + J_\mu^{IV}(x) \ , \\ J_\mu^{IS}(x) &\sim \frac{1}{2}\overline{\Psi}(x)\gamma_\mu\Psi(x) \ , \qquad J_\mu^{IV}(x) \sim \overline{\Psi}(x)\gamma_\mu\frac{\tau_z}{2}\Psi(x) \ , \end{aligned} \tag{66}$$

since $\tau_+ = (\tau_x + \mathrm{i}\tau_y)/2$, we can note that $J_\mu^{IV}(x)$ and $V_\mu(x)$ are linear combinations of the following "isospin current",

$$\vec{J}_\mu(x) = \overline{\Psi}(x)\gamma_\mu\vec{\tau}\Psi(x) \ . \tag{67}$$

This is actually the current derived from the Noether theorem from the $SU(2)_V$ isospin symmetry. Given that the isospin symmetry is almost exact in Nuclear Physics, usually it is assumed that the current $\vec{J}_\mu(x)$ is conserved (Vector Current Conservation - CVC) and that $V_\mu(x)$ can be inferred from $J_\mu^{IV}(x)$ in a formal way by a rotation in isospin space [6]. Since $J_\mu^{em}(x)$ is well known due to the many experiments with electromagnetic probes (see for example Ref [6]), the current $V_\mu(x)$ is rather well determined from the $J_\mu^{IV}(x)$ as mentioned above, with a rotation in isospin space.

The axial part of the weak current rather is not so well-known. In practice, in a first approximation we can assume

$$A_\mu(x) = g_A \overline{\Psi}(x)\gamma_\mu\gamma^5\tau_+\Psi(x) + \frac{f_\pi}{\sqrt{2}}\partial_\mu \left[\phi^{(+)}(x)\right]^\dagger + \cdots \ , \tag{68}$$

where g_A, f_π are constants to be fixed with the experimental data, while the dots "$\cdots$" indicates that there may be (many) other terms in the expression of $A_\mu(x)$. The constant g_A is fixed in order to reproduce the experimental data of neutron lifetime, giving $g_A = 1.267 \pm 0.01$ [2]. The term proportional to $\partial_\mu\phi^{(+)}(x)$ describes the charged pion decay into muons. The value of f_π is fixed by the average lifetime of charged pions, resulting in $f_\pi \approx 92.4$ MeV [2]. This constant is known as the "pion decay constant".

Finally, as noted by Goldberger and Treiman [8], it appears to be a relation between the values of the constants g_π, g_A, f_π and the nucleon mass, in fact it is possible to verify that

$$\frac{f_\pi}{g_A} \approx \frac{M}{g_\pi} \ , \tag{69}$$

is verified to 2%. The expression above relates constants which enter the strong interaction (M and g_π) with constants which enter the weak interaction (f_π and g_A) and it is rather difficult to explain. However, assuming that the axial current it is related (via the Noether's theorem) to the symmetry $SU(2)_A$, it is possible to show that starting with a Hamiltonian density which is invariant (approximately) under the group $G = SU(2)_V \otimes SU(2)_A$, the Goldberger and Treiman follows rather naturally [2].

3.3 The PCAC

In the pion decay $\pi^+ \to \mu^+ + \nu_\mu$, the relevant matrix element to be calculated is the following

$$\langle 0|A_\mu(x)|\pi^+; k\rangle \ , \tag{70}$$

where $|\pi^+; k\rangle$ is the state of a pion with four-momentum k^μ. The matrix element (70) should transform like a four-vector under Lorentz transformations and therefore it must be proportional to k^μ, being the only four-vector available. Since this matrix element should be also invariant under translations, we have

$$\langle 0|A_\mu(x)|\pi^+; k\rangle = \mathrm{i}k_\mu f_\pi(k^2)e^{-\mathrm{i}k\cdot x} \ , \tag{71}$$

where $f_\pi(k^2)$ give a possible dependence on $k^2 = k_\mu k^\mu$ (this function id the so-called pion form factor). For free pions ("on-mass-shell") $k^2 = m^2$. Comparing with Eq. (68), we can see that $f_\pi(m^2) \equiv f_\pi$, the pion decay constant. From Eq. (71) we can deduce that

$$\langle 0|\partial^\mu A_\mu(x)|\pi^+;k\rangle = m^2 f_\pi e^{-\mathrm{i}k\cdot x} \ . \tag{72}$$

If the mass of the pion would be zero, $\partial^\mu A_\mu(x) = 0$ and the axial current would be conserved. In general it is assumed that

$$\partial_\mu A^\mu(x) = m^2 f_\pi \phi^{(+)}(x) \ , \tag{73}$$

known as the hypothesis of "partial conservation of the axial current" (PCAC) [4]. In the sixties the study of processes with low energy pions led to the "discovery" of equalities that put in relation the probability to the emission of " soft" $n+1$ pions (i.e. of low energy), with the probability of the emission of n such pions. These identities were known as the "soft pion theorems." These theorems can be derived by assuming valid the relation (73) [4]. Even the validity of PCAC is closely linked to chiral symmetry in the hadronic Hamiltonian density.

3.4 Pion-pion scattering length

Finally, recently there has been a considerable effort to measure the pion-pion scattering length. The latest results were provided by experiment E865 at Brookhaven [9] and the NA48 at CERN [10], which combined with other experiments have provided the result $a_{\pi\pi} = 0.217 \pm 0.01\, m^{-1}$. The importance of this measure is related to the fact that the theoretical estimate based on the chiral Lagrangian can be done without any free parameters. So the comparison between this experimental data and the theoretical estimate is a really important test of the theory.

The latest calculation [11] based on the chiral Lagrangian provides the estimate $a_{\pi\pi} = 0.220 \pm 0.005 m^{-1}$, where the theoretical error is related to the uncertainty in the knowledge of the parameters f_π, and so on. This result is in very good agreement with the experimental value mentioned above.

4 The effective field theory

At the end of the sixties, models were derived in order to write Lagrangians with pion and nucleon degrees of freedom satisfying chiral symmetry (linear sigma model, the non-linear sigma model, etc., see for example [2, 4]). These models incorporated also the $SU(2)_A$ spontaneously broken symmetry. The most significant example it is the non-linear sigma model, which it is a non-linear realization of chiral symmetry.

In the following, we'll use the Lagrangian formalism (see, for example Cap. 7 of Ref. [1]), since it is easy to discuss the implementation of symmetries. The Hamiltonian density can be obtained then using the so-called Legendre transformation, namely

$$\mathcal{H}(x) = \sum_\ell \frac{\delta \mathcal{L}(x)}{\delta \dot{\psi}_\ell} \dot{\psi}_\ell - \mathcal{L}(x) \ , \tag{74}$$

where ℓ runs over all the different fields ψ_ℓ present in the Lagrangian (in our case, nucleon and pion fields). However, in practice, the interaction part can be obtained simply as $\mathcal{H}_I(x) = -\mathcal{L}_I(x)$ plus some high order term.

The pion field is chosen to enter the Lagrangian only in terms of the 2×2 matrix

$$u = e^{i\vec{\phi}\cdot\vec{\tau}/2f_\pi} \ , \tag{75}$$

which can be shown [4] to transform under G as

$$u' = Luh^\dagger = huR^\dagger \ , \qquad hh^\dagger = I_{2\times 2} \ , \tag{76}$$

L, R being the left and right rotation matrices given in Eqs. (35) and (36). Above, h is a 2×2 matrix depending in a complicate way on L, R, and $\vec{\phi}$. The matrix h is called the "compensator field". In case of vector transformation (i.e. isospin) $L = R = V$, we have $u' = VUV^\dagger$, and so in this case the compensator field will become independent on the pion field and coinciding with V.

To construct the Lagrangian with nucleonic fields, it is convenient to introduce a new field $N = u\Psi_R + u^\dagger \Psi_L$ that under the chiral transformations transforms as

$$N' = (hur^\dagger)\, R\Psi_R + (hu^\dagger L^\dagger)\, L\Psi_L = hN \tag{77}$$

as follows from the relations (76). So under vector transformations $h = V$, the field N transforms as a doublet of isospin. Under axial transformations h becomes a complex function of $\vec{\theta}_A$ and $\vec{\phi}_\pi$ and the transformation is non-linear.

In this case, it is convenient to introduce some quantities that transform under G in the following way: $\mathcal{O}_i' = h\mathcal{O}_i h^\dagger$ and then write down all possible terms of the form: $\overline{N}\mathcal{O}_1 \cdots \mathcal{O}_n N$. One of this quantity is the covariant derivative of the field u, given by:

$$u_\mu = i\Big(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger\Big) = -\frac{\vec{\tau}\cdot\partial_\mu\vec{\phi}}{f_\pi} + \mathcal{O}\left(\vec{\phi}^3\right) \ , \tag{78}$$

which under chiral transformations transforms precisely as

$$u'_\mu = h u_\mu h^\dagger \ . \tag{79}$$

The 4-gradient of the field N, namely $\partial_\mu N$, does not transforms in the simple way $(\partial_\mu N)' = h\partial_\mu N$ since also the compensator field depends in general on x. It is convenient to introduce the "covariant derivative"

$$D_\mu N \equiv (\partial_\mu + \Gamma_\mu) N, \quad \text{with} \quad \Gamma_\mu \equiv \frac{1}{2}\left(u^\dagger \partial_\mu u + u\partial_\mu u^\dagger\right) \ . \tag{80}$$

Exercise 6: show that $D_\mu N \to (D_\mu N)' = hD_\mu N$.

Then the most general Lagrangian written in terms of pion and nucleon fields which results to be invariant under Lorentz transformations, G, parity, and charge conjugation, up to terms with a four-gradient, is

$$\mathcal{L}_{\sigma \ \text{non-lin.}} = \overline{N}\left(i\gamma^\mu D_\mu - M + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu\right) N + \mathcal{L}_{\text{mes.}} \ , \tag{81}$$

where M and g_A are respectively the nucleon mass and coupling axial-vector constant.

Let us now discuss the physical content of Lagrangian (81). We note that the pion-nucleon interaction terms are two: $\overline{N} i\gamma^\mu \Gamma_\mu N$ and $\overline{N} i\gamma^\mu\gamma^5 u_\mu N$. Expanding Γ_μ and u_μ in powers of the pion field we have

$$\mathcal{L}_I = \overline{N}\left(-\frac{g_A}{2f_\pi}\gamma^\mu\gamma_5 \vec{\tau}\cdot\partial_\mu\vec{\phi}\right) N + \mathcal{O}(\pi^2) \ . \tag{82}$$

The pion-nucleon interaction is threfore predicted to be of the pseudo-vector type (see (59). As we know, with this type of interaction we obtain the OPEP (the tail of the long-range potential NN) and a reasonable agreement for the $\pi - N$ scattering lengths. The pseudo-scalar interaction is not allowed since it violates chiral symmetry.

The axial current can be directly derived from Noether's theorem

$$\vec{A}_\mu(x) = g_A \overline{N}(x)\gamma_\mu\gamma^5\vec{\tau}N(x) + f_\pi \partial_\mu\vec{\phi}(x) + \mathcal{O}(\phi^2) + \dots \ . \tag{83}$$

Comparing with Eq. (68), we can justify the choice we have made to define u with the factor $1/f_\pi$ and g_A constant to multiply the third term in Lagrangian (81). In addition, by comparing Eq. (82) with $\mathcal{H}_{\mathcal{PV}}$ given in Eq. (59) we can observe that:

$$\frac{g_A}{2f_\pi} \equiv \frac{f}{m} = \frac{g_\pi}{2M} \longrightarrow \frac{g_A}{f_\pi} = \frac{g_\pi}{M}, \tag{84}$$

namely the Goldberger–Treiman relation (69). The Lagrangian (82) includes also many other "vertices" with two, three, etc, pion fields. In addition, it is possible also to write down many other "chiral symmetric" terms, with two, three four-gradiens etc. (see, for example Ref. [12]).

In addition to pion-nucleon interaction terms, also many "contact" terms of the type $\overline{N}\ldots N\overline{N}\ldots N$ can be included in principle in the Lagrangian, where "..." may be for example combinations of gamma matrices, four-gradients, etc. These terms simulate the exchange of more massive mesons (as ρ mesons). In practice, each of these terms is multiplied by a unknown coupling constant (also called "low-energy constant"), which can be fixed by comparing with some experimental data. Also these terms can be organized in term of the number of four-gradients. Moreover, many of the possible terms can be shown to be linearly dependent. It results that there exist 2 independent terms without four-gradients, 7 with one four gradient, etc. (see, for example, Ref. [13]).

This model allows to study systems of pions and nucleons with good results until low order perturbative contributions are considered. Considering larger orders of the perturbative expansion one encounters divergent expressions involving integrations over one or more momenta ("loops"), so it arises the problem how to obtain finite results.

This problem can be overcome if one is interested in the study of low energy processes, compared to a high energy scale, hereafter indicated with Λ_X. In our case for low energy we mean processes involving pions and nucleons having momenta Q of the order (at most) of the mass of the pion, which occurs for the standard problems of Nuclear Physics. The scale of high energy is represented by the typical scale of the strong interactions $\Lambda_X \approx 1$ GeV. The idea is to use this scale difference to calculate the quantities of interest as a sum over powers of Q/Λ_X. To give a practical example, consider the calculation of an matrix element T between an initial state $|\mathrm{IN}\rangle$ and the final $|\mathrm{FIN}\rangle$, related to the probability of a generic transition between the two states. In our theory the calculation of $\langle FIN|T|\mathrm{IN}\rangle$ can be arranged as a sum of terms, each proportional to the factor $(Q/\Lambda_X)^\nu$, where ν typically is an integer index. The theory only works if ν cannot assume arbitrarily large negative values (better if $\nu \geq 0$). In this case, we can organize the calculation of $\langle\mathrm{FIN}|T|\mathrm{IN}\rangle$ first considering the terms with lowest index $\nu = \nu_{\mathrm{min}}$ (leading order – LO). The terms of order $\nu = \nu_{\mathrm{min}} + 1$ (next-to-leading order – NLO) will bring the first corrections to the LO results and so on. In this way one has a systematic calculation of $\langle\mathrm{FIN}|T|\mathrm{IN}\rangle$ which in principle can be arbitrarily improved. It can be proved, in particular since the pions couple to the nucleons only via a derivative coupling, that for the chiral Lagrangian discussed above there exist always a minimal index ν_{min}. This calculation technique is called chiral perturbation theory (CPT). The index ν is often called the chiral " index ".

There is still the problem of the divergent loop contributions. As discussed by Weinberg [14], in principle, the Lagrangian must contain all possible terms consistent with the symmetries that are believed to be important, not only the simplest terms, as for example in the nonlinear sigma model (81). Since generally there will exist an infinite number of terms compatible with the symmetries of the theory, there are also infinitely many terms in the Lagrangian ("vertices"), each of them coming with a unknown coupling constant, which we can use to absorb all the divergences related to the diagrams with loops. These collections of coupling constants are known as "low-energy coupling constants" (LEC's).

The theory remains predictive since, as said above, at each order in $(Q/\Lambda_X)^\nu$ only a *finite* number of terms contribute. Suppose you want to make a calculation up to the order ν_0, i.e. including only the terms up to $(Q/\Lambda_X)^{\nu_0}$. In the Lagrangian one has to consider all terms compatible with the symmetries of the system which can give contributions up to the chosen order ν_0. Suppose that there are N_0 of such terms. Then, there will be N_0 arbitrary LEC's, which will can absorb all the "ultraviolet" divergences associated with the diagrams with loops. In fact, even the divergent loops should verify the symmetries of the Lagrangian and they can be at most N_0. One then assumes that the LEC's have a infinite part, which cancel exactly the divergences, and a finite part, which then can be fixed by comparing the (now finite) results of the theoretical calculations to (a minimum of) N_0 experimental data. At this given order then the theory has not anymore free parameters and can be used to make predictions of other observables. A theory so organized it is called an "effective field theory" (EFT), and it is used in many fields of physics, in particular when one is only interested it the low-energy regime. If one wants to do a calculation to the next order, it is necessary 1) find in the Lagrangian all new terms which can contribute (each of one will be accompanied by a new LEC), 2) perform the calculation including all the contributions of order $(Q/\Lambda_X)^{\nu_0+1}$ absorbing all the divergences with the new LEC's; 3) use an additional set of experimental data to fix the values of the (finite part of) the new LEC's; 4) at this point one has again a a predictive theory and so on. Note that in principle, the LEC's may be calculated using the underlying fundamental theory (in our case, Q.C.D.), but if this is not possible, in practice one uses set of experimental data.

This program has been carried out in many fields. In particular for a system of pions there are virtually no free parameters, and then it is possible to use the theory to do predictions, as the above-mentioned calculation of the $\pi - \pi$ scattering length. Regarding the study of the force between two nucleons, the program was carried out in a series of works [12], culminating in calculations up to order Q^4 [16, 15]. (see also the review article [17]).

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